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On the Kronecker Product of matrices and their applications to linear systems Via modified QR- algorithm

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Abstract: This paper studies and supplements the proofs of the properties of the Kronecker Product of two matrices of different orders. We observe the relation between the singular value decomposition of the matrices and their Kronecker product and the relationship between the determinant, the trace, the rank and the polynomial matrix of the Kronecker products. We also establish the best least square solutions of the Kronecker product system of equations by using modified QR-algorithm.

Keywords: Kronecker product of matrices, first order system, rank of the Kronecker product of the matrices, QR-algorithm, singular value decomposition.

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1. Introduction

The Kronecker product named after the German Mathematician Leopold Kronecker is an interesting and current research and a great deal of work has been done by many distinguished Mathematicians like Don Fausett, K. N. Murty, Kasi Viswanadh, Lakshmi N. Vellanki to mention a few [7-9, 14-18]. The importance of Kronecker product linear systems gained momentum in recent years in linear algebra, systems theory, matrix calculus and their special fields. In fact the variation of parameters formula established by Kasi Viswanadh, DivyaNethi et.al [10, 11, 13, 17] created a new area of research in differential equations. The techniques adopted are new and can be applied to various problems on Spectral Theory, Method of Lines and Systems Analysis. For mathematical analysis on matrix theory, we refer to [1-4]. If A is an (nxn) matrix and B is an (mxm) matrix, then their Kronecker product of A and B is denoted by $A \otimes B$, is defined as,

$$A \otimes B = a_{ij}B, \text{ i, j} = 1, 2, ..., n. \qquad 1.1$$

and is in fact an (nmxnm) matrix.

The solution of the Sylvester system and Lyapunov system of equations is a hotspot area. Recently innovative and numerical algorithms developed to solve Kronecker product three point boundary value problems by Kasi Viswanadh, K. N. Murty, Lakshmi N. Vellanki, SriramBhagavatula paved a way for further development in differential equations[5, 6, 10,12]. This paper is organized as follows. Section 2 presents a criteria on the singular value of the Kronecker product and gives a definition of the permutation matrix. In addition, we prove the mixed product theorems and the conclusions on the vector operator in a different method. Section 3 is concerned with the best least squares solution in linear system of equations of the form

$$(A \otimes B)(x \otimes y) = (\alpha \otimes \beta), \qquad 1.2$$

where a is an (mxn) matrix , B is a (pxq) matrix and all scalars are assumed to be real.

Let F be a vector field. For any two matrices $A \in F^{mxn}$ and $B \in F^{pxq}$, we define their Kronecker product as

$A \otimes B = a_{ij}B$, i = 1, 2, ..., m, j= 1, 2, ..., n

$$=\begin{pmatrix} a_{11}B & a_{12}B & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & a_{2n}B \\ . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & , & , & , & a_{mn}B \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}I_p & a_{12}I_p & . & . & . & a_{1n}I_p \\ a_{21}I_p & a_{22}I_p & . & . & . & a_{2n}I_p \\ . & . & . & . & . & . \\ a_{m1}I_p & a_{m2}I_p & , & , & , & a_{mn}I_p \end{pmatrix}\begin{pmatrix} B & 0 & . & . & . & 0 \\ 0 & B & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & , & , & , & B \end{pmatrix}$$

$=(A \otimes I_p)(\mathrm{I_n} \otimes B)$.

Similarly,

$(A \otimes B) = (\mathrm{I_m} \otimes B)(A \otimes I_q)$.

Therefore

$(A \otimes B) = (A \otimes I_n)(\mathrm{I_m} \otimes B)$

$= (\mathrm{I_m} \otimes B)(A \otimes \mathrm{I_n})$, if A is an mxm matrix and B is an nxn matrix.

This means that $(\mathrm{I_m} \otimes B)$ and $(A \otimes \mathrm{I_n})$are commutative for square matrices A and B. Thus we have the following theorem.

