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Stability Analysis of Linear Sylvester System of First Order Differential Equations

[1]**Kasi Viswanadh V. Kanuri**, [2]**SriRam Bhagavathula***, [3]**K.N. Murty**

[1]3669 Leatherwood, Dr. Frisco, TX 75033 USA
[2]Conns Appliances Inc, *sriram.bhagavatula@gmail.com*
[3]Department of Applied Mathematics, Andhra University, Visakhapatnam, A.P, India

Abstract
In this paper, we establish stability criteria of the linear Sylvester system of matrix differential equation using the new concept of bounded solutions and deduce the existence of Ψ-bounded solutions as a particular case.

Keywords: Sylvester linear systems, Lyapunov systems, bounded solutions, Stability and asymptotic stability, Fundamental matrix solutions, Variation of parameters formula.

AMS (MOS) classifications: 34A30, 34B14, 93B05, 93B07, 93E42.

1. Introduction

In this paper, we shall be concerned with the Sylvester system of first order linear differential non-homogeneous equation and establish a necessary and sufficient condition for the existence of (Φ, Ψ)bounded solutions and deduce the results of Lyapunov systems as a particular case. We establish variation of parameters formula and use it as a tool to establish our main results. Sylvester system of first order linear non-homogeneous equation is an interesting area of current research and the general form of its solution in two fundamental matrices is only established by Murty and Prasad in the year of 1989 [9]. The paper attracted many eminent mathematicians like Richard Bellman, Don Fausett, Lakshmikantham to mention a few. Recent results established by Viswanadh, V. Kanuri, *et. al.*, is the main motivation behind our results. The concept of Ψ-bounded solutions for linear system of differential equations is due to T. G. Halam [14]. The variation of parameters formula we established is new and will have significant contributions on control engineering problems. The novel idea adopted by Viswanadh, Wu and Murty [8] on the existence of $(\Phi \otimes \Psi)$bounded solutions and on the existence of Ψ-bounded solutions by Kasi Viswanadh, V. Kanuri, *et. al.* [4-7,11,12,13] on time scale dynamical systems is a useful and significant contribution to the theory and differential and difference equations. Further these ideas have been extended by Kasi Viswanadh V. Kanuri to fuzzy differential equations in a novel concept, and is very interesting and useful contribution to the theory of differential equations and also in applications to control systems. The results established on stability, controllability criteria established on state scale dynamical systems on first order linear systems [9] can be generalized to (Φ, Ψ)bounded solutions to Sylvester linear system of differential equations. This paper is organized as follows: section 2 presents a criterion for the existence of Φ-bounded solution of the matrix linear system $T' = AT$and Ψ-bounded solution of the linear system $T' = B^*T$(where * refers to the transpose of the complex conjugate). By super imposing these two solutions, we establish the general solution of the linear matrix Sylvester system

$$T' = A(t)T + TB(t) \tag{1.1}$$

where T is a square matrix of order (n ×n) and $A(t)$, $B(t)$are also n ×n matrices. We present our basic results that are available in literature [4, 5, 6, 7, 8, 10, 11]. Our main results are established in section 3. This section also presents criteria for the Sylvester system (1.1) to be stable, asymptotically stable, and establishing the result on controllability. Throughout this paper, $Y(t)$ stands for a fundamental matrix solution of the linear system

$$T' = A(t)T \tag{1.2}$$

and $Z(t)$stands for a fundamental matrix solution of the linear system

$$T' = B^*T. \tag{1.3}$$

2. Preliminaries

In this section, we shall be concerned with establishing general solution of the Sylvester linear system and present ΦY-bounded solution of the linear system (1.2) and then ΨZ-bounded solution of the system (1.3).

Theorem 2.1 *T* is a solution of (1.1) if and only if $T = YCZ^*$, where *C* is a constant square matrix and Y is a fundamental matrix solution of (1.2) and Z is a fundamental matrix solution of (1.3).

