

Three Point Boundary Value Problems Associated with First Order Fuzzy Difference Systems-Existence and Uniqueness via the Best Least Square Solution

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Abstract:

This paper presents a criteria for the existence and uniqueness of solutions to first order fuzzy difference system using QR-algorithm. Modified QR-algorithm is presented for fuzzy linear systems using singular value decomposition.

Keywords: Fuzzy Difference Systems, Modified QR-algorithm, Fundamental matrix, Decode algorithm.

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1. Introduction:

Existence and uniqueness of solutions to initial value problems have a long mathematical history going back to Picards. The mere fact that f is continuous on R ensures existence of at least one solution to the initial value problem

$$y^1 = f(t, y), \quad y(t_0) = y_0 \quad (1.1)$$

on R . The situation is different for boundary value problems. Length of interval estimates are necessary to prove existence and uniqueness of (1.1). If f satisfies a lipschitz condition in the second variable, then (1.1) has a unique solution. The situation is different for first – order difference system..

$$y_{n+1} = A(n)y_n + f_n, \quad y(n_0) = y_0, \quad (1.2)$$

where A is an $p \times p$ continuous matrix, whose elements $a_{ij}(n)$ are all real or complex valued functions defined on $N_{n_0}^+$ and $y_n \in R^p(C^p)$ with components $y_1(n), y_2(n), \dots, \dots, y_p(n)$, defined on $N_{n_0}^+$. The corresponding homogeneous equation corresponding to (1.2) is

$$y_{n+1} = A(n)y_n, \quad y(n_0) = y_0 \quad (1.3)$$

(1.3) possess a unique solution on $N_{n_0}^+$ as can easily be seen by induction.

This paper presents a criteria for the existence and unicity of solutions to the three point boundary values problems associated with first order matrix difference systems.

$$y_{n+1} = A(n)y_n + f_n \tag{1.4}$$

$$My_{n_0} + Ny_{n_m} + Ry(n_f) = \alpha, \tag{1.5}$$

where M , N and R constant matrices of order $(m \times p)$ and y is a $(p \times 1)$ vector and α is a constant $(p \times 1)$ vector. The corresponding homogeneous boundary value problems

$$y_{n+1} = A(n)y_n \tag{1.6}$$

$$My_{n_0} + Ny_{n_m} + Ry(n_f) = 0. \tag{1.7}$$

Throughout this paper we assume that e_1, e_2, \dots, e_d be standard base vectors in R^d and $y(n, n_0, e_i)$ $i=1,2,\dots,d$ be a linearly independent solutions having $e_i(i = 1,2, \dots, d)$ as standard base vectors. Let S be the solution space of (1.5). It may be noted that any element of S can be expressed as a linear combination of the set of n linearly independent solutions of $y(n, n_0, e_i), i = 1,2, \dots, p$ if $Z(n)$ is any solution of (1.5) then

$$Z(n) = \sum_{i=1}^p c_i y(n, n_0, e_i) \quad i = 1,2, \dots, p$$

We define Wronskian of functions $y_i(n), n = 1,2, \dots, p$ on $N_{n_0}^+$ as

$$W(n) = \begin{bmatrix} y_1(n) & y_2(n) \cdots & y_p(n) \\ y_1(n+1) & y_2(n+1) \cdots & y_p(n+1) \\ \cdots & \cdots & \cdots \\ y_1(n+p-1) & y_2(n+p-1) \cdots & y_p(n+p-1) \end{bmatrix}$$

