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Quotient of Ideals of a Pythagorean Fuzzy Lattice

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Abstract:
In this paper, we defined operations on Pythagorean fuzzy ideals and the Pythagorean fuzzy ideal of a Pythagorean fuzzy lattice is introduced. Certain characterizations of these are provided. The residuals of Pythagorean fuzzy ideals are defined, and it is demonstrated that these residuals also form Pythagorean fuzzy ideals. Furthermore, it is shown that they are the largest Pythagorean fuzzy ideal of P **.**

Keywords: Pythagorean fuzzy sublattices, Pythagorean fuzzy ideal, Quotient of ideals

1. Introduction

The concept of Pythagorean fuzzy sets was originally introduced by [7,8] and has found applications in various algebraic structures such as groups, rings, semigroups, and lattices, as explored by multiple authors.

In paper [2] presented the notions of rough Pythagorean fuzzy ideals in semigroups and is extended to the lower and upper approximations of Pythagorean fuzzy left ideals, bi-ideals, interior ideals etc.

In the paper [2], it is defined the new notion of interval-valued Pythagorean fuzzy ideals in semigroups and established the properties. Within lattice theory, [4] was a pioneer in developing fuzzy sets and delving into fuzzy sublattices. Additionally, [6] introduced intuitionistic fuzzy sets as a powerful tool for addressing uncertainty, thereby extending the idea of fuzzy sets. Subsequently, [13] expanded upon this foundation by introducing the theory of Pythagorean fuzzy sublattices and ideals.

The concept of ideal of a fuzzy subring was introduced by Mordeson and Malik in [10]. subsequenty N Ajmal and A.S Prajapathi introduced the concept of residual ideals of an L-Ring in [11] and in [12] defined the intuitionistic fuzzy ideal of an intuitionistic fuzzy lattice and quotients (residuals) of ideals of an intuitionistic fuzzy sublattice and studied their properties. Motivated by this, in this paper we first defined the pythagorean fuzzy ideal of a Pythagorean fuzzy lattice and certain characterizations are given. Lastly, we defined quotients (residuals) of ideals of an intuitionistic fuzzy sublattice and studied their properties.

2. Preliminaries

In this section we go through the concept of Pythagorean Fuzzy sets (PFS)

2.1. Definition

See [7, 8] Let X be a nonempty set. A Pythagorean fuzzy set (PFS) P in X is an object of the form $\{< x, \varphi_P(x), \eta_P(x) > / x \in X\}$ where $\varphi_P(x): X \to [0,1]$ and $\eta_P(x): X \to [0,1]$ represents degree of the membership function and non-membership function resp, with the condition that $\varphi_P^2(x) + \eta_P^2(x) \leq 1$

2.2. *Definition*

See [8,9] Let $\{< x, \varphi_{P_1}(x), \eta_{P_1}(x) >/x \in X\}$ and $\{< x, \varphi_{P_2}(x), \eta_{P_2}(x >/x \in X\}$ are two Pythagorean fuzzy sets on X, then for all $x, y \in X$

1. $P_1 \subseteq P_2 \Rightarrow \varphi^2_{P_1}(x) \leq \varphi^2_{P_2}(x)$ and $\eta^2_{P_1}(x) \leq \eta^2_{P_2}(x)$
2. $P_1 = P_2 \Rightarrow P_1 \subseteq P_2$ & $P_2 \subseteq P_1$
3. ${P_1}^c = \{< x, \eta_P(x), \varphi_P(x) >/x \in X\}$
4. $P_1 \cap P_2 = \{< x, \varphi^2_{P_1 \cap P_2}(x), \eta^2_{P_1 \cap P_2}(x) > /x \in X\}$ where $\varphi^2_{P_1 \cap P_2}(x) = min\{\varphi^2_{P_1}(x), \varphi^2_{P_2}(x)\}$ and $\eta^2_{P_1 \cap P_2}(x) = max\{\eta^2_{P_1}(x), \eta^2_{P_2}(x)\}$
5. $P_1 \cup P_2 = \{< x, \varphi^2_{P_1 \cup P_2}(x), \eta^2_{P_1 \cup P_2}(x) > /x \in X\}$ where $\varphi^2_{P_1 \cup P_2}(x) = max\{\varphi^2_{P_1}(x), \varphi^2_{P_2}(x)\}$ and $\eta^2_{P_1 \cup P_2}(x) = min\{\eta^2_{P_1}(x), \eta^2_{P_2}(x)\}$
6. $[P] = \{< x, \varphi_P(x), (1 - \varphi^2_P(x))^{0.5} >/x \in X\}$
7. $< P >= \{< x, (1 - \eta^2_P(x))^{0.5}, \eta_P(x) > /x \in X\}$

2.3. *Definition*

See [13] Let L be a lattice and $P = \{< x, \varphi_P, \eta_P > /x \in L\}$ be a PFS of L. Then P is called a Pythagorean fuzzy sublattice (PFL) if the following conditions are satisfied for all $x,\ y \in L$

1. $\varphi^2_P(x \vee y) \geq min\{\varphi^2_P(x), \varphi^2_P(y)\}$
2. $\varphi^2_P(x \wedge y) \geq min\{\varphi^2_P(x), \varphi^2_P(y)\}$
3. $\eta^2_P(x \vee y) \leq max\{\eta^2_P(x), \eta^2_P(y)\}$
4. $\eta^2_P(x \wedge y) \leq max\{\eta^2_P(x), \eta^2_P(y)\}$

2.4. *Definition*

See [13] Let L be a lattice and $P = \{< x, \varphi_P, \eta_P > /x \in L\}$ be a PFS of L. Then P is called a Pythagorean fuzzy ideal (PFI) if the following conditions are satisfied for all $x,\ y \in L$

1. $\varphi^2_P(x \vee y) \geq min\{\varphi^2_P(x), \varphi^2_P(y)\}$
2. $\varphi^2_P(x \wedge y) \geq max\{\varphi^2_P(x), \varphi^2_P(y)\}$
3. $\eta^2_P(x \vee y) \leq max\{\eta^2_P(x), \eta^2_P(y)\}$
4. $\eta^2_P(x \wedge y) \leq min\{\eta^2_P(m), \eta^2_P(y)\}$

3. Operations on Pythagorean fuzzy ideals

In this section, we discuss some operations on Pythagorean fuzzy sets. We apply these operations on Pythagorean fuzzy ideals of a lattice denoted by $PFI(L)$ and examine their lattice structures.