Proof: It can easily be verified that *T* defined by YCZ^* is a solution of (1.1). For

$T' = Y'CZ^* + YCZ^{*\prime} = A(t)YCZ^* + YCZ^*B = AT + TB.$

Hence, YCZ^* is a solution of (1.1). Now, to prove that every solution is of this form, let *T* be a solution, and *K* be a matrix defined by $K = Y^{-1}T$. Then, $Y'K + YK' = AYK + YKB^*$ or $YK' = YKB^*$ or $K' = KB^*$ or $K^{*\prime}$ = BK^*. Since $Z$is a fundamental matrix solution of (1.3), it follows that there exists a constant square matrix *C* such that $K^* = ZC^*$or $K = CZ^*$. Since $T = YK = YCZ^*$. □

In [4], Kasi Viswanadh, V. Kanuri, *et. al.* presented a novel concept on Ψ-bounded solutions of linear differential systems on time scales. We use these ideas as a tool to establish (Φ, Ψ)bounded solutions of the Sylvester system (2.1). If B is replaced by A*, we get Lyapunov system. In this case the general solution is given by YCY*.

Definition 2.1 A function $Y: \mathbb{R}^+ \to \mathbb{R}^{n^2}$is said to be Φ-bounded solution on$\mathbb{R}$ if ΦY is bounded on $\mathbb{R}$.

Definition 2.2 A function $Z: \mathbb{R}^+ \to \mathbb{R}^{n^2}$ is said to be Ψ-bounded solution on$\mathbb{R}$ if $Z^*\Psi^*$is bounded on $\mathbb{R}$.

Definition 2.3 A function $Y: \mathbb{R}^+ \to \mathbb{R}^{n^2}$ is said to be Φ-Lebesgue integrable on $\mathbb{R}^+$if $Y(t)$is measurable and $\Phi(t)Y(t)$ is Lebesgue integrable on $\mathbb{R}^+$.

Definition 2.4 A function $Z: \mathbb{R}^+ \to \mathbb{R}^{n^2}$ is said to be Ψ-Lebesgue integrable on $\mathbb{R}^+$if $Z^*(t)$is measurable and $Z^*(t)\Psi^*(t)$ is Lebesgue integrable on $\mathbb{R}^+$.

Let $\Phi_i: \mathbb{R}^+ \to \mathbb{R}^n$, $i = 1,2,\dots,n$, be continuous and let $\Phi(t) = (\Phi_1(t), \Phi_2(t), \dots, \Phi_n(t))$be linearly independent so that $\Phi$is invertible and also we assume$\Psi$is invertible.

By a solution of the linear system (1.1), we mean $Y(t)CZ^*(t)$, which is an absolutely continuous function and satisfies (1.1) for almost all $t \geq 0$.

Let Ybe a fundamental matrix solution of (1.2) satisfying $Y(0) = I_n$ and Zbe a fundamental matrix solution of (1.3) satisfying$Z(0) = I_n$. Let X_1denote the subspace of $\mathbb{R}^n$consisting of all vectors whose values are of Φ-bounded solutions of (1.2) for $t = 0$and X_2be the arbitrary fixed subspace of $\mathbb{R}^n$ supplementary toX_1. Further, let P_1be the projection matrix of $\mathbb{R}^n$onto X_1 ($P_1^2 = P_1$ and $P_1: \mathbb{R}^n \to \mathbb{R}^n$) and let $P_2 = I - P_1$be the projection matrix on X_2

Definition 2.5 A function $f: \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be Φ-bounded on $\mathbb{R}$if $\Phi(t)f(t)$ is bounded on $\mathbb{R}$, i.e. $\sup_{t\in\mathbb{R}} \|\Phi(t)f(t)\| < \infty$.

Definition 2.6 A matrix $Y: \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be Φ-bounded on $\mathbb{R}$if the matrix $\Phi(t)Y(t)$ is bounded on $\mathbb{R}$, i.e. there exists an $M > 0$ such that $\sup_{t\in\mathbb{R}} \|\Phi(t)Y(t)\| \leq M$

Definition 2.7 A matrix $Y: \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be Φ-integrable on $\mathbb{R}$ component-wise if $\Phi(t)Y(t)$ is integrable on $\mathbb{R}$, i.e. $\int_0^\infty \|\Phi(t)Y(t)\| dt < \infty$.

Definition 2.8 A matrix function (Φ, Ψ): $\mathbb{R} \to \mathbb{R}^{n^2\times n^2}$is said to be (Φ, Ψ)-bounded if the matrix

$\|\Phi Y Z^* \Psi^*\|$ is bounded on $\mathbb{R}$.

