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Kronecker Product of Matrices and Applications to Two-Point Boundary Value Problems Associated With First Order Matrix Difference Systems

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Abstract
This paper presents a criteria for the existence and uniqueness of solutions to two-point boundary value problems associated with a system of Kronecker product first order differential equation. Some of the natural questions on Kronecker product of matrices about seemingly“simple” cases are still unansweredin spite of their increasing interest and hence some significant results on Kronecker product of matrices are answered in this paper. We mainly explain a set of necessary and sufficient conditions on the decomposition of a matrix into Kronecker product of two matrices.

Keywords: First order difference system, Kronecker product decomposition, Fundamental matrix solution, Cholesky decomposition and QR-algorithms AMS (MOS): 34 B14, 93 B05, 93 B15, 65 L09, 15 A06

Introduction

1. In recent years difference equations ariseas a natural description of observed evolution phenomena and found many applications to Science and Engineering problems [1-5]. For this reason, we continue our attention to

$$x(n+1) = A(n)x(n) + f(n) \qquad (1.1)$$

$$M_1 x(n_0) + N_1 x(n_f) = \alpha$$

And

$$y(n+1) = B(n)y(n) + g(n) \qquad (1.2)$$

$$M_2 y(n_0) + N_2 y(n_f) = \beta \,,$$

Where$A(n)$ and $B(n)$ are square matrices of order (mxm) and (nxn) respectively and all scalars are assumed to be real. The above equations can be put in the form

$$(x(n+1) \otimes y(n+1)) = [A(n) \otimes I_p + I_m \otimes B(n)][x(n) \otimes y(n)] \qquad ,(1.3)$$

And the boundary condition matrices can be written as

$$(M_1 \otimes I_p + I_m \otimes N_1)(x(n) \otimes y(n)) +$$

$$(M_2 \otimes I_n + I_p \otimes N_2)(x(n) \otimes y(n)) = \alpha \otimes \beta \,, \qquad (1.4)$$

where A is an $(m \times m)$ matrix and B is $(p \times p)$ square matrices, $x(n)$ and $y(n)$ are column matrices of order $(m \times 1)$ and $(p \times 1)$ respectively. Recently Kronecker product of matrices played an important role in multi variate analysis and in the construction of fast and practical algorithms to solve system of linear equations. These algorithms are very useful in Signal processing, Image Processing, computer vision, quantum computingto mention a few. We now mention some of the basic properties of the Kronecker product of two matrices. Let $A = (a_{ij})$ be an $(m \times n)$ matrix and $B = (b_{ij})$ be a $(p \times q)$ matrix then their Kronecker product $(A \otimes B)$ is defined as

$$(A \otimes B) = \left(a_{ij}B\right) \text{for all } i = 1,2,\dots,m;\ j = 1,2,\dots,n$$

and is in fact an $(mp \times nq)$ matrix. The kronecker product of matrices defined above has the following properties:

1. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ (Provided AC and BD are defined)
2. $(A \otimes B)^T = A^T \otimes B^T$ (A^T stands for transpose of the matrix A)
3. $(A + B) \otimes C = (A \otimes B) + (B \otimes C)$
4. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (provided A and B are invertible)
5. $(A \otimes B)(n) = (A(n) \otimes B(n))$ (For discrete systems)
   $(A \otimes B)(n+1) = (A(n+1) \otimes B(n+1))$ (For discrete systems)
6. $\|A \otimes B\| = \|A\|\|B\|$

For more information on Kronecker Product of matrices, we refer to a recent contributions of Murty, Fausett[8], Divya, Yan Wu, Dileep [12], Y. Wu and
K. N. Murty [9,10]. For basic results on Kronecker Product of Matrices and Linear Systems we refer to Kasi Viswanadh, SriramBhagavatulaet. al [6,7] and Divya et.al in [11].

$$(M_1 \otimes I_p + I_m \otimes N_1)(x \otimes y)(n_0) + (M_2 \otimes I_p + I_m \otimes N_2)(x \otimes y)(n_f) = (\alpha \otimes \beta)$$