3.1. *Definition*

Let $P, Q \in PFI(L)$, then define

$$P + Q = \{< z, \varphi_{P+Q}(z), \eta_{P+Q}(z) >/z \in L\}$$

with

$$\varphi^2_{P+Q}(z) = sup_{z=x \vee y} min\{\varphi^2_P(x), \varphi^2_Q(y)\}$$

$$\eta^2_{P+Q}(z) = inf_{z=x \vee y} max\{\eta^2_P(x), \eta^2_Q(y)\}$$

$$PQ = \{< z, \varphi_{PQ}(z), \eta_{PQ}(z) >/z \in L\}$$

with

$$\varphi^2_{PQ}(z) = sup_{z=x \wedge y} min\{\varphi^2_P(x), \varphi^2_Q(y)\}$$

$$\eta^2_{PQ}(z) = inf_{z=x \wedge y} max\{\eta^2_P(x), \eta^2_Q(y)\}$$

$$P.Q = \{< z, \varphi_{P.Q}(z), \eta_{P.Q}(z) >/z \in L\}$$

with

$$\varphi^2_{P.Q}(z) = sup_{z=\vee_{i=1}^{n}(x_i \wedge y_i)}(min_i\{min(\varphi^2_P(x_i), \varphi^2_Q(y_i)\})$$

$$\eta^2_{P.Q}(z) = inf_{z=\vee_{i=1}^{n}(x_i \wedge y_i)}(max_i\{max(\eta^2_P(x_i), \eta^2_Q(y_i)\})$$

$$P \circ Q = \{< z, \varphi_{P \circ Q}(z), \eta_{P \circ Q}(z) >/z \in L\}$$

with

$$\varphi^2_{P \circ Q}(z) = sup_{z \geq x \wedge y}(min(\varphi^2_P(x), \varphi^2_Q(y))$$

$$\eta^2_{P \circ Q}(z) = inf_{z \geq x \wedge y}(max(\eta^2_P(x), \eta^2_Q(y))$$

$$P \oplus Q = \{< z, \varphi_{P \oplus Q}(z), \eta_{P \oplus Q}(z) >/z \in L\}$$

with

$$\varphi^2_{P\oplus Q}(z) = sup_{z\geq x\wedge y}(min(\varphi^2_P(x), \varphi^2_Q(y))$$

$$\eta^2_{P\oplus Q}(z) = inf_{z\geq x\wedge y}(max(\eta^2_P(x), \eta^2_Q(y))$$

3.2. *Lemma*

Let $, Q, R \in PFI(L)$, then

1. $PQ = QP, P + Q = Q + P, P.Q = Q.P$
2. $PQ \subseteq P.Q \subseteq P \circ Q$
3. $R(P + Q) \subseteq RP + RQ$
4. $(R + Q)P \subseteq RP + QP$
5. $(P \cap Q)R \subseteq PR \cap QR$
6. $P \subseteq Q \Longrightarrow PR \subseteq QR$ and $P.R \subseteq Q.R$
7. $P + Q \subseteq P \bigoplus Q$ and $PQ \subseteq P \circ Q$ equality holds if L is distributive.
8. $P \subseteq P + P, P \subseteq PP, P \subseteq P \bigoplus P, P \subseteq P \circ P \& P \subseteq P.P$

3.3. *Lemma*

Let P be a Pythagorean fuzzy lattice of L, then

1. $P + P = P$
2. $PP = P$

4. Ideal of a Pythagorean fuzzy lattice

In this section, we define the ideal of a PFL and give some characterization of these ideals in terms of operations on PFI(L). Here we assume that L is a lattice $(L,\vee,\wedge)$ with zero element 0 and unit element 1. $\vee$ and $\wedge$ denote maximum and minimum respectively.

4.1. *Definition*

Let P be a Pythagorean fuzzy lattice of L and Q is a Pythagorean fuzzy set of L with $Q \subseteq P$. Then Q is called Pythagorean fuzzy ideal of P if the following conditions are satisfied

1. $\varphi^2_Q(x \vee y) \geq \varphi^2_Q(x) \wedge \varphi^2_Q(y)$
2. $\varphi^2_Q(x \wedge y) \geq [\varphi^2_P(x) \wedge \varphi^2_Q(y)] \vee [\varphi^2_Q(x) \wedge \varphi^2_P(y)]$
3. $\eta^2_Q(x \vee y) \leq \eta^2_Q(x) \vee \eta^2_Q(y)$
4. $\eta^2_Q(x \wedge y) \leq [\eta^2_P(x) \vee \eta^2_Q(y)] \wedge [\eta^2_Q(x) \vee \eta^2_P(y)], \forall x, y \in L$

4.2. *Example*

Consider the lattice "divisors of 10" $L = \{1,2,5,10\}$ and $P = \{<1,0.8,0.3>, <2,0.7,0.4>, <5,0.6,0.5> \quad <10,0.7,0.7>\}$ be a PFS of L. $Q = \{<1,0.9,0.2> \quad <2,0.8,0.3>, <5,0.7,0.4>, <10,0.8,0.6>\}$.
Clearly Q is a Pythagoean fuzzy ideal of P

