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Existence and Uniqueness of Solutions to Three-point Boundary Value Problems associated with Third Order Non-Linear Fuzzy differential Equations

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INTRODUCTION:

Existence and unicity of solutions of third order non-linear differential equations is an interesting area of current research and agreat deal of work has been done by many authors during the last few years. The idea of matching solutions of two-point boundary value problems at the common point of solutions is very fascinating. The first result in this direction is due to Dennis Bar and Tom Sherman during the year 1973, published in the journal of differential equations and has been extended by many eminent mathematicians like Johnny Henderson, R. P. Agarwal, K. N. Murty etc.
However, the study of three-point boundary value problems associated with third order non-linear differential equation warants investigations due to its tremendous application in Cryptography and Control systems. Recently Sriram Bhagavatula, Dileep and K. N. Murty [3] obtained self-adjoint criteria for Kronecker product boundary value problems. Further, Sriram Bhagavatula et.al [4, 5, 6, 7] used iterative literatures for existence and uniqueness of solution to third order differential equation via fixed point methods. They also obtained (F, Y) bounded solution of linear Sylvester systems. Further they obtain stability criteria for non-linear Sylvester systems [4-7]. However the use of Fuzzy differential linear systems needs special attention. . Recently, Lakshmikantham, Murty and Turner obtained existence criteria for two-point boundary value problems associated with second order Fuzzy differential equations by using Green's function and an application of Banach fixed point theorem [9].Our paper is organized as follows: Section2, presents preliminary results on Haussdorff metric and basic results on Fuzzy sets and systems. First and second order fuzzy differential systems are also presented in section 2. Our main results are presented in section 3.
In recent years, the theory of fuzzy differential equation associated with non-linear second order differential equations got more attention because of their application in dynamical modeling and army research wings under resticted conditions. We present some of the basic notions on fuzzy sets and systems in the next section. For results on stability analysis we refer [6, 11].

1. PRELIMINARIES:

Let $P_k(R^n)$ denotes the set of all non-empty compact convex subsets of R^n. We define the addition and scalar multiplication in $P_k(R^n)$ in the usual way i.e. if $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$ then $\alpha(A+B) = \alpha A + \alpha B, \alpha(\beta A) = (\alpha\beta)A$ and $1.A = A$.

If $\alpha, \beta \in R$ and $\alpha, \beta \geq 0$ then $(\alpha+\beta)A = \alpha A + \beta A$.

Then the distance between A and B is defined by the Haussdorff metric as

$$d(A,B) = inf\{\epsilon: A \subset N(B,\epsilon), B \subset N(A,\epsilon)\}$$

where

$$N(A,\epsilon) = \{x \in R^n: \|x-y\| < \epsilon \, for\, some\, y \in R^n\, and\, y \in B\}$$

Let $I = [a,b] \subset R$ be a compact subinterval of R and we denote

$$E^n = \{u \in R^n \to [0,1] / u\, satisfies\, the\, following\, conditions\}:$$

(i) u is normal, that is there exists an $x_0 \in R^n$ such that $u(x_0)1$

(ii) u is fuzzy convex, i.e. for $x, y \in R^n$ and $0 < \lambda < 1$, $u(\lambda x + (1-\lambda)y) \in R^n$

(iii) u is upper semi continuous

(iv) $[u]^\alpha = \{x \in R^n: u(x) \geq \alpha\}$

$[u]^0 = \{x \in R^n: u(x) > 0\}$is compact.

Definition 2.1 :A fuzzy number in parametric form is represented by (u_α^-, u_α^+), where

$u_\alpha^- = min[u]^\alpha$ and $u_\alpha^+ = max[u]^\alpha$, $0 \leq \alpha \leq 1$ and has the following properties:

(i) u_α^- is bounded left-continuous monotonically increasing function of α over the interval [0,1].

(ii) u_α^+ is bounded right-continuous monotonically decreasing function of α over the interval [0,1].