Theorem 1.1: Let $A \in F^{mxn}$ and $B \in F^{pxq}$, then

$$(A \otimes B) = (A \otimes I_p)(\mathrm{I_n} \otimes B) = (\mathrm{I_m} \otimes B)(A \otimes I_q) .$$

2. Preliminaries

For a Kronecker product of two matrices, defined above has the following properties,

i. $(\mathrm{I_m} \otimes A)$ = diag[A, A, ..., A]

ii. If $\alpha = (\alpha_1, \alpha_2, \dots \alpha_m)^T$and $= (\beta_1, \beta_2, \dots \beta_n)^T$, then $\alpha \otimes \beta^T = \beta^T \otimes \alpha \epsilon F^{mxn}$.

iii. $(\mu A \otimes B) = (A \otimes \mu B) = \mu(A \otimes B)$, for any scalar μ.

iv. $(A + B) \otimes C = A \otimes C + B \otimes C$

v. $A \otimes (B + C) = A \otimes B + A \otimes C$

vi. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

vii. $(A \otimes B)^T = A^T \otimes B^T$

viii. $(A \otimes B)^* = A^* \otimes B^*$ (* refers to the transpose of the complex conjugate of the matrix)

ix. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (provided A and B are square non-singular matrices).

Theorem 2.1: Let $A \in F^{mxn}$, $C \in F^{nxp}$, $B \in F^{qxr}$ and $D \in F^{rxs}$, then

$(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

Proof: We have

$$(A \otimes B)(C \otimes D) = (A \otimes I_q)(I_n \otimes B)(C \otimes I_r)(I_p \otimes D)$$

$$= (A \otimes I_q)(C \otimes B)(I_p \otimes D)$$

$$= (A \otimes I_q)(C \otimes I_q)(I_p \otimes B)(I_p \otimes D)$$

$$= (AC \otimes I_q)(I_p \otimes BD)$$

$$= (AC \otimes BD).$$

The proof of the theorem is complete.

Theorem 2.2: If A and B are non-singular matrices , then

i. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
ii. $(A \otimes B)$is a normal matrix if A and B are normal matrices.
iii. $(A \otimes B)$is an orthogonal (unitary) matrix, if A and B are orthogonal matrices.

Proof: To prove that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$,

consider

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1}$$

$=I_n \otimes I_m$(If A and B are nxn and mxm matrices respectively)

$=I_{nm}$.

Similarly, we have $(A^{-1} \otimes B^{-1})(A \otimes B) = I_n \otimes I_m = I_{mn}$.

A square matrix A is said to be unitary, if $AA^* = A^*A = I$. For,

consider $(A \otimes B)(A^* \otimes B^*) = AA^* \otimes BB^*$

$= I_n \otimes I_m$

$=I_{nm}$.

A square matrix A is said to be normal if $AA^* = A^*A = I$.

We next turn our attention to the vector valued operator and a vec-permutation matrix.

3. Vector operator and vec-permutation matrix

Let $A \in F^{mxn}$, then the vector col[A] is defined as

$$\text{Col[A]}=\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \vdots \\ a_n \end{bmatrix} \epsilon F^{mxn} \ .$$

Theorem 3.1: Let $A \in F^{mxn}$, $B \in F^{nxp}$ and $C \in F^{pxn}$, then

i. $(I_p \otimes A)$col[B]=col[AB]
ii. $(A \otimes I_p)$col[c]=col[CA^T].

Proof: Let B_i denotes the i^th column of the matrix B, then

$$(I_p \otimes A)\text{col[B]}= \begin{bmatrix} A & 0 & \cdot & \cdot & . & 0 \\ 0 & A & \vdots & \vdots & \vdots & 0 \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & . & . & . & A \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \vdots \\ B_p \end{bmatrix}$$

$$= \begin{bmatrix} A(B_1) \\ A(B_2) \\ \cdot \\ \vdots \\ A(B_p) \end{bmatrix} = \begin{bmatrix} AB_1 \\ AB_2 \\ \cdot \\ \vdots \\ AB_p \end{bmatrix} = \text{col[AB]}.$$