Let $X(n, n_0, e_i)$, $i = 1,2,\dots,m$ be m linearly independent solutions having e_i as initial vectors, and let S be the solution space of $x(n+1) = A(n)x(n)$. It may be noted that any element of S can be expressed as a linear combination of the set of n solutions of $X(n, n_0, e_i) i = 1,2,\dots,m$ i.e. if $x(n)$ is any solution of (1.1) with $f(n) = 0$, then

$$x(n) = \sum_{i=1}^{m} c_i\, X(n, n_0, e_i),\ i = 1,2,\dots,m$$

We define K functions $f(n)$ on $N_{n_0}^{+}$ as

$$K(n) = \begin{bmatrix} f_1(n) & f_2(n) & . \ . \ . & f_m(n) \\ f_1(n+1) & f_2(n+1) & . \ . \ . & f_m(n+1) \\ . & . & . \ . \ . & . \\ f_1(n+k-1) & f_2(n+k-1) & . \ . \ . & f_m(n+k-1) \end{bmatrix}. \qquad (1.5)$$

If $f_i(n)$, $i = 1,2,\dots,m$ are m solutions of

$$x(n+1) = A(n)x(n),$$

then $\det[K(n)] \neq 0$. If $x(n)$ is any solution of (1.1) and $\bar{x}(n)$ be a particular solution of(1.5) then any solution of (1.5) can be written as

$$x(n) = \bar{x}(n) + \sum_{i=1}^{m} \alpha_i\, X(n, n_0, e_i), i = 1,2,\dots,m.$$

Similarly, if $y(n)$ is any solution of (1.2) with $g(n) = 0$ and $\bar{y}(n)$ be a particular solution of (1.2), then

$$y(n) = \bar{y}(n) + \sum_{i=1}^{p} \beta_i Y(n, n_0, e_i),\ i = 1,2, \dots, p.$$

We have the following interesting result:

Theorem 1.1:

If $X(n)$ is a fundamental solution of the homogeneous difference system

$$x(n+1) = A(n)x(n)$$

and$Y(n)$ is fundamental matrix of

$$y(n+1) = B(n)y(n)$$

then$\big(X(n) \otimes Y(n)\big)$ is a fundamental matrix of

$$(x \otimes y)(n+1) = [A(n) \otimes B(n)][x(n) \otimes y(n)]$$

Proof: Now consider

$$(X \otimes Y)(n+1) = X(n+1) \otimes Y(n+1)$$

$$= A(n)X(n) \otimes Y(n)B(n)$$

$$= [A(n) \otimes B(n)][X(n) \otimes Y(n)]$$

This clearly shows that $(x(n) \otimes y(n))$ is a solution of

$$(x(n+1) \otimes y(n+1)) = [A(n) \otimes B(n)][x(n) \otimes y(n)]$$

This paper is organized as follows:

Section 2 presents the general solution of the Kronecker Product of non-homogeneous difference system. $(x(n+1) \otimes y(n+1)) = [A(n) \otimes B(n)][x(n) \otimes y(n)] + (f(n) \otimes g(n))$, (2.1) and then present the general solution of the Kronecker Product boundary value problems.

Section 3 presents three algorithms when the Kronecker Product boundary value problem has a singular characteristic matrix.

Now, any solution of the non-homogeneous Kronecker Product system (2.1) is of the form

$$y(n) = \bar{y}(n) + [\Phi(n, n_0) \otimes \Psi(n, n_0)][C_1 \otimes C_2]$$

whereC_1 is a constant m-vector and C_2 is a constant p-vector and $\Phi(n, n_0)$ is a fundamental matrix of $x(n+1) = A(n)x(n)$ and $y(n)$ is a particular solution of the system (2.1) and $\Psi(n, n_0)$ is also a fundamental matrix of $y(n+1) = B(n)y(n)$.