(iii) $u_\alpha^- \leq u_\alpha^+$ for all $\alpha \in [0,1]$.

If $f: R^n \to R^n$ is a continuous function, then according to Zadeh's extension principle, we can extend $f: E^n \times E^n \to E^n$ by defining

$f(u,v)(z) = sup_{z=f(x,y)}\{\min u(x), u(y)\}$ and

$f[(u,v)]^\alpha = f[[u]^\alpha, [v]^\alpha]$ for $0 \leq \alpha \leq 1$

for all $u, v \in E^n$, $\lambda \in R$ and for $0 \leq \alpha \leq 1$, the sum $u+v$ and the product λu are defined as

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha$$
$$[\lambda u]^\alpha = \lambda[u]^\alpha$$

where $[u]^\alpha + [v]^\alpha$means the usual addition of R^n and $\lambda[u]^\alpha$ means the usual product between a scalar and $\subset R^n$.

Definition 2.2: we define $D: E^n \times E^n \to R_+ u\{0\}$ by

$$D(u,v) = sup_{0\leq\alpha\leq1} d_H([u]^\alpha, [v]^\alpha)$$

where d_H is the Haussdorff metric defined in $P_k(R^n)$ i.e.

$D(u,v) = sup_{0\leq\alpha\leq1} max\{[u_\alpha^-, v_\alpha^-], [u_\alpha^+, v_\alpha^+]\}$.

It can easily be verified that $(E^n, D)$is a complete metric space and D has the following properties:

For any u, v and $w \in P_k(R^n)$ and $\lambda \in R$

(i) $D(u+w, v+w) = D(u,v)$

(ii) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$

(iii) $D(u,v) \leq D(u,w) + D(w,v)$

Definition 2.3: Let $x, y \in E^n$. If there exists a $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x with respect of y and is denoted by $X \ominus y$.

Definition 2.4: Let $f: T \to E^n$ and $t_0 \in T$. We say that f is differentiable at t_0, if there exists an element $f' \in E^n$ such that for all $h > 0$, the H-difference

$f(t_0+h) \ominus f(t_0), f(t_0) \ominus f(t_0-h)$ and

$\lim_{h\to0^+} \frac{f(t_0+h)-f(t_0)}{h}$, $\lim_{h\to0^-} \frac{f(t_0)-f(t_0-h)}{h}$ exists and each equals to $f'(t_0)$. At the end points, we take one sided derivative.

Theorem 2.1: Let $f: T \to E'$ be differentiable. Denote

$F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$ for $\alpha \in [0,1]$. Then $f_\alpha(t)$ and $g_\alpha(t)$ are differentiable and

$$[F'(t)]^\alpha = [f_\alpha'(t), g_\alpha'(t)]$$

Proof: For any $\alpha \in [0,1]$, we have

$$[F(t+h) - F(t)]^\alpha = [f_\alpha(t+h) - f_\alpha(t), g_\alpha(t+h) - g_\alpha(t)]$$

$$\lim_{h\to0^+}\left[\frac{F(t+h)-F(t)}{h}\right]^\alpha = \lim_{h\to0^+}\left[\frac{f_\alpha(t+h)-f_\alpha(t)}{h}, \frac{g_\alpha(t+h)-g_\alpha(t)}{h}\right]$$

and

$$\lim_{h\to 0^-}\left[\frac{F(t)-F(t-h)}{h}\right]^\alpha$$
$$=\lim_{h\to 0^-}\left[\frac{f_\alpha(t)-f_\alpha(t-h)}{h},\frac{g_\alpha(t)-g_\alpha(t-h)}{h}\right]$$
Hence
$$[F'(t)]^\alpha=[f'_\alpha(t),g'_\alpha(t)]$$

