www.ijecs.in
International Journal Of Engineering And Computer Science ISSN:2319-7242
Volume 2 Issue 4 April, 2013 Page No. 1192-1199

Quantification Analysis of Chaotic Fractal Dimensions

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Abstract

Fractals appear from a various sources and have been observed in nature .One of the substantial characteristics of fractals is that they can be described by a non-integer dimension. The geometry and the mathematics of fractal dimension have contributed useful tools for a variety of scientific speciality. The fractal dimension quantifies its dimension across the curves and trajectories. In recent years, various numerical methods have been developed for quantifying the dimension directly from the observations of the natural system. The purpose of this paper is to quantify dimensions of fractals that arise in nature by two fractal quantifiers to quantify the dimensions i.e. compass dimension and box counting dimension thereby deducing an algorithm of chord length and the number of solution steps used in computing fractacality. Results demonstrate that trajectory's fractal dimension can be nearly approximated. We expect this paper could make the fractal theory understood absolutely, and could expand fractal application in numerous fields.

Keywords:-Fractal Dimension, Box dimension , Divider Dimension, Natural Fractals

I. INTRODUCTION

The twin subjects of fractal geometry and chaotic dynamics have been behind an enormous change in the way scientists and engineers perceive, and subsequently model, the world in which we live. Chemists, biologists, physicists, physiologists, geologists, economists, and engineers (mechanical, electrical, chemical, civil, aeronautical etc) have all used methods developed in both fractal geometry and chaotic dynamics to explain a multitude of diverse physical phenomena: from trees to turbulence, cities to cracks, music to moon craters, measles epidemics, and much more. Many of the ideas within fractal geometry and chaotic dynamics have been in existence for a long time, however, it took the arrival of the computer, with its capacity to accurately and quickly carry out large repetitive calculations, to provide the tool necessary for the in-depth exploration of these subject areas. In recent years, the explosion of interest in fractals and chaos has essentially ridden on the back of advances in computer development.

There are many definitions of a fractal. Possibly the simplest way to define a fractal is as an object which appears self-similar under varying degrees of magnification. A diagram is possibly the best way to illustrate what is meant by a fractal object.. As we zoom into the coastline, we find that its ruggedness is repeated on finer and finer scales, and under rescaling looks essentially the same: the coastline is a fractal curve. The person, however, is not a self-similar object. As we zoom into various parts of the body, we see quite different forms. The hand does not resemble the whole body, the fingernail does not look like the hand and so on. Even viewing different parts of the body at the same scale, say the hand and the head, we would see that again they are not similar in form. We conclude that a person is not a fractal object. It is interesting to note at this stage that, although the body as a whole is not a fractal object, recent studies have attempted, with some success, to characterize certain parts of the body using fractal geometry, for example, the branching structure of the lung and the fine structure of the neuron (brain cell).

four natural fractals: the boundary of clouds, wall cracks, a hillside silhouette and a fern. All four possess self-similarity The first three natural fractals possess the same statistical properties (i.e. the same degree of ruggedness) as we zoom in. They possess statistical self-similarity. On the other hand, the fern possesses exact self-similarity. Each frond of the fern is a mini-copy of the whole fern, and each frond branch is similar to the whole frond, and so on. In addition, as we move towards the top of the fern we see a smaller and smaller copy of the whole fern. The fractals require a two-dimensional (2D) plane to 'live in', that is all the points on them can be specified using only two co-ordinates. Put more formally, they have a Euclidean dimension of two.

II. LITERATURE REVIEW

Haudroff dimension

we used the similarity dimension to produce fractal dimensions for fractal objects. There are, however, many more definitions of dimension which produce fractal dimensions. One of the most important in classifying,fractals is the Hausdorff dimension. In fact, Mandelbrot suggested that a fractal may be defined as an object which,has a Hausdorff dimension which exceeds its topological dimension. A complete mathematical description of the,Hausdorff dimension is outside the scope of this text. In addition, the Hausdorff dimension is not particularly useful,to the engineer or scientist hoping to quantify a fractal object, the problem being that it is practically impossible to calculate it for real data. We therefore begin this section by concentrating on the closely related **box counting, dimension** and its application to determining the fractal dimension of natural fractals before coming to a brief explanation of the Hausdorff dimension.(i) *The box counting dimension.* To examine a suspected fractal object for its box counting dimension we cover the,object with covering elements or 'boxes' of side length δ. The number of boxes, *N*, required to cover the object is related to δ through its box counting dimension, *DB.* The method for determining *DB* is illustrated in the simple

example of figure 1.1, where a straight line (a one-dimensional object) of unit length is probed by cubes (3Dobjects) of side length δ. We require *N* cubes (volume δ3) to cover the line.