3. Main results

In this section, we shall be concerned with the existence of (Φ,Ψ)-bounded solution of the linear Sylvester system (1.1), and then present the stability and asymptotic stability of the Sylvester system.

Theorem 3.1 Let A and B be $(n\times n)$ continuous square matrices on $\mathbb{R}$. Then, the system (1.1) has at least one (Φ,Ψ)-bounded solution on $\mathbb{R}$ for every continuous (Φ,Ψ)-bounded function if and only if there exists a positive constant K such that

$\int_{-\infty}^{\infty}\left\|\Phi(t)Y(t)PZ^*(t)\Psi^*(t)\right\| \le K$ for all $t\ge 0$ (3.1)

where $P=P_-$ on $(-\infty,t)$,$P=P_0+P_+$ on $(t,0)$,$P=P_+$ on $(0,\infty)$, $P=P_-$ on $(-\infty,0)$, $P=P_0+P_-$ on $(0,t)$, $P=P_+$ on (t,∞).

Proof: First, suppose the linear Sylvester system has at least one (Φ,Ψ)-bounded solution on $\mathbb{R}$ for every continuous (Φ,Ψ)-bounded function on $\mathbb{R}$. Then, it is claimed that there exists a constant $K>0$ such that the inequality (3.1) holds. Let B be the Banach space of all (Φ,Ψ)-bounded continuous functions $T:\mathbb{R}\to\mathbb{R}^{n^2}$ with norm $\|T\|_{\mathrm{B}} = \sup_{t\in\mathbb{R}}\left\|\Phi(t)Y(t)Z^*(t)\Psi^*(t)T(t)\right\|$, we define

(i) C: the Banach space of all (Φ,Ψ)-bounded continuous functions $T:\mathbb{R}\to\mathbb{R}^{n^2}$ with norm

$$\|T\|_C=\sup_{t\in\mathbb{R}}\left\|\Phi(t)Y(t)Z^*(t)\Psi^*(t)\right\|$$

(ii) B: the Banach space of all $(\Phi,\Psi)-\triangle$ integrable functions $T:\mathbb{R}\to\mathbb{R}^{n^2}$ with norm

$$\|\mathrm{T}\|\mathrm{B}=\int_{-\infty}^{\infty}\|\Phi(t)Y(t)Z^*(t)\Psi^*(t)\|dt$$

(iii) D: the set of all continuous function $T:\mathbb{R}\to\mathbb{R}^{n^2}$ which are absolutely continuous on all intervals $J\subset\mathbb{R}$, (Φ,Ψ)-bounded on $\mathbb{R}$, $T(0)\in X_-\otimes X_+$, and $\boldsymbol{T}'=\boldsymbol{AT}+\boldsymbol{TB}\in$ B.

Step 1: We first claim that $(D,\|\ \ \|_D)$is a Banach space. For, we first note that $(D,\|\ \ \|_D)$ is a vector space. Let $\{T_n\}_{n\in\mathbb{N}}$be a fundamental sequence in B. Then, there exists a continuous (Φ,Ψ)-bounded function on $\mathbb{R}$ such that

$$\lim_{n\to\infty}\Phi_n(t)T_n(t)\Psi_n^*(t)T_n^*(t)=\Phi(t)T(t)\Psi^*(t)T^*(t)$$

Uniformly on$\mathbb{R}$. From the inequality

$$\|T_n(t)-T(t)\|\le\left\|\Phi^{-1}(t)\right\|\left\{\left\|\Phi(t)\Psi^*(t)\Psi^{*^{-1}}(t)\right\|\|\Psi^*(t)\|-\left\|\Phi^{-1}(t)\right\|\|\Phi(t)T(t)\Psi^*(t)\|\left\|\Psi^{*^{-1}}(t)\right\|\right\}$$

Hence, $\lim_{n\to\infty}T_n(t)=T(t)$ uniformly on every compact subset of $\mathbb{R}$. Thus,$T(0)\in X_-\otimes X_+$. Thus, $(D,\|\ \ \|_D)$ is a Banach space.