Lemma 2.1: Let $\Phi_\alpha: T\to E^n$ be a solution. Then Φ is a solution of the initial value problem

$$x'_\alpha(t)=f(t,x_\alpha(t)),\ x_\alpha(t_0)=x_0^\alpha \quad (2.1.1)$$

if and only if Φ_α is a solution of the integral equation

$$x_\alpha(t)=x_0^\alpha+\int_{t_0}^t f(s,x_\alpha(s))ds$$ for all $t\in T$ and $\alpha\in[0,1]$. (2.1.2)

Proof: Fix $\alpha\in[0,1]$. Suppose $x_\alpha(t)$ is a solution of the integral equation (2.1.2).
Then
$$x'_\alpha(t)=\lim_{h\to t_0^+}\frac{x_\alpha(t_0+h)-x_\alpha(t_0)}{h}$$
$$=\lim_{h\to t_0^+}\int_{t_0}^t f\left(s,\frac{x_\alpha(s_0+h)-x_\alpha(s_0)}{h}\right)ds$$
$=f(t,x_\alpha(t))$ and $x_\alpha(t_0)=x_0^\alpha$

Thus $x_\alpha(t)$ is a solution of the differential system (2.1.1).

Conversely, suppose $x_\alpha(t))$ is asolution of the fuzzy differential equation (2.1.1). Then
$$x'_\alpha(t)=f(t,x_\alpha(t)),\ x_\alpha(t_0)=x_0^\alpha$$
Integrating in between the limits t_0 to t yields
$$\int_{t_0}^t\lim_{h\to 0}\frac{x_\alpha(t+h)-x_\alpha(t)}{h}$$
$$=\int_{t_0}^t\lim_{h\to 0}\frac{f(t,x_\alpha(t+h))-f(t,x_\alpha(t))}{h}$$
$$x_\alpha(t)=x^\alpha(t_0)+\int_{t_0}^t f(s,x_\alpha(s))ds$$
Thus $x_\alpha(t)$ is a solution of the fuzzy integral equation (2.1.2).

Similarly, if $t<t_0$, we can prove the same result by using the definition of
$$\lim_{h\to 0^-}\frac{f(t_0)-f(t_0-h)}{h}$$

Definition 2.5: We say that $f(t,x_\alpha(t))$ satisfies a Lipschitz condition with the Lipschitz constant $K>0$ if for any $(t,x_\alpha(t))$ and $(t,y_\alpha(t))\in T\times E^n$
$$D(f(t,x_\alpha(t),f(t,y_\alpha(t)))\le KD(x_\alpha,y_\alpha)$$ for each $\alpha\in[0,1]$.

Theorem 2.1.2: Let $f: T\times E^n\to E^n$ be continuous and satisfies a Lipschitz condition with Lipschitz constant $K>0$. Then the initial problem (2.1.1) has one and only one solution for each $\alpha\in[0,1]$.

Proof: Fix $\alpha\in[0,1]$. Now for any $[\Phi_\alpha(t),\Psi_\alpha(t)]\in(J,E^n)$ define
$$H(\Phi_\alpha,\Psi_\alpha)(t)=sup_{t\in J}D(\Phi_\alpha(t),\Psi_\alpha(t)).$$
Since (E^n,D) is a complete metric space, it follows that $C(J,(E^n)$ is a complete metric space. Then consider
$$T\Phi_\alpha(t)=y_0+\int_{t_0}^t f(s,\Phi_\alpha(s))ds$$
and $$T\,\Psi_\alpha(t)=y_0+\int_{t_0}^t f(s,\Psi_\alpha(s))ds$$
Hence
$$H(T\Phi_\alpha(t),T\,\Psi_\alpha(t))$$
$$=sup_{t\in J}D\left(\int_{t_0}^t(f(s,\Psi_\alpha(s)),f(s,\Psi_\alpha(s))ds)\right)$$
$$\le nKH(\Phi_\alpha,\Psi_\alpha)$$
For all $\Phi_\alpha,\Psi_\alpha\in C[J,(E^n]$. Hence by the generalized contractive mapping theorem T has unique fixed point whenever $nK<1$, which is in fact the desired solution of the fuzzy initial value problem for a fixed $\alpha\in[0,1]$. This is true for all $\alpha\in[0,1]$, the result follows:

2.MAIN RESULT

In this section, we consider the non-linear fuzzy differential equation of third order
$$y'''_\alpha(t)=f\left(t,y_\alpha(t),y'_\alpha(t),y''_\alpha(t)\right) \quad (3.1)$$

$$y_\alpha(a)=y_1,y''_\alpha(a)=m,y_\alpha(b)=y_2$$
where f is continuous on the interval $[a,b]$ and $\alpha\in[0,1]$.

If $y_\alpha(t)$ is a solution of the differential equation (3.1) if and only if $y_\alpha(t)$ is a solution of the integral equation

$$y_\alpha(t)$$
$$=y_2+m\frac{(b-a)^2}{2}+2n(b-a)$$
$$+\int_a^t\frac{(t-s)^2}{2}f\left(s,y_\alpha(s),y'_\alpha(s),y''_\alpha(s)\right)ds$$

if and only if $y_\alpha(t)$ is a solution of the non-homogeneous fuzzy boundary value problem

$$y_\alpha'''(t) = f\left(t, y_\alpha(t), y_\alpha'(t), y_\alpha''(t)\right)$$

$y_\alpha(0) = y_1, y_\alpha'(0) = m, y_\alpha''(0) = n$ for all $\alpha \in [0,1]$.

Let $C^2[J, E^n, H]$ denote a complete metric space and for any $\alpha \in [0,1]$,

$\Phi_\alpha \in C^2[J, E^n]$, we define

$$(F\Phi_\alpha)(t) = \int_a^c G_\alpha(t,s) f\left(s, \Phi_\alpha(s), \Phi_\alpha'(s), \Phi_\alpha''(s)\right) ds$$

where $G_\alpha(t,s)$ is the Green's function for the homogeneous boundary problem.

Theorem 3.1.1: Let $y_1, y_2, y_3, x_2, m \in R$ with $x_1 < x_2 < x_3$ and suppose that there exists a positive constant N such that

$|f(x,y,z,w)| \le N$ for all $x \in [x_1, x_3]$, $-\infty < y, z, w < \infty$. Then there exists solution to each of the four point boundary value problems (3.1.3i) and (3.1.4i), $i = 1, 2$

$$y''' = f(t, y, y', y'')$$
$$y(x_1) = y_1, y^i(x_2) = m, y(x_2) = y_2 \quad (3.1.3i)$$

and

$$y''' = f(y, y, y', y'')$$

$$y(x_2) = y_2, y^i(x_2) = m, y(x_3) = y_3 \quad (3.1.4i)$$

Proof: For the proof of the theorem, we refer to Dennis Barr and Tom Sherman [1]

Theorem 3.1.2: Let $f \in C[I \times E^n \times E^n \times E^n, E^n]$ and satisfy the Lipschitz condition with the Lipschitz constant K, L, M and assume that

$$\alpha_1 = \frac{2}{81} K(b-a)_\alpha^3 + \frac{1}{4} L_\alpha (b-a)^2 + \frac{2}{3} M_\alpha (b-a) < 1$$

Then the following two-point boundary value problem

$$y_\alpha'''(t) = f\left(t, y_\alpha(t), y_\alpha'(t), y_\alpha''(t)\right)$$

$$y_\alpha(a) = y_1, y_\alpha'(a) = m, y(b) = y_2 \quad (3.1.5)$$

has one and only solution for all $\alpha \in [0,1]$.