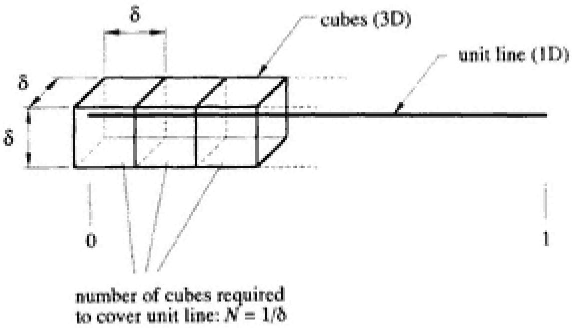

Figure 1 .A line (1D) using boxes (3D).

.obtained by covering the object with *N* hyper cubes of side length δ. Note that the above expression is of rather limited use. It assumes the object is of unit hyper volume and in general will produce erroneous results for large δ The definition of the similarity dimension *DS* given in equation However, do not confuse the two: the calculation of *DS* requires that exactly self-similar parts of the fractal are identified, whereas *DB* requires the object to be covered with self-similar boxes. Hence, *DB* allows us greater flexibility in the type of fractal object that may be investigated.

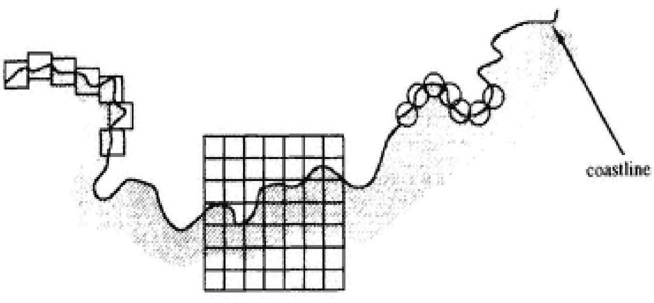

Figure 2.Determining the fractal dimension of a coastline using the box counting method.

$$D_B = \frac{\log(N)}{\log(1/\delta)} \quad (3.1)$$

The general expression for the dimension of an object with a hypervolume (i.e. length, area, volume or fractal hypervolume) not equal to unity, but rather given by *V**, is

$$D_B = \frac{\log(N) - \log(V^*)}{\log(1/\delta)} \quad (3.2)$$

where *N* is the number of hypercubes of side length δ required to cover the object, i.e. *N* = . Rearrange equation (3.2) gives

$$\log(N) = D_B \log(1/\delta) + \log(V^*) \quad (3.3)$$

which is in the form of the equation of a straight line where the gradient of the line, *DB*, is the box counting dimension of the object. This form is suitable for determining the box counting dimension of a wide variety of fractal objects by plotting log(*N*) against log (1/δ) for probing elements of various side lengths, δ. Figure 1.2 illustrates three popular methods of covering a coastline curve using boxes and circles to obtain a box counting dimension estimate. One may place boxes against each other to obtain the minimum number required to cover the curve. Alternatively, one may use a regular grid of boxes and count the number of boxes, *N*, which contain a part of the curve for each box side length δ. Circles of diameter δ may also be used as probing elements to cover the curve, placing them so that they produce the minimum covering of the curve. In this case, δ corresponds to the diameter of the covering circles. Whichever method is used, we obtain the box counting dimension from the limiting gradient (as δ tends to zero) of a plot of log(N) against log(l/δ), i.e. the derivative

$$D_B = \lim_{\delta \to 0} \frac{d(\log(N))}{d((\log(1/\delta))}. \quad (3.4a)$$

. In practice, the box counting dimension may be estimated by selectingtwo sets of [log(1/δ) log(*N*)] co-ordinates at small values of δ (i.e. large values of log(1/δ)). An estimate of *DB* is then given by

$$D_B = \frac{\log(N_2) - \log(N_1)}{\log(1/\delta_2) - \log(1/\delta_1)} \quad (3.4b)$$