$$(A \otimes I_p)\text{col[c]}= \begin{bmatrix} a_{11}C_1 + a_{12}C_2 + \cdots + a_{1n}C_n \\ a_{21}C_1 + a_{22}C_2 + \cdots + a_{2n}C_n \\ \dots\dots\dots\dots\dots\dots\dots . \\ a_{n1}C_1 + a_{n2}C_2 + \cdots + a_{nn}C_n \end{bmatrix}$$

$$= \begin{bmatrix} C(A_1^T) \\ C(A_2^T) \\ \cdot \\ C(A_n^T) \end{bmatrix} = \text{col}(CA^T).$$

Thus the proof of the theorem is complete.

Theorem 3.2: Let A be an (mxn) matrix, B be an (nxp) matrix and C be an (pxq) matrix . Then

$$Col\left[ABC\right]=\left(C^T\otimes A\right)Col\left[B\right]. \quad 3.1$$

Proof: We have

Col[ABC]=COL[(AB)C]

$=(C^T\otimes I_m)(I_p\otimes A)colB$

$$=\left(C^T\otimes A\right)Col\left[B\right].$$

The above Theorem is useful to solvelinear system of equations and in control of Linear systems.

Let e_{in} denote an n-dimensional column vector which has 1 in the i^{th} place and 0 elsewhere i.e.,

$$e_{in}=\left[0,0,......,1,0.......0\right]^T. \quad 3.2$$

Let P be the permutation matrix denoted by $P_{mn}{}^T=\begin{bmatrix} I_m\otimes e_{1n}{}^T \\ I_m\otimes e_{2n}{}^T \\ .... \\ I_m\otimes e_{nn}{}^T \end{bmatrix}$

Then $P_{mn}{}^T P_{mn}=\left[I_m\otimes e_{1n}{}^T, I_m\otimes e_{2n}{}^T,.........I_m\otimes e_{nn}{}^T\right]\begin{bmatrix} I_m\otimes e_{1n}{}^T \\ I_m\otimes e_{2n}{}^T \\ .... \\ I_m\otimes e_{nn}{}^T \end{bmatrix}$

$$= \mathrm{I}_m\otimes\begin{bmatrix} I_m\otimes e_{1n}{}^T \\ I_m\otimes e_{2n}{}^T \\ .... \\ I_m\otimes e_{nn}{}^T \end{bmatrix}$$

$$= \mathrm{I}_m\otimes\sum_{i=1}^{n} e_{in}e_{in}{}^T$$

$$= \mathrm{I}_m\otimes I_n = I_{mn} = \mathrm{I}.$$

For any matrix A of order (mxn),

$$Col\left[A\right]=P_{mn}Col\left[A^T\right].$$

If A Є F^{mxn}, B Є F^{nxm},then

$$B\otimes A=P_{mn}(A\otimes B)P_{nm}{}^T. \quad 3.3$$

Theorem3.3: Let $A\in F^{m\times m}, B\in F^{n\times n}$ then

$(i)\exp\left[A\otimes B\right]=\left[\exp A\otimes\exp B\right]$

$(ii) Sin\left[A\otimes B\right]=SinA\otimes CosB+CosA\otimes SinB$

$(iii) Cos\left[A\otimes B\right]=CosA\otimes CosB-SinA\otimes SinB$

Proof: Proof of (i) is obvious. To prove (ii) and (iii)
We consider

$$\exp\left[iA\otimes iB\right]=e^{iA}\otimes e^{iB}=\exp(iA)\otimes\exp(iB)$$

$$\exp\left[i(A\otimes B)\right]=\exp(iA)\otimes\exp(iB)$$

$$Cos(A\otimes B)+iSin(A\otimes B)=(CosA+iSinA)\otimes(CosB+iSinB)$$

$$=(CosA\otimes CosB-SinA\otimes SinB)+i(SinA\otimes CosB+CosA\otimes SinB)$$

Equating real and imaginary parts,we get (iii) and (ii).