We now verify $[\Phi(n, n_0) \otimes \Psi(n, n_0)]$ is a solution of the homogeneous system.

Theorem 2.1: If $\Phi(n, n_0)$is a fundamental matrix of the homogeneous system

$x(n+1) = A(n)x(n)$and$\Psi(n,n_0)$ is fundamental matrix of $y(n+1) = B(n)y(n)$ then $[\Phi(n,n_0) \otimes \Psi(n,n_0)]$ is a fundamental matrix of homogeneous Kronecker product system (2.1)

Proof: Consider$[\Phi(n,n_0) \otimes \Psi(n,n_0)]$. It can be easily seen that

$$\Phi(n+1,n_0) \otimes \Psi(n+1,n_0) = A(n)\Phi(n_0) + B(n)\Psi(n_0)$$

$$= [A(n) \otimes B(n)][\Phi(n_0) \otimes \Psi(n_0)]$$

$$= [A(n) \otimes B(n)][C_1 \otimes C_2]$$

where$C_1 = \Phi(n_0)$ and $C_2 = \Psi(n_0)$. Hence $[\Phi(n) \otimes \Psi(n)]$is a fundamental matrix of the homogeneous Kronecker Product system (2.1)

Theorem 2.2: A particular solution $\bar{y}(n)$ of non-homogeneous Kronecker Product system (2.1) is given by

$$\bar{y}(n) = \sum_{j=n_0}^{n-1}[\Phi(n,j+1) \otimes \Psi(n,j+1)][f(n) \otimes g(n)]$$

Proof: Any solution of the homogeneous Kronecker product homogeneous system is of the form

$$[x(n) \otimes y(n)] = [\Phi(n,n_0) \otimes \Psi(n,n_0)][C_1 \otimes C_2]$$

whereC_1 and C_2constant m and p- vectors respectively.But a solution cannot be a solution of the Kronecker Product non-homogeneous system unless $[f(n) \otimes g(n)] = 0$. So we seek a particular solution of the system (2.1) in the form

$$[x(n) \otimes y(n)] = [\Phi(n,n_0) \otimes \Psi(n,n_0)][C_1 \otimes C_2](n)$$

Then

$$[x(n+1) \otimes y(n+1)] = [\Phi(n+1,n_0) \otimes \Psi(n+1,n_0)][C_1(n+1) \otimes C_2(n+1)]$$

Therefore

$$[\mathrm{A(n)}\Phi(n,n_0) \otimes B(n)\Psi(n,n_0)][C_1(n+1) \otimes C_2(n+1)] =$$

$$\mathrm{A(n)}\Phi(n,n_0) \otimes B(n)\Psi(n,n_0)[C_1(n) \otimes C_2(n)] + [f(n) \otimes g(n)]$$

Thus

$$[\mathrm{A(n)}\Phi(n,n_0) \otimes B(n)\Psi(n,n_0)][C_1(n+1) - C_1(n) \otimes C_2(n+1) - C_2(n)] = [f(n) \otimes g(n)]$$

$$[\mathrm{A(n)}\Phi(n,n_0) \otimes B(n)\Psi(n,n_0)]\Delta\, C(n) = [f(n) \otimes g(n)]$$

$$\Delta\, C(n) = [\mathrm{A(n)}\Phi(n,n_0) \otimes B(n)\Psi(n,n_0)]^{-1}[f(n) \otimes g(n)]$$

Or

$$C(n) = C(n_0) + [\mathrm{A(n)} \otimes B(n)]^{-1}[\Phi(n,n_0) \otimes \Psi(n,n_0)]^{-1}[f(n) \otimes g(n)]$$

$$C(n) = C(n_0) + [\mathrm{A(n)} \otimes B(n)]^{-1} \sum_{j=n_0}^{n-1} \Phi(n_0, j+1) \otimes \sum_{j=p_0}^{p-1} \Psi(n_0, j+1)\, [f(n) \otimes g(n)] \qquad \textbf{(2.2)}$$