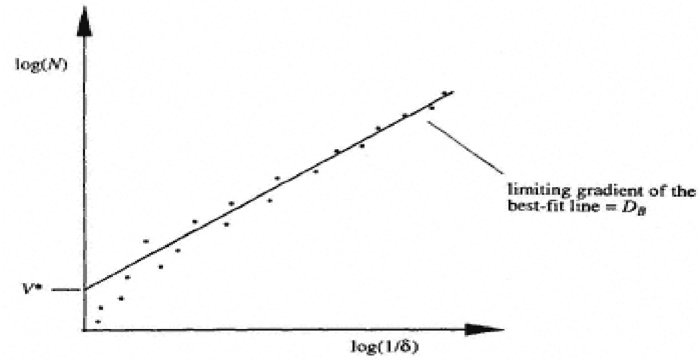

Figure 3.Estimating the box counting dimension of experimental data.

Alternatively, a more refined estimate may be obtained by drawing a best fit line through the points at small values of δ and calculating the slope of this line (see figure1.3). For this case the *N* and δ values of equation (3.4*b*) are taken from two points on the best-fit line. This is particularly advisable where the data fluctuate at the limits of resolution. The box counting dimension is widely used in practice for estimating the dimension of a variety of fractal objects. The technique is not confined to estimating the dimensions of objects in the plane, such as the coastline curve. It may be extended to probe fractal objects of high fractal dimension in multi-dimensional spaces, using multidimensional covering hypercubes. Its popularity stems from the relative ease by which it may be incorporated into computer algorithms for numerical investigations of fractal data. The grid method lends itself particularly to encoding within a computer program. By covering the data with grids of different box side lengths, δ, and counting the number of boxes, *N*, that contain the data, the box counting dimension is easily computed using equation (3.4*b*).

When using the box counting dimension we are asking, 'How many boxes or hypercubes do we need to cover the object?': in this case we only need use probing elements, or hypercubes, which have an integer dimension equal to or exceeding that of the object. In contrast, when using the Hausdorff dimension we are asking instead, 'What is the 'size' of the object?', that is we are trying to measure it. To measure its size or hypervolume we need to use the appropriate dimension of covering hypercubes, this appropriate dimension being the Hausdorff dimension *DH*. In the rest of this section a brief overview of the Hausdorff dimension is given for completeness of the text.

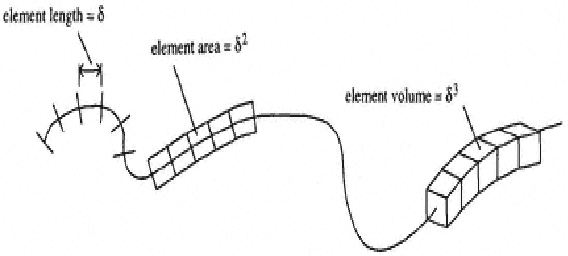

Figure 4.Measuring a smooth curve.

The Structured Walk Technique and the Divider Dimension

A commonly used method for determining a fractal dimension estimate of fractal curves in the plane is the structured walk technique, illustrated in figure 5 The technique is much faster to perform by hand than the box counting dimension and requires the use of a compass or a set of dividers. (A ruler may also be used if neither of the first two pieces of equipment is available, but this does result in a more laborious task.) The method is outlined as follows.