Theorem 3.4: If a and B are (mxm) and (nxn) matrices respectively, then

$$(A \otimes B) = (A \otimes I_n)(I_m \otimes B)$$
$$= (I_m \otimes B)(A \otimes I_n)$$

Note that $(A \otimes I_n)$ and $(I_m \otimes B)$ are commutative for square matrices.
The power of Kronecker product of matrices is defined as

$$A^{(k+1)} = A^k \otimes A = A \otimes A^k, \ k = 1, 2, \dots \ .$$

If the following matrix products, $A_1 A_{2\dots} A_K$ and $B_1 B_{2\dots} B_K$ exist, then

$$(A_1 \otimes B_1)(A_2 \otimes B_2) \dots (A_k \otimes B_k) = (A_1 A_2 \dots A_k \otimes B_1 B_2 \dots B_k).$$

Theorem 3.5: Let $A \in F^{mxp}$, $B \in F^{nxq}$, then the system of equations

$$(A \otimes B)(x \otimes y) = (\alpha \otimes \beta) \qquad 3.4$$

has a least squares solution $(\overline{x} \otimes \overline{y})$, if and only if it is a solution of the augmented matrix system

$$(A \otimes B)^T (A \otimes B)(x \otimes y) = (A \otimes B)^T (\alpha \otimes \beta), \text{ where mp>nq.}$$

Proof: Assume that mn>pq and the columns of $(A \otimes B)$ are linearly independent. Let $(x \otimes y) \in F^{pq}$. Then $(A \otimes B)(x \otimes y)$ is an arbitrary vector in the column space of $(A \otimes B)$, which we can write as $F(A \otimes B)$. Let

$$r(x \otimes y) = (\alpha \otimes \beta) - (A \otimes B)(x \otimes y),$$

is a minimum if $(A \otimes B)(x \otimes y)$is the orthogonal projection of $(\alpha \otimes \beta)$ onto the space of $F(A \otimes B)$. Since $F(A \otimes B)^T = (A \otimes B)^T (\alpha \otimes \beta) - (A \otimes B)(\overline{x} \otimes \overline{y}) = 0$
which is equal to the system of normal equations of the form

$$(A \otimes B)^T (A \otimes B)(x \otimes y) = (A \otimes B)^T (\alpha \otimes \beta).$$

For the solution to be unique, the matrix $(A \otimes B)$ must have full column rank of $(A \otimes B)$.
Theorem 3.6: Consider a system of equations

$(A \otimes B)(x \otimes y) = (\alpha \otimes \beta)$and the associated normal system of equations

$(A \otimes B)^T (A \otimes B)(x \otimes y) = (A \otimes B)^T (\alpha \otimes \beta)$. Then the following are equivalent.

a. The least squares problem has a unique solution.
b. The linear system $(A \otimes B)(x \otimes y) = 0$ has only the trivial solution.
c. The columns of $(A \otimes B)$ are linearly independent.

Theorem 3.7: If $A \in F^{mxm}$, $B \in F^{nxn}$, then

i. trace $(A \otimes B)$=trace(A).trace(B)
ii. $|A \otimes B| = |A|^m |B|^n = |B \otimes A|$
iii. rank$(A \otimes B)$ = rank (A). Rank (B)
iv. $|(A \otimes B)(C \otimes D)| = |A \otimes B||C \otimes D|$
v. Rank$[(A \otimes B)(C \otimes D)]$=rank(AC).rank(BD).

Proofs are elementary and hence left to the reader.