4.3. *Definition*

Let P be a Pythagorean fuzzy lattice of L and Q is also a Pythagorean fuzzy lattice of L with $Q \subseteq P$. Then Q is called Pythagorean fuzzy sublattice of P

4.4. *Lemma*

The intersection of two Pythagorean fuzzy ideals of P is again a Pythagorean fuzzy ideal of P
proof
Let Q, R be two Pythagorean fuzzy ideal of P. We have to show that $Q \cap R$ is again a Pythagorean fuzzy ideal of P.

$$\varphi^2_{Q\cap R}(x \vee y) = min\{\varphi^2_Q(x \vee y), \varphi^2_R(x \vee y)\}$$

Since Q and R are Pythagorean fuzzy ideals of P

$$\varphi^2_{Q\cap R}(x \vee y) \geq min\{\varphi^2_Q(x) \vee \varphi^2_Q(y), \varphi^2_R(x) \vee \varphi^2_R(y)\}$$
$$\geq min\{\varphi^2_Q(x) \vee \varphi^2_R(x), \varphi^2_Q(y) \vee \varphi^2_R(y)\}$$
$$\geq \varphi^2_{Q\cap R}(x) \wedge \varphi^2_{Q\cap R}(y)$$

$$\varphi^2_{Q\cap R}(x \wedge y) = min\{\varphi^2_Q(x \wedge y), \varphi^2_R(x \wedge y)\}$$

Since Q and R are Pythagorean fuzzy ideals of P

$$\varphi^2_{Q\cap R}(x \wedge y) \geq \min\{[\varphi^2_Q(x) \wedge \varphi^2_P(y)] \vee [\varphi^2_Q(y) \vee \varphi^2_P(x)$$
$$\wedge \varphi^2_P(y)] \vee [\varphi^2_R(y) \vee \varphi^2_P(x)]\}$$
$$\geq \min\{\varphi^2_Q(x), \varphi^2_R(x)\} \wedge \varphi^2_P(y) \vee$$
$$min\{\varphi^2_Q(y), \varphi^2_R(y)\} \wedge \varphi^2_P(x)$$
$$\geq [\varphi^2_P(y) \wedge \varphi^2_{Q\cap R}(x)] \vee [\varphi^2_{Q\cap R}(y) \wedge \varphi^2_P(x)]$$

Also

$$\eta^2_{Q\cap R}(x \vee y) = max\{\eta^2_Q(x \vee y), \eta^2_R(x \vee y)\}$$

Since Q and R are Pythagorean fuzzy ideals of P

$$\begin{aligned}\eta^2_{Q\cap R}(x \vee y) &\leq max\{\eta^2_Q(x) \vee \eta^2_Q(y), \eta^2_R(x) \vee \eta^2_R(y)\}\\ &\leq max\{\eta^2_Q(x) \vee \eta^2_R(x), \eta^2_Q(y) \vee \eta^2_R(y)\}\\ &\geq \eta^2_{Q\cap R}(x) \vee \eta^2_{Q\cap R}(y)\end{aligned}$$

$$\eta^2_{Q\cap R}(x \wedge y) = max\{\eta^2_Q(x \wedge y), \eta^2_R(x \wedge y)\}$$

Since Q and R are Pythagorean fuzzy ideals of P

$$\begin{aligned}\eta^2_{Q\cap R}(x \wedge y) &\leq \max\{[\eta^2_Q(x) \vee \eta^2_P(y)] \wedge [\eta^2_Q(y) \vee \eta^2_P(x)], [\eta^2_R(x) \vee \eta^2_P(y)]\\ &\qquad \wedge [\eta^2_R(y) \vee \eta^2_P(x)]\}\\ &\leq \max\{\eta^2_Q(x), \eta^2_R(x)\} \vee \eta^2_P(y) \wedge\\ &\qquad max\{\eta^2_Q(y), \eta^2_R(y)\} \vee \eta^2_P(x)\\ &\leq [\eta^2_P(y) \vee \eta^2_{Q\cap R}(x] \wedge [\eta^2_{Q\cap R}(y) \vee \eta^2_P(x)]\end{aligned}$$

Hence $Q \cap R$ is a Pythagoean fuzzy ideal of P

4.5. *Theorem*

Let P be a Pythagorean fuzzy lattice and $Q \subseteq P$ is a Pythagorean fuzzy set of L. Then Q is a Pythagorean fuzzy ideal of P iff $\forall x, y \in L$

1. $\varphi^2_Q(x \vee y) \geq \varphi^2_Q(x) \wedge \varphi^2_Q(y)$
2. $\eta^2_Q(x \vee y) \leq \eta^2_Q(x) \vee \eta^2_Q(y)$
3. $PQ \subseteq Q$

Proof
Assume that 1, 2 and 3 holds. We will prove that Q is a Pythagorean fuzzy ideal of P. Let $x, y \in L$

$$\begin{aligned}\varphi^2_Q(x \wedge y) &\geq \varphi^2_{PQ}(x \wedge y)\\ &= sup_{x\wedge y = x_i \wedge y_i}(\varphi^2_P(x_i) \wedge \varphi^2_Q(y_i))\\ &\geq \varphi^2_P(x) \wedge \varphi^2_Q(y)\end{aligned}$$

Since $QP = PQ \subseteq Q$, $\varphi^2_Q(x \wedge y) \geq \varphi^2_Q(x) \wedge \varphi^2_P(y)$.
Hence

$$\varphi^2_Q(x \wedge y) \geq (\varphi^2_P(x) \wedge \varphi^2_Q(y)) \vee (\varphi^2_Q(x) \wedge \varphi^2_P(y))$$