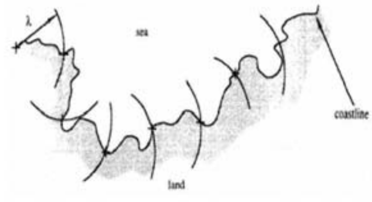

Figure 5 Structured walk technique

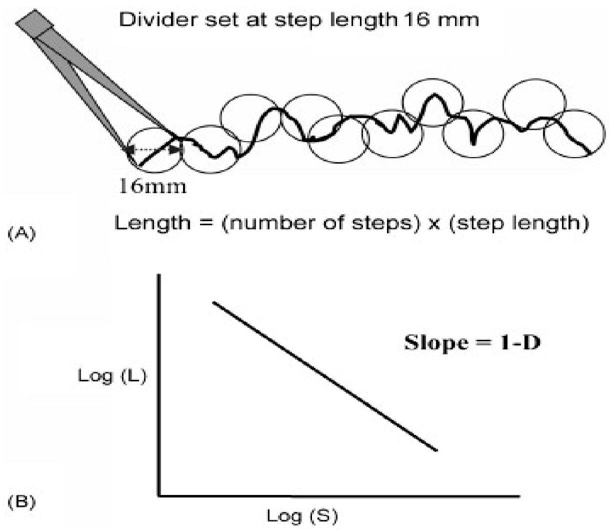

Figure 6 Divider Dimension

Hence, the dimension of the curve may be found by measuring S from the best fit line of the plotted points of steps. The slope of the Richardson plot is negative, i.e. the best fit line falls from left to right, thus DD > 1. Note that when drawing successive arcs one may obtain slightly different dimension estimates depending upon the direction of approach of the arc. One may repeatedly swing clockwise into the coastline from the 'sea' (the inswing method), anti-clockwise out of the coastline from the 'land' (the outswing method), or alternate between the two (the alternate method). It is good practice to try all three methods and, in addition, to use various starting locations on the curve.

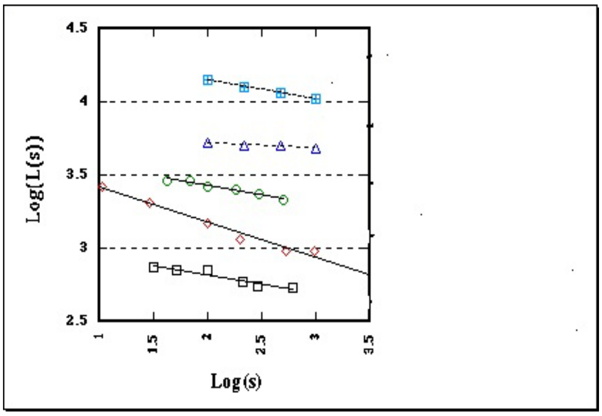

Figure 7 Richardson plot of country boundaries.

Figure.7 contains a Richardson plot of original data by L F Richardson who noted that reported lengths of the border between two countries were often claimed to be different by the two countries involved. For example, he noted that the Spanish–Portuguese border was stated as being 987 km and 1214 km by Spain and Portugal respectively, and similarly the Dutch–Belgian border was stated as being 380 km and 449 km respectively by the two countries. After investigation he reasoned that the differences could be attributed to the length of the measuring stick used in the calculation of the boundary length. The smaller the measuring stick length, λ, the longer the measured length L was found to be. On the Richardson plot of figure 3.11 the data points for a circle are also plotted. Notice that the circle boundary slope tends to zero for small values of λ, as the divider dimension, DD, tends to the topological dimension, DT (= 1). This implies that the circle boundary is not a fractal, and more, in that it is a smooth curve with measurable length.

Figure8(a) contains a simulated soot particle made up of circles connected tangentially. The outer boundary of the particle reproduces the general features of profiles typically found in agglomerations of soot particles from exhaust emissions. Figure 8(b) contains the Richardson plot of the particle boundary in figure 8(a). Notice the two distinct slopes associated with the structure and texture of the particle.
The divider dimension is therefore an extremely useful tool. However, its main shortcoming is that it is limited to the investigation of curves in the plane. Should we wish to measure fractal objects other than curves, say for example the surface of a cloud or fractal landscape, the box counting dimension should be used; this, due to its versatility, may be used to probe all manner of fractal objects occurring in multi-dimensional spaces.

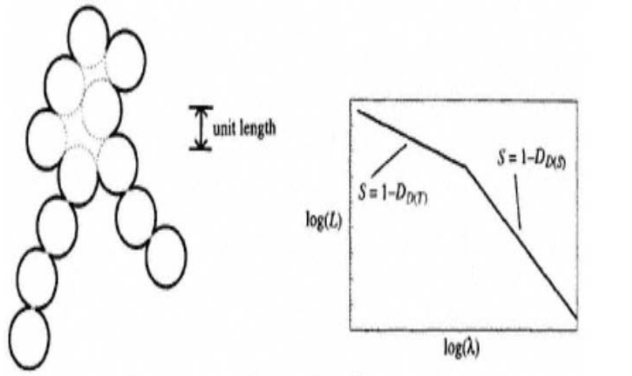

Figure 8. Richardson plot of a synthetic particle boundary.(a) Synthetic particle. (b) Richardson plot.


Figure 9.A selection of natural fractal objects.