4. Linear system of equations and Modified QR-algorithms.

In this section, we shall be concerned with the two linearsystems of equations of the form

$$Ax=\alpha \qquad 4.1$$

and $$By=\beta, \qquad 4.2$$

where A is an (mxp) matrix ,B is an (nxq) matrix ,x is a column vector of order (px1) and y is also a column vector of order (1xq). Equations (4.1) and (4.2) can be conveniently recast in the form

$$(A \otimes B)(x \otimes y) = (\alpha \otimes \beta). \qquad 4.3$$

Result 4.1: Let A be an (mxp) given matrix with rank r≤ min{m,p}.Then there exists a factorization of the form AP=QR with the following properties:

(I) P is a (pxp) permutation matrix with the first p columns of P form a basis of

$$I_m(A) = \{ Ax \in R^m / x \in R^p \}$$

(II) Q is an m x r matrix with orthonormal columns and R is an (r x p) upper trapezoidal matrix of the form $R=(R_1,R_2)$,where R_1 is non-singular (r x r) upper triangular matrix and R_2 is a (r x p-r) matrix. Similar results holds for the linear system (4.2).

Suppose A and B are QR decomposed as

$$A = Q_1 R_1 \text{ and } B = Q_2 R_2,$$

where Q_1 is (m x m) and Q_2 is (nxn) are both matrixes with orthonormal columns and R_1 is (m x p) and R_2 is (n x q) upper trapezoidal matrices. Now

$$(A\otimes B) = (Q_1R_1\otimes Q_2R_2)$$
$$= (Q_1\otimes Q_2)(R_1\otimes R_2).$$

Assume that rank (A) = p≤m and rank (B) =q≤n. Then the general structures of R_1 and R_2 takes the form

$$R_1 = \begin{bmatrix} r_{11}^{(1)} & r_{12}^{(1)} & . & . & . & r_{1p}^{(1)} \\ 0 & r_{22}^{(1)} & . & . & . & r_{2p}^{(1)} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & r_{pp}^{(1)} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix} \text{ and } R_2\begin{bmatrix} r_{11}^{(2)} & r_{12}^{(2)} & . & . & . & r_{1q}^{(2)} \\ 0 & r_{22}^{(2)} & . & . & . & r_{2q}^{(2)} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & r_{qq}^{(2)} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix},$$

where R_1 is (n x n) sub matrix and R_2 is a (q x q) sub matrix are $O^{(1)}$ and $O^{(2)}$ are the null matrix of appropriate order.

Theorem 4.2: Let $A = Q_1R_1$ and $B = Q_2R_2$ where Q_1 and Q_2 are square matrixes with orthonormal columns ; then $(Q_1 \otimes Q_2)$ is orthonormal.

Proof: $(A\otimes B) = (Q_1R_1\otimes Q_2R_2) = (Q_1\otimes Q_2)(R_1\otimes R_2)$

Consider $(Q_1\otimes Q_2)^T(Q_1\otimes Q_2) = (Q_1^T\otimes Q_2^T)(Q_1\otimes Q_2)$

$$= (Q_1^T Q_2\otimes Q_2^TQ_2)$$
$$= (I_m\otimes I_n)$$
$$= I_{mn.}$$

This implicates that $(Q_1\otimes Q_2)$ is orthonormal and $Z(R_1\otimes R_2) = \binom{\tau}{o}$ where Z is an (pq x pq) square matrix and O is a null matrix of order (mn-pq x pq) matrix .

Theorem 4.3: $\tau^T\tau$ is the Cholesky's factorization of $(A\otimes B)^T (A\otimes B)$ where $\tau=(R^{(1)}\otimes R^{(2)})$

Proof: Consider

$$(A\otimes B)^T (A\otimes B) = (A^T\otimes B^T)(A\otimes B)$$
$$= (A^TA)\otimes(B^TB)$$
$$=\left(R_1^TQ_1^T \otimes R_2^TQ_2^T\right)\left(Q_1R_1 \otimes Q_2R_2\right)$$
$$= (R_1\otimes R_2)^T(Q_1\otimes Q_2)^T(Q_1\otimes Q_2)(R_1\otimes R_2)$$
$$= (R_1\otimes R_2^T)Z^TZ(R_1\otimes R_2)$$
$$= [Z(R_1\otimes R_2)]^T[Z(R_1\otimes R_2)]$$
$$= \tau^T\tau$$
$$= G G^T,$$

Where G is upper triangular.