Note that $|W(n)| \neq 0$ for all $n \in N_{n_0}^+$ If $y_i(n), i = 1,2, \dots, p$ be p linearly independent solution of (1.5) , then $|W(n)| \neq 0$ for all $n \neq n_0$ if $y(n)$ is any solution of (1.4) and $\bar{y}(n)$ is a particular solution of (1.4) then $y(n) - \bar{y}(n)$ is a solution of (1.5) and any solution $y(n)$ of (1.1) is given by

$$y(n) = \bar{y}(n) + \sum_{i=1}^p \alpha_i y(n, n_0, e_i)$$

In the year 2009, Murty, Balaram, Viswanadh [6] established existence and uniqueness of kroneck product initial value problems by using tensor based harness of the shortest vector problem . Further, Murty, Yan Wu and Viswanadh Kanuri [3] established that Metrics that suit for Dichotomy, Well conditioning and object oriented design, on measure chains. Charyulu, Anand and Deekshitulu[12] established controllability , observability criteria on fuzzy matrix discrete dynamical systems. Further Kasi Viswanadh and Murty [5] established existence and uniqueness criteria for three point boundary value problems using shortest and closest lattice vector methods. In the year 2020, Kasi Viswanadh [4] established existence of ψ^α bounded solutions of linear fuzzy differential equation. We make use of these results to establish existence and uniqueness criteria for the fuzzy first order difference system.

$$y^\alpha(n+1) = A(n)y_n^\alpha + f(n), \tag{1.8}$$

satisfying boundary conditions

$$My^\alpha(n_0) + Ny^\alpha(n_m) + Ry^\alpha(n_f) = \alpha. \quad (1.9)$$

By using modified QR-algorithm we develop QR-algorithm for fuzzy linear systems. Section 2 presents preliminary results on fuzzy differential discrete systems and establishes main result by using modified QR-algorithm for fuzzy linear systems. These are results in fact generalize all existing results on linear systems and includes them as a particular case. The algorithm we present is a centrally crucial problems and is helpful in solving many least square problems in numerical linear algebra.

2) Preliminaries:

We present in this section some of the basic results and definitions on fuzzy systems. The family of all non-empty compact convex subsets of R^d is denoted by $P_k(R^d)$. If $\alpha, \beta \in R$ and $A, B \in P_k(R^d)$, we define

$$\alpha(A + B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A \text{ and } 1.A = A$$

If $\alpha, \beta > 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. Let $T = [a, b]$ be a compact subinterval of R . we have the following :

Definition 2.1:

Let $E^n = \{u \in R^d \rightarrow [0, 1]\}$, $u \in E^n$ is called a fuzzy number, if it satisfies the following axioms

- i) u is normal, that is there exists an $x_0 \in R^d$ such that $u(x_0) = 1$
- ii) u is fuzzy convex, that is for any $x, y, \in R^d$ and $0 < \lambda < 1$, $u(\lambda x + (1 - \lambda)y) \in R^d$
- iii) u is upper semi continuous
- iv) $u^0 = cl\{x \in R^d / u(x) \geq 0\}$ is compact.

For $\alpha \in [0, 1]$ the α - level set $\{u\}^\alpha \in P_k(R^d)$.

Definition 2.1(fuzzy set). Let X be a non-empty set. A fuzzy set A in X characterized by its membership function $A: X \rightarrow [0, 1]$ and $A(x)$ is interpreted as the degree of membership of elements of x in every fuzzy set A to each $x \in X$

The value of zero is used to represent complete non-membership, the value of one is used to represent complete membership and the values between 0 and 1 are used to represent intermediate degrees of membership.

Example 2.1: The membership function of the fuzzy set of real numbers close to one is defined as $A(x) = e^{-\beta(x-1)^2}$, where $\beta > 0$

Example 2.2: Let the membership functions for the set of real numbers close to zero is defined as $B(x) = \frac{1}{1+x^3}$

Using this function, we can determine the membership grade of real number in the fuzzy set, which signifies the degree to which that membership is close to zero. For instance the number 1 a grade of 0.5 and the number zero is a grade of 1. Mostly the results available in literature are of zero number and is of grade one only.

Definition 2.2: A map $f: [0, 1] \rightarrow E^d$ is strongly measurable if for all $\alpha \in [0, 1]$ the multivalued map $f_\alpha: [0, 1] \rightarrow P_k(R^d)$ is defined as $f_\alpha(t) = [f(t)]^\alpha$

is Lebesgue measurable, when $P_k(R^d)$ is endowed with the topology by the Hausdorff metric d .