Both the divider and box counting dimensions have been used to measure the fractal dimension of many natural fractals, including of course real coastlines and fine particle boundaries, other examples include (see figure 9) cloud boundaries, smoke plume boundaries, chromatograph diffusion fronts, landscape profiles, and so on. Both dimension estimates are also useful in the estimation of the fractal dimension of crossing curves, such as fBm, We leave this section by looking at the relationship between the box counting dimension and divider dimension on a fractal curve. First, we consider the box counting dimension. Rearranging equation (3.2) for non-unit hypervolumes we obtain where D is the box counting dimension. As an aid to clarity in the following discussion, we omit the subscript B of
the box counting dimension and include δ in parenthesis to denote that N is a function of the box size δ.

II. PROPOSED WORK

DEVELOPMENT OF THE FRACTAL TRAJECTORY ALGORITHM

The following original algorithm is based on the earlier empirical work performed by Richardson (1961) and later extended by Mandelbrot (1967).Richardson measured the lengths of several frontiers by manually walking a pairof dividers along the outline so as to count the number of steps. The opening of the dividers (n) was fixed in advance and a fractional side was estimated at the end of the walk. The main purpose in this section of Richardson's research was to study the broad variation of In with n.
Richardson produced a scatterplot in which he plotted log total length against log step size for five land frontiers and a

circle. Mandelbrot (1967) discovereda relationship between the slope (8) of the lines and fractal dimension (D). To Richardson the slope had no theoritical meaning, but to Mandelbrot it could beused as an estimate of 1-D, which leads to:

$$D=1-\beta \qquad (1)$$

The algorithm simulates walking a pair of dividers along a curve and counts the number of steps. In cases where more than one intersection occurs, the intersection which comes first in order forward along the curve is selected. To be more accurate, step size (prescribed opening of the dividers) is called chordlength (cl) and the number of steps is called the number of chord lengths.

In order to begin walking the dividers along the curve, the dividers must be set to some opening. The curves used in this research are not infinitely subdivided fractal curves so that selection of the initial chord length must be based on some attribute of the curve. For a very contorted curve it would be meaningless to choose a chord length many times shorter than the shortest line segment. If an extremely short chord length is selected, an attempt to examine the fractal character of a curve would extend beyond the primitive sub elements used to represent the geometry of the resulting form. In other words, beyond this lower limit of primitive subelements, the curve's fractal dimension behaves as if it is a straight line. A suggested initial chord length is determined by calculating the distance between each two consecutive points on the curve and taking 1/2 the average distance. The average distance is divided by 2 because the sampling theorem states one should sample at 1/2 the wavelength so that no significant variation escapes. This presents an approximate lower limit as to the selection of the initial chord length. Although the accuracy of this method is dependent on the manner in which the curve is digitized, the form of the curve often dictates this manner.

After the initial chord length is determined, the algorithm computes the distance between the first two points on the curve using the standard distanceformula. If the distance is greater than chord length (L), a new point is interpolated between points 1 and 2 using the following interpolation equations:

$$D = (L-Dl) / (D1-DA) \qquad (2)$$
$$Anew = A\ 1 + DP^* (A2-Aj) \qquad (3)$$
$$Bnew = BJ + DP^* (B\ Z - B\ I\ ) \qquad (4)$$

where D = distance proportion

D1 = distance between the present point and the first forward point on the curve

DA = distance between the present point and the second forward point on the curve

Anew = new A- coordinate

Bnew = new B- coordinate

A,B = A and B coordinates of point 1 and 2.

If the distance is less than the chord length, the distance between points 1 and 3 is computed. If the distance is less than the chord length is greater than the chord length, it is known that the chord length segment intersects between points 2 and 3 and that the distance between these points is determined .

The point of intersection is computed using trigonometric functions. An angle C is Determined using the law of cosines.

$$C= \cos^{-1} (DISTY^2 + DISTX^2 - DISTZ^2)/2^* DISTY * DISTX \qquad (5)$$

Since angle C is known, an angle A, which is the angle the chord length intersects between points 2 and 3, can be computed.

$$A=SIN^{-1} ((DISTX^*sinZC)/L) \qquad (6)$$

Now that two angles are known, angle 3 is easily computed. Because angles A and B are known, a side (DISTY') can be calculated;

$$DISTY' = (DISTX^*sinY)/sinX \qquad (7)$$

DISTY' provides the distance, from point 2, in which the chord length's intersection is located on the segment between points 2 and 3. A distance proportion D is calculated using

$$D=DISTY'/DISTY \qquad (8)$$

Since the distance proportion and the A,B coordinates for points 2 and 3 are known, the equations used to interpolate for a straight line segment can be used to determine the new coordinates. After the new point is located, this new point becomes point 1 and the next two forward points on the curve become points 2 and 3. Each time a chord length's intersection is determined, 1 is added to the number of chord lengths.