Also

$$\begin{aligned}\eta^2_Q(x \wedge y) &\leq \eta^2_{PQ}(x \wedge y)\\ &= inf_{x\wedge y = x_i \wedge y_i}(\eta^2_P(x_i) \vee \eta^2_Q(y_i))\\ &\leq \eta^2_P(x) \vee \eta^2_Q(y)\end{aligned}$$

Since $QP = PQ \subseteq Q$, $\eta^2_Q(x \wedge y) \leq \eta^2_Q(x) \vee \eta^2_P(y)$.
Hence

$$\eta^2_Q(x \wedge y) \leq (\eta^2_P(x) \vee \eta^2_Q(y)) \wedge (\eta^2_Q(x) \vee \eta^2_P(y))$$

Therefore Q is a Pythagorean fuzzy ideal of P. Conversly assume that Q is a Pythagorean fuzzy ideal of P. Then condition 1 and 2 holds from the definition.
Also we have

$$\varphi^2_Q(x \wedge y) \geq \varphi^2_P(x) \wedge \varphi^2_Q(y)$$

and

$$\eta^2_Q(x \wedge y) \leq \eta^2_P(x) \vee \eta^2_Q(y) \;\; \forall x, y \in L$$

So $\forall z \in L$ with $z = x \wedge y$, we have

$$\begin{aligned}\varphi^2_Q(z) &\geq \bigvee_{z = x\wedge y} \varphi^2_P(x) \wedge \varphi^2_Q(y)\\ &= \varphi^2_{PQ}(z)\end{aligned}$$

Also

$$\begin{aligned}\eta^2_Q(z) &\leq \bigvee_{z = x\wedge y} \eta^2_P(x) \wedge \eta^2_Q(y)\\ &= \eta^2_{PQ}(z)\end{aligned}$$

Therefore $PQ \subseteq Q$

4.6. *Theorem*

Let P be a Pythagorean fuzzy lattice and Q, R are pythagorean fuzzy ideals of P, then $Q + R$ is a pythagorean fuzzy ideal of P

Proof
Let $Q, R \in PFI(L)$

$$Q + R = \{< z, \varphi^2_{Q+R}(z), \eta^2_{Q+R}(z) > / z \in L\}$$

with

$$\varphi^2_{Q+R}(z) = sup_{z = x\vee y} min\{\varphi^2_Q(x), \varphi^2_R(y)\}$$

$$\eta^2_{Q+R}(z) = inf_{z = x\vee y} max\{\eta^2_Q(x), \eta^2_R(y)\}$$

We have to show that

$$\varphi^2_{Q+R}(z) \geq \varphi^2_{Q+R}(x) \wedge \varphi^2_{Q+R}(y)$$

Let

$$min\{\varphi^2_{Q+R}(x), \varphi^2_{Q+R}(y)\} = k$$

Then for any $\epsilon > 0$,

$$k - \epsilon < \varphi_{Q+R}^2(x) = sup_{x=a\vee b} min\{\varphi_Q^2(a), \varphi_R^2(b)\}$$

and

$$k - \epsilon < \varphi_{Q+R}^2(y) = sup_{y=c\vee d} min\{\varphi_Q^2(c), \varphi_R^2(d)\}$$

so there exist representatives $x = a \vee b$ and $y = c \vee d$ such that

$$k - \epsilon < min\{\varphi_Q^2(a), \varphi_R^2(b)\}$$

and

$$k - \epsilon < min\{\varphi_Q^2(c), \varphi_R^2(d)\}$$

Then

$$k - \epsilon < \varphi_Q^2(a), \varphi_R^2(b), \varphi_Q^2(c), \varphi_R^2(d)$$

Therefore

$$k - \epsilon < min\{\varphi_Q^2(a), \varphi_Q^2(c)\} = \varphi_Q^2(a \vee c),$$

Also

$$k - \epsilon < min\{\varphi_R^2(b), \varphi_R^2(d)\} = \varphi_R^2(b \vee d),$$

since Q and R are pythagorean fuzzy ideals of L. So that

$$k - \epsilon < \min\{\varphi_Q^2(a \vee c), \varphi_R^2(b \vee d)\}$$

$$= x \vee y$$

$$= (a \vee b) \vee (c \vee d)$$

$$= (a \vee c) \vee (b \vee d$$

$$\varphi_{Q+R}^2(x \vee y) = sup_{x\vee y=u\vee v} \min\{\varphi_Q^2(a \vee c), \varphi_R^2(b \vee d)\}\}$$

$$\geq min\varphi_Q^2(\mathrm{u}), \varphi_R^2(v)$$

$$\geq k - \epsilon$$

Since $\epsilon > 0$ is arbitrary

$$\varphi_{Q+R}^2(x \vee y) > k = min\{\varphi_{Q+R}^2(x), \varphi_{Q+R}^2(y)\}$$

Let

$$max\{\varphi_{Q+R}^2(x), \varphi_{Q+R}^2(y)\} = q$$

Then for any $\epsilon > 0$,

$$q + \epsilon > \eta_{Q+R}^2(x) = inf_{x=a\vee b} max\{\eta_Q^2(a), \eta_R^2(b)\}$$

and

$$q + \epsilon > \eta_{Q+R}^2(y) = inf_{y=c\vee d} max\{\eta_Q^2(c), \eta_R^2(d)\}$$

so there exist representatives $x = a \vee b$ and $y = c \vee d$ such that

$$q + \epsilon > max\{\eta_Q^2(a), \eta_R^2(b)\}$$

and

$$q + \epsilon > max\{\eta_Q^2(c), \eta_R^2(d)\}$$

Then

$$q + \epsilon > \eta_Q^2(a), \eta_R^2(b), \eta_Q^2(c), \eta_R^2(d)$$

Therefore

$$q + \epsilon > max\{\eta_Q^2(a), \eta_Q^2(c)\} > \eta_Q^2(a \vee c),$$

Also

$$q + \epsilon > max\{\eta_R^2(b), \eta_R^2(d)\} > \eta_R^2(b \vee d),$$

since Q and R are pythagorean fuzzy ideals of L. So that

$$q + \epsilon > \max\{\eta_Q^2(a \vee c), \eta_R^2(b \vee d)\}$$

$$= x \vee y = (a \vee b) \vee (c \vee d)$$

$$= (a \vee c) \vee (b \vee d) = \eta_{Q+R}^2(x \vee y)$$
$$= inf_{x\vee y=u\vee v} \max\{\eta_Q^2(u), \eta_R^2(v)$$
$$\leq max\varphi_Q^2(a \vee b), \varphi_R^2(b \vee d)$$
$$\leq q + \epsilon$$