Note that $\Phi(n, n_0) = \Phi(n)\Phi^{-1}(n_0)$

$$[\Phi(n, n_0)]^{-1} = \Phi(n_0)\Phi^{-1}(n)$$

$$= \Phi(n_0, n)$$

The above formula is known as Variation of Parameters formula. We also assume that $[A(n) \otimes B(n)]$ is invertible. Thus any solution $[x(n, n_0, C_1) \otimes y(n, n_0, C_2)]$is given by

$$\begin{aligned} &[x(n, n_0, C_1) \otimes y(n, n_0, C_2)] \\ &\qquad = [\Phi(n, n_0) \otimes \Psi(n, n_0)][C_1 \otimes C_2](n_0) \\ &\qquad + \sum_{j=n_0}^{n-1} \Phi(n, n_0)\Phi(n-j+1) f_j(n) \otimes \sum_{j=p_0}^{p-1} \Psi(n, n_0)\Psi(n-j+1)\, g_j(n) \end{aligned}$$

(2.3)

Section 3: Two-Point Boundary Value Problems

In this section, we shall be concerned with the two-point boundary value problem satisfying two-point boundary conditions

$$(M_1 \otimes N_1)(y \otimes x)n_0 + (M_2 \otimes N_2)(y \otimes x)n_f = (\alpha \otimes \beta) \qquad (3.1)$$

wheren_0, $n_f \in N_{n_0}^{+}, n_0 < n_f$. Substituting the general form of the solution given in (2.3) in the boundary condition matrix (3.1) we get

$$\begin{aligned} &(M_1 \otimes N_1)[\Phi(n, n_0) \otimes \Psi(n, n_0)] + (M_2 \otimes N_2)[\Phi(n, n_0) \otimes \Psi(n, n_0)][C_1 \otimes C_2](n_0) \\ &\qquad = (\alpha \otimes \beta) - \left[(M_1 \otimes N_1) \sum_{j=n_0}^{n-1} \Phi(n_0, j+1) f_j \otimes \sum_{j=p_0}^{p-1} \Psi(n_0, j+1)\, g_j \right] [f \otimes g] \end{aligned}$$

If we assume that the homogeneous boundary value problems has only the trivial solution, it follows that the characteristic matrix D given by

$$D = (M_1 \otimes N_1)[\Phi(n_0, n_0) \otimes \Psi(n_0, n_0)] + (M_2 \otimes N_2)\left[\Phi\left(n_f, n_0\right) \otimes \Psi\left(n_f, n_0\right)\right]$$

is non-singular. Hence

$$\begin{aligned} [C_1 \otimes C_2](n_0) = D^{-1} \Bigg[&(\alpha \otimes \beta) \\ &- \sum_{j=n_0}^{n-1} (M_1 \otimes N_1)\Phi(n_0 + j + 1) \otimes \Psi(n_0 + j + 1) \\ &- \sum_{j=p_0}^{p-1} (M_2 \otimes N_2)\Phi(n_0 + j + 1) \otimes \Psi(n_0 + j + 1) \Bigg] \end{aligned}$$

Note that,$\Phi(n_0, n_0) = I$. The right hand side of the system of equations can be written as

$D\ C = [f \otimes g](n-1)$

where$f = D^{-1}\Big[(\alpha \otimes \beta) - \sum_{j=n_0}^{n-1}(M_1 \otimes N_1)\Phi(n_0+j-1) \otimes \Psi(n_0+j-1) - \sum_{j=p_0}^{p-1}(M_2 \otimes N_2)\Phi(n_0+j-1) \otimes \Psi(n_0+j-1)\Big]$

Now we solve system of equations by using the following algorithms, when D is singular or deficient rank.

Section 4: Algorithms:

In this section, we first develop, two new kinds of Kronecker Product decompositions will be developed i.e. Kronecker Product general decomposition and KroneckerProduct isomer decomposition.