In the case where DISTZ is less than the chord length, the third point is incremented by 1 (fourth point) and the distance again checked. This continues until the distance is greater than the chord length or the end of the curve is encountered

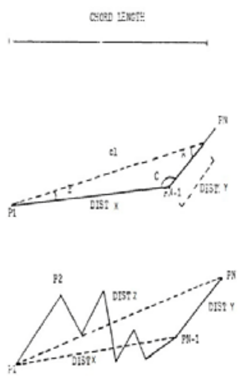

Figure 10. More than three point interpolation

When the distance does become greater than the chord length, the chord length's point of intersection is determined by using the same trigonometric equations as discussed above. The only difference is the sides of the triangles may

be longer. At the end of the curve, if the chord length is greater than DISTX, the portion of the remaining chord length is added to the number of chord lengths.
After the dividers are walked along the curve with the initial chord length, the dividers are opened to another distance. This distance is a geometric adding of the first chord length. For example, if the initial chord length is 2, then the subsequent chord lengths would be 4, 8, 16, 32, 64, and so on. This eliminates biasing when using linear regression because on a logarithmic scale, geometric adding provides equal spacing between the chord lengths.

After each time the dividers are walked along the trajectory, the number of chord lengths and the corresponding chord lengths are saved. These are used in the linear regression where log line length (number of chord lengths *chord length) is regressed against log chord length. A trajectory 's fractal dimension is determined by using equation 1.

III. RESULTS

The length of an island can be quantifying by the map of an island and a pair of a divider withan opening of d and counting no of steps as M(d) required for one circumference as given in figure.

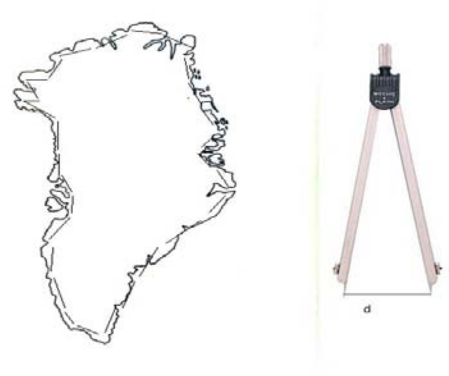

Figure 11. Measuring the length of Greenland

Table shows the number M of steps and the length L of the outline for different openings d.It is natural that the measured length increases as the opening of the divider is reduced.

S No.	d (Km)	M	L=M. d[Km]	ln d	ln L
1.	300	8	2400	5.7	7.78
2.	150	18	2700	5.01	7.9
3.	120	28	3360	4.7	8.1
4.	60	62	3720	4.1	8.22
5.	30	150	4500	3.4	8.4

Table 1: length of Greenland's boundary

If the outline has a well defined length, then L should refer a constant value L_0 as d=0,The double algorithmic plot shows that L does not reach a fixed value as d is reduced

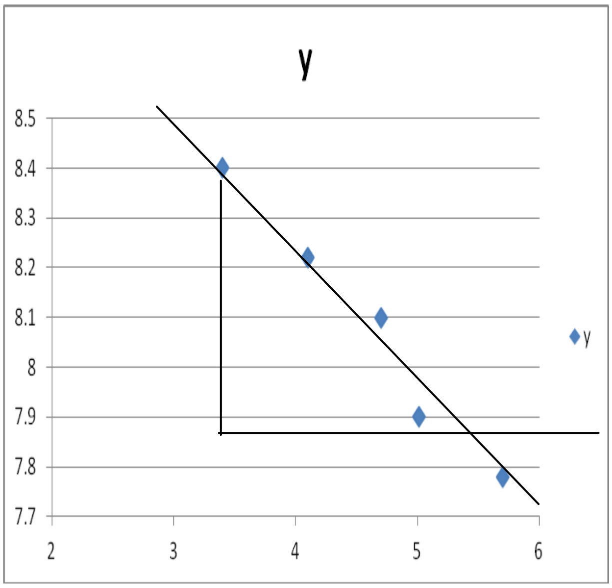

Figure 12 graph for d and L as x-axis & y-axis

Thus we are about to know length of the coastline , that depends upon someone wants to do. If somebody wants to build a fence around the boundary with fence posts at every ten meters , for him the boundary is longer than for someone who wants to place lighthouses every fifty kilometers , Since the values are lying on a straight line a power law is properly fit.