Since $\epsilon > 0$ is arbitrary

$$\eta_{Q+R}^2(x \vee y) > k = max\{\eta_{Q+R}^2(x), \eta_{Q+R}^2(y)\}$$

Since$P(Q + R) \subseteq PQ + PR \subseteq Q + R$
$Q + R$ is a Pythagorean fuzzy ideal of P by theorem 4.5

5. Quotient of Ideals

We define residuals of ideals of the Pythagorean fuzzy ideal and prove that the residuals of ideal is again a Pythagorean fuzzy ideal of the Pythagorean fuzzy lattice

5.1. *Definition*

Let P be a Pythagorean fuzzy lattice of L and Q and R are pythagorean fuzzy ideals of P. Then the quotient of Q by R denoted by $Q|R$ is defined as

$$Q|R = \cup\ \{S|\ \ S\Delta P \quad and \quad SR \subset Q\}$$

where $S\Delta P$ denote S is a pythagorean fuzzy ideal of P

5.2. *Theorem*

Let P be a Pythagorean fuzzy lattice of L and Q and R are Pythagorean fuzzy ideals of P. Then $Q|R$ is a Pythagoean fuzzy ideal of P

Proof: Let $\eta = \{S|S\Delta P\ \ and\ \ SR \subset Q\}$. Suppose $S$and $S' \in \eta$ Then S and S' are Pythagorean fuzzy ideals of P such that $SR \subset Q$ and $S'R \subset Q$. Then by theorem 4.6 $S + S'$ is Pythagorean fuzzy ideal of P.

Also $(S + S')R = SR + S'R \subset Q + Q = Q$.

Hence $S + S' \in \eta$.
Let $x, y \in L$, then

$$\begin{aligned}\varphi^2_{Q|R}(x) \wedge \varphi^2_{Q|R}(y) &= (\bigvee_{S\in\eta} \varphi^2_S(x)) \wedge (\bigvee_{S'\in\eta} \varphi^2_S(y)) \\ &= \vee\{\varphi^2_S(x) \wedge \varphi^2_{S'}(y))|S, S' \in \eta\} \\ &\le \vee\{\varphi^2_{S+S'}(x \vee y)|S, S' \in \eta\} \\ &\le \varphi^2_{Q|R}(x \vee y)\end{aligned}$$

Also,

$$\begin{aligned}\varphi^2_{Q|R}(x \wedge y) &= \bigvee_{S\in\eta} \varphi^2_S(x \wedge y) \\ &\ge \bigvee_{S\in\eta} \varphi^2_S(x) \wedge \varphi^2_P(y) (since\ \ S\Delta P) \\ &= [\bigvee_{S\in\eta} \varphi^2_S(x)] \wedge \varphi^2_P(y) \\ &= \varphi^2_{Q|R}(x) \wedge \varphi^2_P(y), \forall x, y \in L\end{aligned}$$

Similarly

$$\varphi^2_{Q|R}(x \wedge y) \ge \varphi^2_{Q|R}(y) \wedge \varphi^2_P(x), \forall x, y \in L$$

Thus

$$\varphi^2_{Q|R}(x \wedge y) \ge [\varphi^2_{Q|R}(x) \wedge \varphi^2_P(y)] \vee [\varphi^2_{Q|R}(y) \wedge \varphi^2_P(x)]$$

Now

$$\begin{aligned}\eta^2_{Q|R}(x) \vee \eta^2_{Q|R}(y) &= (\bigwedge_{S\in\eta} \eta^2_S(x)) \vee (\bigwedge_{S'\in\eta} \eta^2_S(y)) \\ &= \wedge\{\eta^2_S(x) \vee \eta^2_{S'}(y))|S, S' \in \eta\} \\ &\le \wedge\{\eta^2_{S+S'}(x \vee y)|S, S' \in \eta\} \\ &\le \eta^2_{Q|R}(x \vee y),\ \ Since\ S + S' \in \eta\end{aligned}$$

Also,

$$\begin{aligned}\eta^2_{Q|R}(x \wedge y) &= \bigwedge_{S\in\eta} \eta^2_S(x \wedge y) \\ &\le \bigwedge_{S\in\eta} \eta^2_S(x) \vee \eta^2_P(y) (since\ \ S\Delta P) \\ &= [\bigwedge_{S\in\eta} \eta^2_S(x)] \vee \eta^2_P(y) \\ &= \eta^2_{Q|R}(x) \vee \eta^2_P(y), \forall x, y \in L\end{aligned}$$

Similarly

$$\eta^2_{Q|R}(x \wedge y) \le \eta^2_{Q|R}(y) \vee \eta^2_P(x), \forall x, y \in L$$

Thus

$$\eta^2_{Q|R}(x \wedge y) \le [\eta^2_{Q|R}(x) \vee \eta^2_P(y)] \wedge [\eta^2_{Q|R}(y) \vee \varphi^2_P(x)]$$