Theorem 2.1 if $u \in P_k(R^d)$, then

- (1) $[u]^\alpha \in P_k(N_{n_0}^+)$ for all $\alpha \in [0,1]$
- (2) if $[u]^{\alpha_2} < [u]^{\alpha_1}$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$
- (3) if $\{\alpha_k\}$ is a non decreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \lim_{k \rightarrow \infty} [u]^{\alpha_k}$
conversely, if $\{A^\alpha: 0 \leq \alpha \leq 1\}$ and $[u]^0 = u_0$ and $A^\alpha \subset A^0$

Definition 2.3: we define $D: E^d \times E^d \rightarrow R_+ \setminus \{0\}$ by $D(u, v) = \sup_{0 \leq \alpha \leq 1} d_H\{[u]^\alpha, [v]^\alpha\}$ where d_H is the Hausdorff metric defined in $P_k(R^d)$ For any $u, v, w \in P_k(R^d)$ and $\lambda \in R$, we have

- (1) $D(u + w, v + w) = D(u, v)$
- (2) $D(\lambda u, \lambda v) = |\lambda|D(u, v)$ and
- (3) $D(u, v) = D(u, w) + D(w, v)$

Definition 2.4: Let $f: T \rightarrow E^d$ for $t_0 \in R$, we say that f is differentiable at t_0 (Hausdorff differentiable) if there exists an element $f'(t_0) \in R^d$ such that for all $h > 0$ the H-difference $f(t_0 + h) - f(t_0)$ and $F(t_0) - F(t_0 - h)$ exists and the limit (in the metric)

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

are all exists and each equal to $f'(t_0)$. At the end points we only take one sided derivative.

We now turn our attention to the existence and uniqueness of three point boundary value problems when the characteristic matrix D is non-invertible. Let $y^\alpha(n_0, n_0, e_i), i = 1, 2, \dots, \alpha$ be the n -linearly independent solutions of (1.7) having e_i as its initial base vector. Then any solution of (1.7) is of the form

$$y(n) = \sum_{j=n_0}^{d-1} \varphi^{-\alpha}(n, j+1) f_j + \varphi^{-\alpha}(n, n_0) C \quad (2.1)$$

where c is an constant n vector. Now substituting the general form of solution (2.1) in the boundary condition matrix (1.8)

We get

$$[M \varphi^{-\alpha}(n, n_0) + N \varphi^{-\alpha}(n_m, n_0) + R \varphi^{-\alpha}(n_f, n_0) C] =$$

$$\alpha - [\sum M \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_0, j+1) f_j + N \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_m, j+1) f_j + R \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_f, j+1) f_j].$$

We assume that for each $\alpha \in [0,1]$, the characteristic matrix D^α defined by

$D^\alpha = M \varphi^\alpha(n, n_0) + N \varphi^\alpha(n_m, n_0) + R \varphi^\alpha(n_f, n_0)$ is non-singular (here we assume that M, N and R are constant square matrices). In a way we are assuming that the homogeneous boundary value problem (with $f = 0$ and $\alpha = 0$) has only the trivial solution. In this case

$$C_{n_0} = D^{\alpha-1} [\alpha - [\sum M \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_0, j+1) f_j + N \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_m, j+1) f_j + R \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_f, j+1) f_j]]$$

Note that $\varphi(n_0, n_0) = \varphi(n_0)\varphi^{-1}(n_0) = I$ If D^α is non singular for each $\alpha \in [0,1]$, (2.2) determines the unique solution of the boundary value problem. If D^α is singular then, we can only determine best least square solution of the three point boundary value problem. In this case using (1.9), we get a system of equations $D^\alpha C = f$ where

$$f = D^{\alpha^{-1}}[\alpha - [\sum M \sum_{j=d_0}^{d-1} \varphi(n_0, j + 1) f_j + N \sum_{j=d_0}^{d-1} \varphi(n_m, j + 1) f_j + R \sum_{j=d_0}^{d-1} \varphi(n_{n_0}, j + 1) f_j]]$$

We solve the system of equation (2.3) by using modified QR-algorithm.