Theorem 4.1: Let A be an(mxm) matrix and B be a (pxp) matrix. Then the Kronecker Product decomposition for a matrix $M = (A \otimes B) \in R^{mpXmp}$ is given by

$M = M_{ij} \in R^{mp\times np}$can be decomposed to the form $M = (A \otimes B)$

Proof: Let $A = (a_1, a_2, \dots, a_m) \in R^{m\times m}$ with $a_i \in R^m, 1 \le i \le m$ then denote $Vect(A) = (a_1', a_2', \dots, a_m')'$ then the matrix $M = (A \otimes B)$ if and only if

$rank\{Vector(M_{11}), Vector\ (M_{12}), \dots, Vector(M_{1m}), \dots., Vector\ (M_{mm})\} = 1$

In general, the Kronecker product decomposition is not unique.

The following algorithm describes the general program of Kronecker product decomposition.

Algorithm 1:

Step 1: Input M, m, p, the size of the matrix $M \in R^{mp\times mp}$ and $M! = 0$

Step 2: Define $M_{ij}, i = 1,2, \dots, m, j = 1,2, \dots, p$

Step 3: Calculate $Vect(M_{ij})$

Step 4: If $rank\{Vec(M_{11}), Vec\ (M_{12}), \dots, Vec(M_{1m}), \dots., Vec(M_{mm})\} = 1$

go to step 5

else output "Cannot be decomposed"; end

Step 5: look for the first $M_{ij}! = 0$, define $B = M_{ij}$

Step 6: Calculate a_{ij}: $Vect\left(M_{ij}\right) = a_{ij}\ Vect(B), i = 1,2, \dots, m, j = 1,2, \dots, p$

Step 7: Define $A = \left(a_{ij}\right), i = 1,2, \dots, m, j = 1,2, \dots, p$; output$A, B$; end

Theorem 4.2: Let A be an $(m \times n)$ matrix with rank r and B be a $(p \times q)$ matrix with rank s. Then

$rank(A \otimes B) = \rho(A \otimes B) = \rho(A).\rho(B)$where$\rho$ stands for rank of $(.)$

Proof: Since the matrix A is of rank $r \le \min(m, n)$, there exist invertible matrices $P_A$andQ_A such that

$P_A A Q_A = I^r$ ($I$is an $m \times n$ matrix) and similarly

$P_B B Q_B = I^s$ ($I$is an $p \times q$ matrix)

Applying mixed property, we therefore have

$(P_A \otimes P_B)(A \otimes B)(Q_A \otimes Q_B) = I^r \otimes I^s$

Since $(P_A \otimes P_B)$and $(Q_A \otimes Q_B)$ are invertible matrices, we therefore have

$$\rho(A \otimes B) = \rho(I^r \otimes I^s)$$

But the matrix has precisely rs-non-zero elements each on a different row and column by construction. Therefore

$\rho(A \otimes B) = \rho(A).\rho(B)$

The following algorithm presents the general program for Kronecker product gemel decomposition (KPGD) problem that include whether a matrix can be decomposed or not and how to get the results of KPGD.

Algorithm 2:

Step 1: input D, p, q; verify the size of D is $p^2 \times q^2$ and $M! = 0$

Step 2: define $flag = 0$, D_{ij}, $i = 1,2, \dots, p, j = 1,2, \dots, q$

Step 3: for (i, j), $i = 1,2, \dots, p, j = 1,2, \dots, q$

if:$D_{ij} = 0$ continue

else if: $D_{ij}! = 0$and$D_{ij} \leq 0 flag = 1$, break;

else: define $B = D_{ij} / \sqrt{m_{i,j}^{ij}} < 0 flag = 2$, break;

Step 4: if $flag == 2 \&\& D = B \otimes B$, A=B; output A, end

else output "Cannot be decomposed"; end

Section 5: QR-Factorization

Let $A \in R^{m \times m}$ and $B \in R^{p \times p}$ matrices and let P_A, L_A and P_B, L_B be the matrices corresponding to their LU-factorizations with partial pivoting. Then we can derive LU-factorization to the Kronecker product