Now, applying these results to our main problem(4.3),we get the best least square solution $\overline{x} \otimes \overline{y}$ as follows:

$$(A\otimes B)(x\otimes y)(t_0) = (\alpha\otimes\beta)$$

$$(A\otimes B)^T(A\otimes B)(\overline{x} \otimes \overline{y})(t_0) = (A\otimes B)^T(\alpha\otimes\beta)$$

$$\tau^T\tau(\overline{x} \otimes \overline{y})(t_0) = (A\otimes B)^T(\alpha\otimes\beta) = (h_1\otimes h_2) \quad \text{(say)}$$

Since the coefficient matrix is the product of the upper and lower triangular matrices,the solution of the Kronecker Product Linear system can be computed in the following procedure.

(I) Solve the system of equation by forward substitution.

(II) Solve $\tau(\overline{x} \otimes \overline{y})(t_0) = (h_1\otimes h_2)$ by backward substitution.

If the dimension of $\tau(pq\otimes pq)$ is too large to permit the directsolution of τ, the above two step method can be further refined. Partition each of the vector $(h_1\otimes h_2)$ into p-sub matrices and proceed. If $(h_1\otimes h_2)$ and the solution vectors $(\overline{x} \otimes \overline{y})(t_0)$partitioned as in step (1) as

$$\begin{bmatrix} r_{11}^{(i)}\left[R^2\right]^T ....O_q..............O_q \\ r^{(i)}{}_{12}\left[R^2\right]^T .....r^{(i)}{}_2\left[R^2\right]^T .....O_q \\ .... \\ r_{1p}\left[R^2\right]^T ....r_{pq} \end{bmatrix}\begin{bmatrix} h^{(j)}{}_1 \otimes h_2^{(j)} \\ . \\ . \\ . \\ h^{(p)}{}_1 \otimes h_2^{(j)} \end{bmatrix} = \begin{bmatrix} \alpha_1^{(j)} \otimes \alpha_2^{(j)} \\ .... \\ \alpha_1^{(p)} \otimes \alpha_2^{(j)} \end{bmatrix}$$

$j=1,2,......q$

Since $\left[R^{(2)}\right]^T$ is a lower triangular matrix,the forward substitution solves the system.

Note that,if the columns of A (p≤m) are linearly independent and columns of B (q ≤ n) are linearly independent, then unique solution of the system of equations are obtained as follows.

$$(A\otimes B)(x\otimes y)=(\alpha\otimes\beta).$$

Multiplying both sides with $A^T\otimes B^T$

$$\left(A^T\otimes B^T\right)(A\otimes B)(x\otimes y)=\left(A^T\otimes B^T\right)(\alpha\otimes\beta)$$

$$\left[A^TA\otimes B^TB\right](x\otimes y)=\left(A^T\otimes B^T\right)(\alpha\otimes\beta)$$

Or $(x\otimes y)=\left[(A^TA)^{-1}\otimes(B^TB)^{-1}\right][\alpha\otimes\beta]$.

Similarly,if the rows of A and B are linearly independent then the transformation of the form

$$x=A^Tx_1\ ,\qquad y=B^Ty_1,$$

transforms the system (4.1) in the form

$$\left[\left(AA^T\right)\otimes\left(BB^T\right)\right](x_1\otimes y_1)=(\alpha\otimes\beta)$$

Or $(x_1\otimes y_1)=\left[\left(AA^T\right)^{-1}\otimes\left(BB^T\right)^{-1}\right][\alpha\otimes\beta],$

and hence a unique solution of (4.1) is given by

$$(x\otimes y)=\left(A^T\otimes B^T\right)(x_1\otimes y_1).$$

Thus $(x\otimes y)=\left(A^T\otimes B^T\right)\left[\left(AA^T\right)^{-1}\otimes\left(BB^T\right)^{-1}\right][\alpha\otimes\beta]$.

Note that the solutions given above are unique.

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