3.The Least squares problem :

The least squares (1.5) problem is one of the central problems in numerical linear algebra. Suppose we have a system of equations of the form $D^\alpha C = f$

Where $D^\alpha \in R^{m \times n}$, and $m > n$ meaning R is long and thin matrix and $f \in R^{m \times 1}$. We wish to find C for any fixed $\alpha \in [0,1]$ such that $D^\alpha C = f$. In general, we can never expect to find a solution C such that $D^\alpha C \approx f$.

Formally (Ls) problem can be defined as

$$\arg \min_c \|D^\alpha C - f\|_2$$

Let Q be an orthogonal matrix and $Q^\alpha \in R^{m \times m}$. For each $\alpha \in [0,1]$ then Q does not change the norm of a vector . If we rotate or reflect a vector, then the vectors length won't change. Consider why

$$\|Q^\alpha y\|_2^2 = (Q^\alpha y)^T (Q^\alpha y) = y^T (Q^\alpha)^T (Q^\alpha y) = y^T y = \|y\|_2^2$$

With this idea is min , consider now an orthogonal matrix can be used for an LS problem.

$$\begin{aligned} &= \min_c \|D^\alpha C - f\|_2 \\ &= \min_c \|(Q^\alpha)^T (D^\alpha C - f)\|_2 \\ &= \min_c \|(Q^\alpha)^T (QR^\alpha C - f)\|_2 \\ &= \min_c \|(R^\alpha C - Q^T f)\|_2 \end{aligned}$$

Our goal is to find a Q such that $Q^\alpha . D^\alpha = Q^\alpha . R^\alpha$ where R^α is upper triangular for each $\alpha \in [0,1]$. QR factorization for solving least square problems

In fact QR - decomposition exists for any matrix . Given a matrix our goal is to find two matrices $Q^\alpha . R^\alpha$ such that Q is orthogonal and R^α is upper triangular. Here D^α is a $m \times p$ matrix and hence

$$D^\alpha = Q^\alpha \begin{bmatrix} R^\alpha \\ 0 \end{bmatrix} = [Q_1^\alpha \quad Q_2^\alpha] \begin{bmatrix} R^\alpha \\ 0 \end{bmatrix}$$

Note that the matrix R^α will be always square say $p \times p$

$$\text{Consider } D^\alpha C = f \tag{3.2}$$

If D^α is an $m \times p$ matrix with columns linearly independent then

$$(D^{\alpha T} D^\alpha) C = D^\alpha f \text{ for each } \alpha \in [0,1]$$

Now $(D^{\alpha T} D^\alpha)$ is a square matrix of order p and hence $C = (D^{\alpha T} D^\alpha)^{-1} D^\alpha f$

Thus , using the (QR) decomposition yields a better least square estimate than the normal equations in terms of solution quality . In case D^α is a $m \times p$ matrix with rows of D^α are linearly independent, then the transformation $C = (D^{\alpha T})^y$ gives

$$(D^\alpha D^{\alpha T} y) = D^\alpha f$$

$$y = (D^\alpha D^{\alpha T})^{-1} D^\alpha f$$

Since $C = D^{\alpha T} y$, We have $y = D^{\alpha T} (D^\alpha D^{\alpha T})^{-1} D^\alpha f$ is the unique solution.