Hence $Q|R$ is a Pythagorean fuzzy ideal of P. Clearly $|R \subset P$. Clearly $Q|R \subset P$. Since Q is Pythagorean fuzzy ideal of , $QP \subset Q$, by theorem 2.5. Since $R \subset P$, $QR \subset QP \subset Q$ and hence $Q \in \eta$. So $Q \subset Q|P$. Thus we have $Q \subset Q|R \subset P$

5.3. *Theorem*

Let P be a Pythagorean fuzzy lattice of L and Q and R are pythagorean fuzzy ideals of P. Then $Q|R$ is the largest pythagorean fuzzy ideal of P with the property $(Q|R).R \subset Q$

Proof:
Let

$$\eta = \{S|S\Delta P\ \ and\ \ SR \subset Q\}$$

$$Q|R = \bigcup_{S\in\eta} S$$

Let $x \in L$ such that $x = \vee^n_{i=1} a_i \wedge b_i$, where $a_i, b_i \in L$. Then

$$\begin{aligned}\varphi_Q^2(a_i \wedge b_i) &\geq \varphi_{SR}^2(a_i \wedge b_i)\\ &\geq \varphi_S^2(a_i) \wedge \varphi_R^2(b_i)\\ &\geq \bigvee_{S\in\eta} \varphi_S^2(a_i) \wedge \varphi_R^2(b_i)\\ &= [\bigvee_{S\in\eta} \varphi_S^2(a_i)] \wedge \varphi_R^2(b_i)\\ &= \varphi_{Q|R}^2(a_i) \wedge \varphi_R^2(b_i)\end{aligned}$$

Hence

$$\begin{aligned}\varphi_Q^2(x) &= \varphi_Q^2(\bigvee_{i=1}^{n}(a_i \wedge b_i)\\ &\geq \bigwedge_{i=1}^{n} \varphi_Q^2(a_i \wedge b_i), since\ Q\ is\ a\ PFIofP\\ &\geq \{\bigwedge_{i=1}^{n} \varphi_{Q|R}^2(a_i) \wedge \varphi_R^2(b_i)\ \mid\ x = \bigvee_{i=1}^{n} a_i \wedge b_i\}\\ &= \varphi_{(Q|R).R}(x)\end{aligned}$$

Similiarly

$$\begin{aligned}\eta_Q^2(a_i \wedge b_i) &\leq \eta_{SR}^2(a_i \wedge b_i)\\ &\leq \eta_S^2(a_i) \vee \eta_R^2(b_i)\\ &\leq \bigvee_{S\in\eta} \eta_S^2(a_i) \vee \eta_R^2(b_i)\\ &= [\bigvee_{S\in\eta} \eta_S^2(a_i)] \vee \eta_R^2(b_i)\\ &= \eta_{Q|R}^2(a_i) \vee \eta_R^2(b_i)\end{aligned}$$

Hence

$$\begin{aligned}\eta_Q^2(x) &= \eta_Q^2(\bigvee_{i=1}^{n}(a_i \wedge b_i)\\ &\leq \bigvee_{i=1}^{n} \eta_Q^2(a_i \wedge b_i), since\ Q\ is\ a\ PFIofP\\ &\leq \{\bigvee_{i=1}^{n} \eta_{Q|R}^2(a_i) \vee \eta_R^2(b_i)\ \mid\ x = \bigvee_{i=1}^{n} a_i \wedge b_i\}\\ &= \eta_{(Q|R).R}(x)\end{aligned}$$

Hence $Q|R \subset Q$. To show that $Q|R$ is the largest such PFI , let S be an ideal of P such that $SR \subset Q$, then $SR \subset S.R \subset Q$. Thus $S \in \eta$. Hence $S \in Q|R$. Thus $Q|R$ is the largest PFI of P such that $(Q|R).R \subset Q$.

5.4. *Theorem*

Let P be a PFI and $Q, R, andS$ are PFI's of P. Then the following conditions are hold

1. $IfQ \subseteq R$ then $Q/S \subseteq R/S$ and $S/R \subseteq S/Q$
2. If $Q \subseteq R$, then $R/Q = P$
3. $Q/Q = P$

Proof:

1. If $Q \subseteq R$. Let $\eta = \{T|\ T\Delta P\ and\ \ TS \subset Q\}$ and $\psi = \{T|\ T\Delta P\ and\ \ TS \subset R\}$. If $\in \eta$, then $T\Delta P$ and $TS \subseteq Q \subseteq R$, then $T \in \psi$, and hence $\eta \subseteq \psi$. So

$$Q|S = \bigcup_{T\in\eta} T \subseteq \bigcup_{T\in\psi} T = R|S$$

Similarly let $\eta_1 = \{T|T\Delta PandTR \subset S\}$ and $\psi_1 = \{T|T\Delta PandTQ \subset S\}$. If $T_1 \in \eta_1$, then $T_1\Delta P$ and $T_1R \subseteq S$, but $Q \subseteq R$, then $T_1Q \subseteq T_1R \subseteq S \Longrightarrow T_1Q \subseteq S$, then $T_1 \in \psi$, and hence $\eta_1 \subseteq \psi_1$. So

$$Q|S = \bigcup_{T_1\in\eta_1} T_1 \subseteq \bigcup_{T_1\in\psi_1} T_1 = S|Q$$

2. Let $\eta = \{T|\ T\Delta P\ and\ \ TQ \subset R\}$. Since $Q\Delta P$, we have $PQ \subseteq Q \subseteq R$ and $P\Delta P$. Thus $P \in \eta$ and hence $P \subseteq \bigcup_{T\in\eta} T = R|Q \subseteq P$, since $R|Q$ is a PFI of P. Therefore $R|Q = P$
3. We have $Q \subseteq Q$, so from (b), $Q|Q = P$