$(A \otimes B)$as follows:

$$(A \otimes B) = (P_A^T L_A U_A) \otimes (P_B^T L_B U_B)$$

$$= (P_A^T \otimes P_B^T)(L_A \otimes L_B)(U_A \otimes U_B)$$

If A and B are positive (semi) definite and L_A, L_B be the matrices corresponding to their Cholesky factorizations. Then Cholesky factorization of their Kronecker product as

$$(A \otimes B) = (L_A L_A^T) \otimes (L_B L_B^T)$$

$$= (L_A \otimes L_B)(L_A^T \otimes L_B^T)$$

$$= (L_A \otimes L_B)(L_A \otimes L_B)^T$$

It may be noted that

$$(A \otimes B)^T = A^T \otimes B^T$$

Proof: $(A \otimes B) = \frac{1}{2}[Q(A \otimes B) + (B \otimes A)Q^T]^T$

$$= \frac{1}{2}[Q(A \otimes B)^T + (B \otimes A)^T Q^T]$$

$$= \frac{1}{2}[Q(A^T \otimes B^T + B^T \otimes A^T)Q^T]$$

$$= A^T \otimes B^T$$

Result 5.1: Let A be an $(m \times n)$ given matrix with rank $s \leq min\{m, n\}$. Then there exists a unique factorization of the form

$$AP = QR$$

with the following properties.

(i) P is an $(n \times n)$ permutation matrix with first n columns of AP form a basis of

$$Im\ (A) = \{Ax \in R^m / x \in R^n\}$$

(ii) Q is an $(m \times s)$ matrix with orthonormal columns and R is an $(s \times n)$ upper trapezoidal matrix of the form

$R = (R_1, R_2)$whereR_1 is a non-singular $(s \times s)$ upper triangular matrix with orthonormal columns and R_2 is a $(s, n - s)$ matrix. Note that columns of A are linearly independent. Then the system of equation

$$Ax = \alpha \qquad (5.1)$$

has a unique solution given by

$$x = (A^T A)^{-1} \alpha$$

Note that $(A^T A)$ is $(n \times n)$non-singular matrix. Similarly if the rows of A are linearly independent, then

$$Ax = \alpha$$

and $x = A^T y$ transforms into

$$AA^T y = \alpha\ ,$$
and$y = (AA^T)^{-1} \alpha$

Or

$x = A^T y = A^T (AA^T)^{-1} \alpha$is the unique solution of the system of equations (5.1). We make use of these results to establish our result on Kronecker product of linear system of equations

$$(A \otimes B)(x \otimes y) = (\alpha \otimes \beta). \quad (5.2)$$

Suppose A is an$(m \times m)$ matrix with full rank and B is $(p \times p)$ matrix with full rank. Suppose A and B are QR-decomposed as

$$AP_1 = Q_1 R_1 \text{and} BP_2 = Q_2 R_2 \text{ i.e.}$$

$$AP_1 = Q_1 \begin{bmatrix} R^{(1)} \\ 0 \end{bmatrix} \text{and} BP_2 = Q_2 \begin{bmatrix} R^{(2)} \\ 0 \end{bmatrix}$$

whereQ_1 and Q_2 are orthonormal matrices and $R^{(1)}$, $R^{(2)}$ are square upper triangular matrices and R_1, R_2 are upper triangular matrices. $O^{(1)}$, $O^{(2)}$ are Zero matrices of appropriate order. P_1andP_2 are permutation matrices arise from column pivoting used to keep the diagonal elements as far as away from zero as possible. If we introduce the partitioning of orthogonal matrices Q_1 and Q_2 in the form

$Q_1 = \begin{bmatrix} Q_1^{(1)} & Q_2^{(1)} \end{bmatrix}$and $Q_2 = \begin{bmatrix} Q_1^{(2)} & Q_2^{(2)} \end{bmatrix}$, then the system of equation can be back solved to obtain the general solution of the Kronecker product system of equations as in [3,4,6].

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