Proof: The boundary value problem

$$y_\alpha''' = 0$$

$$y_\alpha(a) = 0,\ y_\alpha'(a) = 0, y_\alpha(b) = 0$$

for each $\alpha \in [0,1]$ has no non-trivial solution. Therefore if $h_\alpha(t)$ is a continuous function on $[a,b]$ and $\alpha \in [0,1]$, the equation

$$y_\alpha'''(t) = h_\alpha(t)$$

$$y_\alpha(0) = 0, y_\alpha'(0) = 0,\ y_\alpha(b) = 0$$

has unique solution given by

$$G_\alpha(s,t) = \begin{cases} (x-s)^2 - \dfrac{(x-a)(b-s)^2}{2(b-a)}, a \le s \le x \le b \\ -\dfrac{(x-a)(b-s)^2}{2(b-a)}, a \le x \le s \le b \end{cases}$$

It can be shown by elementary methods that

$$max_{a \le x \le b} \int_a^b |G_\alpha(x,s)|\, ds \le \frac{2}{81}(b-a)^3$$

$max_{a \le x \le b} \int_a^b |(G_\alpha)_x(x,s)|\, ds \le \frac{1}{4}(b-a)^2$ and

$$max_{a \le x \le b} \int_a^b |(G_\alpha)_{xx}(x,s)|\, ds \le \frac{2}{3}(b-a)$$

Let B be the Banach space of twice continuously differentiable function on $[a,b]$ with norm

$$\|y_\alpha\| = max_{a \le x \le b}\left[L|y_\alpha(x)| + M|y_\alpha'(x)| + N\|y_\alpha''(x)\|\right]$$

for each $\alpha \in [0,1]$, then can show that if T defined by

$$T(y_\alpha(x)) = \int_a^b G_\alpha(x,s) f(s, y_\alpha(s), y_\alpha'(s), y_\alpha''(s)) ds$$

satisfies

$\|T((y_1)_\alpha) - T((y_2)_\alpha)\| \leq \alpha_1 \|(y_1)_\alpha - (y_2)_\alpha\|$ for each $\alpha \in [0,1]$ and

$$\alpha_1 = [\frac{2}{81}L(b-a)^3 + \frac{M}{4}(b-a)^2 + \frac{2}{3}N(b-a)] < 1$$

One can easily show that T is a contraction mapping by Banach fixed point theorem, T has a unique solution. Now, to obtain a unique solution of the two-point boundary value problem (3.1.5), let $h_\alpha(t)$ be a third degree polynomial satisfying $h_\alpha(a) = y_1$, $h_\alpha'(a) = m$ and $h_\alpha(a) = y_2$ for $\alpha \in [0,1]$, then by repeating the same process, we obtain a unique solution of the two-point boundary value problem (3.1.5).

Definition 3.1.1: we say that the function $f(x, y_\alpha, z_\alpha, w_\alpha)$ satisfies condition at a point β on (α, γ) if there exists α and γ such that for $\alpha \subset \beta \subset \gamma$,

(i) $(y_1)_\alpha \geq (y_2)_\alpha$, $(z_1)_\alpha < (z_2)_\alpha$ and $(w_1)_\alpha = (w_2)_\alpha$ for all $x \in (\alpha, \beta]$ and

(ii) $(y_1)_\alpha \leq (y_2)_\alpha$, $(z_1)_\alpha < (z_2)_\alpha$ and $(w_1)_\alpha = (w_2)_\alpha$ implies

$f(x, (y_1)_\alpha, (z_1)_\alpha, (w_1)_\alpha) < f(x, (y_2)_\alpha, (z_2)_\alpha, (w_2)_\alpha)$ for all $x \in [\beta, \gamma)$.