5.5. *Corollary*

Let P be a PFI of L and Q and R are PFI's of P, then

1. $(Q|R)|Q = P$
2. $(Q|Q)|R = P$
3. $(Q|(Q \cap R) = P$

Proof

1. Since $Q \subseteq Q|R$, by theorem 5.4(b) we have $(Q|R)|Q = P$
2. By (c) of theorem 5.4, $Q|Q = P$, since $R \subseteq P = Q|Q$, $(Q|Q))|R = P$
3. $Q\Delta P$ and $R\Delta P$, so $(Q \cap R)\Delta P$ and $Q \cap R \subseteq Q$. Hence by theorem 5.4(b) $Q|(Q \cap R) = P$

5.6. *Theorem*

Let P be a PFL of L, $Q_i, i = 1,2,3 \ldots m$, R and S are PFI's of P. Then

$$\bigcap_{i=1}^{m} Q|R = \bigcap_{i=1}^{m} (Q_i|R)$$

Proof:

Since $\bigcap_{i=1}^{m} Q_i \subseteq Q_i, i = 1,2,3\ldots m$, so by theorem 5.4 $\bigcap_{i=1}^{m} Q_i|R \subseteq Q_i|R, \quad \forall i = 1,2,3\ldots m$ Hence $(\bigcap_{i=1}^{m} Q_i)|R \subseteq \bigcap_{i=1}^{m} Q_i|R$. Let

$$\eta_1 = \{T|\ T\Delta P\ \ and\ \ \ TR \subset Q_1\}$$

$$\eta_2 = \{T|\ T\Delta P\ \ and\ \ \ TR \subset Q_2\}$$

$$\eta_3 = \{T|\ T\Delta P\ \ and\ \ \ TR \subset Q_1 \cap Q_2\}$$

$$\begin{aligned}\varphi^2_{(Q_1|R)\cap(Q_2|R)}(x) &= \varphi^2_{Q_1|R} \wedge \varphi^2_{Q_2|R} \\ &= (\bigvee_{T\in\eta_1} \varphi^2_T(x)) \wedge (\bigvee_{T'\in\eta_2} \varphi^2_{T'}(x)) \\ &= \vee\{\varphi^2_T(x) \wedge \varphi^2_{T'}(x)/T \in \eta_1, T' \in \eta_2\}\end{aligned}$$

Also

$$\begin{aligned}\eta^2_{(Q_1|R)\cap(Q_2|R)}(x) &= \eta^2_{Q_1|R} \vee \varphi^2_{Q_2|R} \\ &= (\bigwedge_{T\in\eta_1} \eta^2_T(x)) \vee (\bigwedge_{T'\in\eta_2} \eta^2_{T'}(x)) \\ &= \wedge\{\eta^2_T(x) \vee \eta^2_{T'}(x)/T \in \eta_1, T' \in \eta_2\}\end{aligned}$$

Let $T \in \eta_1$ and $T' \in \eta_2$, then $TR \subseteq Q_1$ and $T'R \subseteq Q_2$. Also $T \cap T'$ is a PFI of P, so that

$$(T \cap T')R \subseteq TR \cap T'R \subseteq Q_1 \cap Q_2$$

. Therefore $T \cap T' \in \eta_3$. Hence

$Q_1 \cap Q_2|R = \bigcup_{T\in\eta_3} T \supseteq \bigcup_{T\in\eta_1, T'\in\eta_2} T \cap T'$ implies

$$\begin{aligned}\varphi^2_{Q_1\cap Q_2}|C &\geq \varphi_{T\cap T'}(x) \\ &= \vee\,[\varphi^2_T(x) \wedge \varphi^2_{T'}(x)] \\ &= \varphi_{Q_1|R\cap Q_2|R}\quad from\ \ equation(1)\end{aligned}$$

and

$$\begin{aligned}\eta^2_{Q_1\cap Q_2}|C &\leq \eta_{T\cap T'}(x) \\ &= \wedge\,[\varphi^2_T(x) \wedge \eta^2_{T'}(x)] \\ &= \eta_{Q_1|R\cap Q_2|R}\quad from\ \ equation(2)\end{aligned}$$

Hence $Q_1 \cap Q_2|R \supseteq Q_1|R \cap Q2|R$. In general

$$\bigcap_{i=1}^{m} Q|R \supseteq \bigcap_{i=1}^{m} (Q_i|R)$$

Therefore

$$\bigcap_{i=1}^{m} Q|R = \bigcap_{i=1}^{m} (Q_i|R)$$

5.7. *Theorem*

Let P be a PFI OF L and $Q, R \in PFI(P^*)$, where P denote the set of all PFI's $Q_i, i = 1,2,3\ldots m$ of P that satisfies the property $\varphi_{Q_i}(0) = \varphi_{Q_j}(0)$ and $\eta_{Q_i}(0) = \eta_{Q_j}(0)$, $\forall i, j$ Then we have the following results

1. $Q \subseteq Q + R$ and $R \subseteq Q + R$
2. $Q|R = Q|(Q + R)$
3. $(Q + R)|Q = P$ and $(Q + R)|Q \cap R = P$