Rank-Deficient Least –square problems:

When D^α is a square matrix of order $p \times p$, we use least squares algorithm under the assumption D^α is not of full rank. If it is of full rank then the solution of $D^\alpha C = f$ can be determined uniquely. We can use suitable choices the first one is SVD(singular value decomposition) or its cheaper approximations $Q^\alpha R^\alpha$ with column pivoting. If matrix D^α for each $\alpha \in [0,1]$ is rank deficient, then it is no longer the case that space spanned by the columns of $* Q^\alpha *$ is the same space spanned by columns of $* A *$ i.e.,

$$span D_1^\alpha, D_2^\alpha, \dots, D_p^\alpha \cong span q_1^\alpha, q_2^\alpha, \dots, q_p^\alpha$$

$Q^\alpha R^\alpha$ application: The generalized minimum residual (GMRES) algorithm will be presented for solving very large , sparse linear systems of equations by using $Q^\alpha R^\alpha$ decomposition. This decomposition is well known and was in-fact proposed by Saad and Shultz in 1986[.]. We make use of this method for developing $Q^\alpha R^\alpha$ algorithm to solve our problems in boundary value problems in boundary value problem.. Let $D(\alpha) := D$ and $Q(\alpha): Q$ for any $\alpha \in [0,1]$

Def: *Arnold i_single_iter(D, Q, K):*

$$Q = D.dot(Q[:k])$$

$$h = m p.zeros(K+)$$

For i in(k + 1):

$$h(i)q.T.dot(Q^f[:, i])$$

$$q = h[i] * Q[:, i]$$

$$h(k +) = np.linalg.norm(q^\alpha)$$

$$q /= h[k + 1]$$

Return h,q

Def gmres(DC, x, max_iters):

EPSILON =

n ,_ = D. Shape

assert (D^\alpha, shape[\theta] = D^\alpha.shape[:])

$$r = f - D^\alpha.dot(C^\alpha)$$

$$q^\alpha = m.p.zeros((m, \max_iters))$$

$$Q[:, \theta] = q.squeeze()$$

$$\beta = m.p.linalg.norm(r)$$

$$C_i = n.p.zeros((n, 1))$$

$C_i[\theta] =: \neq e - 1$ standard basis vector,

$$H = mp.zeros((n + 1, n))$$

$$F = mp(zeros(\max_iters, n, n))$$

for i in range (max_iters):

$$F(i) = mp.eye(n)$$

for k in range (max_iters - 1) :

$$H[:k + 2, k], Q^\alpha[:k + 1] = arhold\ i - single_iter(D, Q, k)$$

\neq don't need to this for \emptyset, \dots, m since completed previously

$$c, s = given\ coeffs(H[k, k], H[k + 1, k])$$

$$F[k, k, k] = c$$

$$F[k, k, k + 1] = s$$

$$F[k, k + 1, k] = -s$$

$$F[k, k + 1, k + 1] = c$$

\neq apply the rotation to both of these

$$H[:k + 2, k] = F[k:k + 2:k + 2].dot[H[:k + 2, k]]$$

$$c_i = F[k].dot(X_i)$$

If $\beta = m.p.linalg.norm(C_i[k + 1]) < \epsilon$

STOP

\neq when terminates , solve the least square problem

\neq y must be (k,1)

$$y; -, -, - = nplinalg.lstsq(H[:k + 1],$$

$\neq 0$ k will have dimension (m, k) $C_i[:k + 1]$

$$C - k = C + Q[:, :, k + 1].dot(y)$$

return C - k

Def given_coeffs(a,b)

$$c = a|np.sqrt(a**2 + b**2)$$

$$s = b|np.sqrt(a**2 + b**2)$$

return c, s

Def Arnold i(D, f, k):

$$n = D.shape[\emptyset]$$

$$H = mp.zeros(k, k)$$

$$Q = mp.zeros(m, k)$$

≠ normalize the input vector

≠ use it as the first krylov vector

$$Q[:, \theta] = b|np.linalg.norm(f)$$

for j in range (k - 1):

$$Q[:, j + 1] = a.dot(Q[:, j])$$

for i in range (j):

$$H[i, j] = Q[:, j + 1].dot(Q[:, j])$$

$$Q[:, j + 1] = Q[:, j + 1] - H[i, j] * Q[:, 1]$$

$$H[j + 1, j] = np.linalg.norm(Q[:, j + 1])$$

$$Q[:, j + 1] /= H[j + 1, j]$$

return Q, H

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