Lemma 3.1.1: Let $y_1, y_2, y_3, x_2 \in R$ with $x_1 < x_2 < x_3$ and suppose that $f(x, y, z, w)$ satisfies condition A at x_2 on (x_1, x_3) with $x_1 \leq \alpha < x_2 < \gamma < x_3$. Then for each $m \in R$ there exists at most one solution to each of the boundary value problems:

$$y_\alpha''' = f(x, y_\alpha, y_\alpha', y_\alpha'')$$

(3.1.5i)

$$y_\alpha(x_1) = y_1, y_\alpha(x_2) = y_2, (y_\alpha^i)'(x_2) = m \ (i = 1,2)$$

$$y_\alpha''' = f(x, y_\alpha, y_\alpha', y_\alpha'')$$

(3.1.6i)

$$y_\alpha(x_2) = y_2, (y_\alpha^i)'(x_2) = m, y_\alpha(x_3) = y_3 \ (i = 1,2)$$

Proof: The proof of the uniqueness of solution of (3.1.6$_2$) will be given. An analogous proof will establish uniqueness of solutions of (3.1.5i).

Suppose $(\Phi_1)_\alpha(x)$ and $(\Phi_2)_\alpha(x)$ be two solutions of (3.1.6$_2$) for each $\alpha \in [0,1]$. Then write $\Psi_\alpha(x) = (\Phi_1)_\alpha(x) - (\Phi_2)_\alpha(x)$. Clearly, $\Psi_\alpha(x_2) = 0$, $\Psi_\alpha(x_3) = 0$, $\Psi_\alpha'(x_2) = 0$. If $\Psi_\alpha'(x_2) = 0$, then from uniqueness of initial value problems $(\Phi_1)_\alpha(x) = (\Phi_2)_\alpha(x) \ \forall x \in [x_2, x_3]$. Thus assume that $\Psi_\alpha'(x) \neq 0$. Then there exists $r \in (x_2, x_3)$ such that $\Psi_\alpha'(r) \neq 0$ for all $x \in [x_2, r)$. Assume that $\Psi_\alpha''(x) > 0$. If $\Psi_\alpha'(x) < 0$ then the following arguments can be applied to $(-\Psi_\alpha')(x)$. The above properties of $\Psi_\alpha(x)$ implies that there exists $p \in (x_2, r)$ such that $\Psi_\alpha''(p) < 0$.

Hence either

(i) $\Psi_\alpha'(x) < 0$ for all $x \in (x_2, p]$ or

(ii) there exists a $q \in (x_2, p]$ such that $\Psi_\alpha'(q) = 0$ and $\Psi'(x) < 0$

for all $x \in (q, p]$

Suppose (i) holds. Then

$$\Psi_\alpha'''(x_2) = \lim_{x \to x_2^+} \frac{\Psi_\alpha''(x) - \Psi_\alpha''(x_2)}{x - x_2} = \lim_{x \to x_2^+} \frac{\Psi_\alpha''(x)}{x - x_2} \leq 0$$

However f satisfies condition A at x_2 implies

$$\Psi_\alpha'''(x_2) = f(x_2, (y_1)_\alpha(x_2), (y_1)_\alpha'(x_2), (y_1)_\alpha''(x_2)) - f(x_2, (y_2)_\alpha(x_2), (y_2)_\alpha'(x_2), (y_2)_\alpha''(x_2)) > 0$$

Similar arguments hold if (ii) holds.

Thus either (i) or (ii) holds a contradiction arises. Hence uniqueness is established.

Theorem 3.1.3: Let $y_1, y_2, y_3, x_2 \in R$ with $x_1 < x_2 < x_3$ and suppose that for each $m \in R$ there exist solution (3.1.3i) and (3.1.4i) $(i = 1, 2)$ and f satisfies condition A at x_2 on (x_1, x_3). Then there exists a unique solution of the fuzzy non-linear third order boundary value problem

$$y_{\alpha}''' = f(x, y_{\alpha}, y_{\alpha}', y_{\alpha}'')$$

$y_{\alpha}(x_1) = y_1, y_{\alpha}(x_2) = y_2, y_{\alpha}(x_3) = y_3$ for all $\alpha \in [0,1]$.

Proof: Proof follows from lemma (3.1.1).

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