Proof:

$$\begin{aligned}\varphi_{Q+R}(x) &= \bigcup_{x=y\vee z} \varphi^2_Q(y) \wedge \varphi^2_R(z) \\ &\geq \varphi^2_Q(x) \wedge \varphi^2_R(0)\quad as\quad x = x \vee 0 \\ &= \varphi^2_Q(x) \wedge \varphi^2_Q(0)\quad as\quad \varphi^2_Q(0) = \varphi^2_R(0) \\ &= \varphi^2_Q(x)\end{aligned}$$

$$\begin{aligned}\eta_{Q+R}(x) &= \bigcap_{x=y\vee z} \eta^2_Q(y) \vee \eta^2_R(z) \\ &\leq \eta^2_Q(x) \vee \eta^2_R(0)\quad as\quad x = x \vee 0 \\ &= \eta^2_Q(x) \vee \eta^2_Q(0)\quad as\quad \eta^2_Q(0) = \eta^2_R(0) \\ &= \eta^2_Q(x)\end{aligned}$$

so $Q \subseteq Q + R$. Similiraly we can prove that $R \subseteq Q + R$

1. We have $(Q + R)\Delta P$ and $R \subseteq Q + R$ so $Q|(Q + R) \subseteq Q|R$

$$\eta = \{T|\ T\Delta P\ \ and\ \ \ TR \subset Q\}$$

$$\psi = \{T|\ T\Delta P\ \ and\ \ \ T(Q + R) \subset Q\}$$

Let $T \in \eta$, then $T\Delta P$ and $T \subseteq P$, so $TQ \subseteq PQ$. But $PQ \subseteq Q$ and $Q\Delta P$. So $T(Q + R) \subseteq TQ + TR \subseteq Q + Q = Q$ and hence $T \in \psi$ so $\subseteq \psi$. Thus $Q|R = \bigcup_{T\in\eta} T = \bigcup_{T\in\psi} T = Q|(Q + R)$ and hence $Q|R = Q|(Q + R)$

2. We have $(Q + R)\Delta P$ and $Q \subseteq (Q + R)$. So by theorem $(Q + R)|Q = P$. Also we have $(Q \cap R)\Delta P$ and $(Q \cap R) \subseteq (Q + R)$. Hence by theorem 5.4, $(Q + R)|(Q \cap R) = P$

5.8. *Theorem*

Let P be a PFI of L and $\{Q_i\}, i = 1,2,3\ldots m \in PFI(P^*)$ and R be any PFI of P. Then $R|\Sigma_{i=1}^{m} Q_i =$

$\bigcap_{i=1}^{n} R|Q_i$

Proof:

We have $Q_1 + Q_2 \Delta P$ and $Q_1 \subseteq Q_1 + Q_2$ and $Q_2 \subseteq Q_1 + Q_2$, so by theorem 5.4, $R|Q_1 + Q_2 \subseteq R|Q_1$ and $R|Q_1 + Q_2 \subseteq R|Q_2$ Therefore $R|Q_1 + Q_2 \subseteq R|Q_1 \cap R|Q_2$ Let

$$\eta_1 = \{T|\ T\Delta P\ \ and\ \ \ TQ_1 \subset R\}$$

$$\eta_2 = \{T|\ T\Delta P\ \ and\ \ \ TQ_2 \subset R\}$$

$$\eta_3 = \{T|\ T\Delta P\ \ and\ \ \ T(Q_1 + Q_2) \subseteq R\}$$

Then $\forall x \in L$

$$\begin{aligned}\varphi_{R|Q_1 \cap R|Q_2}(x) &= \varphi_{R|Q_1}(x) \wedge \varphi_{R|Q_2}(x) \\ &= (\bigvee_{T\in\eta_1} \varphi_T^2(x)) \wedge (\bigvee_{T'\in\eta_2} \varphi_{T'}^2(x)) \\ &= \vee\{\varphi_T^2(x) \wedge \varphi_{T'}^2(x)/T \in \eta_1, T' \in \eta_2\}\end{aligned}$$

Similiarly

$$\begin{aligned}\eta^2_{(Q_1|R)\cap(Q_2|R)}(x) &= \eta^2_{Q_1|R} \vee \varphi^2_{Q_2|R} \\ &= (\bigwedge_{T\in\eta_1} \eta_T^2(x)) \vee (\bigwedge_{T'\in\eta_2} \eta_{T'}^2(x)) \\ &= \wedge\{\eta_T^2(x) \vee \eta_{T'}^2(x)/T \in \eta_1, T' \in \eta_2\}\end{aligned}$$

Now let $T \in \eta_1$ and $T' \in \eta_2$, then $TQ_1 \subseteq R$ and $T'Q_2 \subseteq R$ Also $T \cap T'\Delta P$, so that $(T \cap T')(Q_1 + Q_2) \subseteq (T \cap T')Q_1 + (T \cap T')Q_2 \subseteq TQ_1 + T'Q_2 \subseteq R + R = R$. Hence $T \cap T' \in \eta_3$ so $\eta_2 \subseteq \eta_3$, thus $R|Q_1 + Q_2 = \bigcup_{T\in\eta_3} T \supseteq \bigcup_{T\in\eta_1, T'\in\eta_2} T \cap T'$,

Hence $\forall x \in L$, we have

$$\begin{aligned}\varphi^2_{R|Q_1+Q_2}(x) &\geq \vee\, (\varphi_T^2(x) \wedge \varphi_{T'}^2(x)) \\ &= \varphi_{R|Q_1\cap R|Q_2}(x)\ from\ equation (3)\end{aligned}$$

Similiarly

$$\eta^2_{R|Q_1+Q_2}(x) \leq \eta_{R|Q_1\cap R|Q_2}(x)\ from\ equation(4)$$

Therefore $R|Q_1 \cap Q_2 \supseteq R|Q_1 \cap R|Q_2$. Hence $R|Q_1 \cap Q_2 = R|Q_1 \cap R|Q_2$

In general $R|\Sigma_{i=1}^{m} Q_i = \bigcap_{i=1}^{n} R|Q_i$

Conclusion

Basic operations on Pythagorean fuzzy sets are intoduced, and the concept of an ideal in a Pythagorean fuzzy lattice is defined. Various characterizations related to these operations are explored. The residuals of Pythagorean fuzzy ideals are defined, and it is demonstrated that these residuals also form Pythagorean fuzzy ideals. Furthermore, it is shown that they are the largest Pythagorean fuzzy ideal of P.

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