$$L(d) = \text{cont. } r^{1-Dc}$$

D_C is the divider/ compass dimension. It can be calculated from the slope of the regression line. The slope is

$$1-D_C = 8.4-7.9/3.4-5.5 \approx -0.23 \Rightarrow D_C = 1.23$$

With a constant of 8.5 for the man with the fence the coastline would have a length of $L(0.01\text{ km}) = e^{8.5}.\ 0.01^{-0.23} \approx 14000$ Kms for the man with the lighthouses $L(50\text{km}) = e^{8.5}.\ 50^{-0.23} \approx 1800$. The coast length differs by a factor of eight approx.The compass dimension is a measure to compute the fractal dimension of natural objects since it is an estimator of hausdorff dimension .

Box Dimension

An appropriate estimator for the fractal dimension of nearly any autonaomous structure is the box dimension . The length of the coastline was estimated by L=M.d where M is the number of steps needed for a roundtrip along the coast and d is the opening of a pair of compasses. Alternatively the coast can be covered with a grid of square cells with cell size d.
The number M(d) of squares needed to cover the coastline is roughly equal to the number of steps when using a pair of compasses with opening d . This holds for small d.

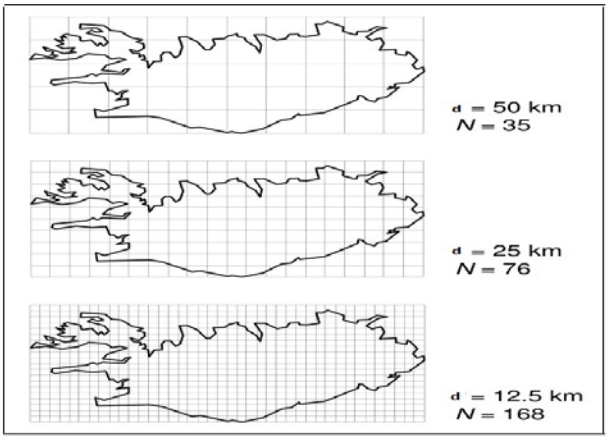

Figure 13 different values of d and N

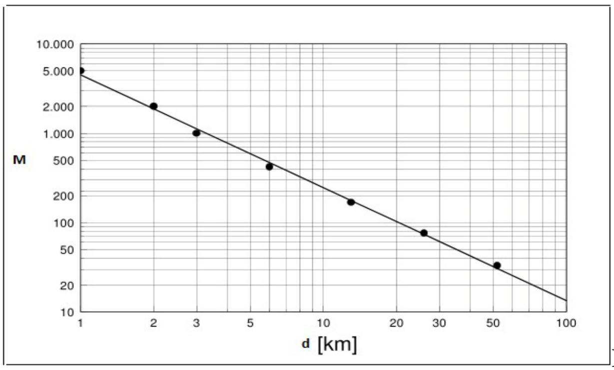

Figure 14 M versus d (logarithmic) graph

The straight line corresponds to the relation :

$$M(d)=const.\ d^{Db}$$

D_b is the box dimension of the coastline and can be ascertained from the slope of regression line as D_b= 1.26 . This matches with the compass dimension of Dc=1.23.
Thus the coastlength can be expressed as :

$$L(d) = M(d).d = 4506.d^{-1.26}.\ d$$

For the calculation of the box dimension it doesn't matter if the number of boxes is counted for the entire structure .
For binary images it is appropriate to choose the grid length as nmber of pixels The box dimension which is an estimator of the hausdorff dimension is Db= 1.77 .

V. CONCLUSION

Box-counting estimators are popular for estimating fractal dimension. However, very little is known of their speculative properties, instead increasing statistical interest in their application. We show that, if the irregular curve to which the estimators are applied , concise formulae may be developed for asymptotic bias and variance of box-counting estimators. These formulae point to crucial differences between a native form of the box-counting estimator, based directly on the capacity definition of fractal dimension.Eminently the box dimension can easily be unearthed if the structure is available in binary form .

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