We now establish variation of parameters formula for the non-homogeneous Sylvester system

$T'=A(t)T+TB(t)+F(t)$ (3.2)

where $F(t)$is a given $(n\times n)$ square matrix. Let T be any solution of (3.2) and $\overline{T}$be a particular solution of (3.2). Then $T-\overline{T}$is a solution of the homogenous system (1.1). Any solution of the homogeneous system is of the form $T(t)=Y(t)CZ^*(t)$, where $Y(t)$ is a fundamental matrix solution of (1.2) and $Z(t)$is a fundamental matrix of (1.3). Such a solution cannot be a solution of (3.1) unless$F(t)=0$.

We seek a particular solution of (3.1) in the form

$$\overline{T}(t)=Y(t)C(t)Z^*(t)$$

and see that $\overline{T}(t)$is a particular solution of (3.1). Now,

$$\overline{T}'(t)=Y'(t)C(t)Z^*(t)+Y(t)C'(t)Z^*(t)+Y(t)C(t)Z^{*'}(t)$$

$$=A(t)Y(t)C(t)Z^*(t)+Y(t)C'(t)Z^*(t)+Y(t)C(t)Z^*B$$

Now on substitution in the eqn (3.1) gives

$$A(t)Y(t)C(t)Z^*(t) + Y(t)C'(t)Z^*(t) + Y(t)C(t)Z^*B$$
$$= A(t)Y(t)C(t)Z^*(t) + Y(t)C(t)Z^*(t)B(t) + F(t)$$

which gives $Y(t)C'(t)Z^*(t) = F(t)$, then $C'(t) = Y^{-1}(t)F(t)Z^{*^{-1}}(t)$, or

$$C(t) = \int_a^t Y^{-1}(s)F(s)Z^{*^{-1}}(s)\ ds$$

and hence, $\overline{T}(t) = Y(t)\int_a^t Y^{-1}(s)F(s)Z^{*^{-1}}(s)\ ds\, Z^*(t)$. Now, it can easily be verified that $\overline{T}(t)$is a solution of (3.2), and the general solution is given by

$T(t) = Y(t)CZ^*(t) + \overline{T}(t) = Y(t)CZ^*(t) + Y(t)\int_a^t Y^{-1}(s)F(s)Z^{*^{-1}}(s)\ ds\, Z^*(t)$.

We now claim that three exists a constant $K_0 > 0$ such that for every $F \in$ B and for corresponding solution of $T \in D$ of (3.2), we have

$$\sup_{t\in\mathbb{R}}\|\Phi(t)T(t)\Psi^*(t)\| \leq K_0 \sup_{t\in\mathbb{R}}\|\Phi(t)F(t)\Psi^*(t)\|$$

or

$$\sup_{t\in\mathbb{R}}\max_{1\leq i\leq n}\|\Phi_i(t)T_i(t)\Psi_i{}^*(t)\| \leq K_0 \sup_{t\in\mathbb{R}}\max_{1\leq i\leq n}\|\Phi_i(t)F_i(t)\Psi_i{}^*(t)\|$$

or

$$\sup_{t\in\mathbb{R}}\max_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}\|\Phi_{ij}(t)T_{ij}(t)\Psi_{ij}{}^*(t)\| \leq K_0 \sup_{t\in\mathbb{R}}\max_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}\|\Phi_{ij}(t)F_{ij}(t)\Psi_{ij}{}^*(t)\| \qquad (3.3)$$

For, define the mapping $R: D \to B$as $RT = T' - AT - TB$. Clearly, R is linear and bounded with $\|R\| \leq 1$. Let $RT = 0$, and the fact T satisfies the differential equation

$T' = AT + TB$ and hence $T \in B$. This shows that T is (Φ, Ψ)-bounded on$\mathbb{R}$ of the system (1.1). Then

$T(0) \in X_0 \cap (X_- \oplus X_+) = \{0\}$.

Thus, $T = 0$so that R is one-to-one. To prove that R is "onto", for any $F \in$ B, let T be a (Φ, Ψ)-bounded on$\mathbb{R}$of the system (1.1) and T be the solution of the Cauchy problem

$T' = A(t)T + TB(t) + F(t)$

satisfying $T(0) = (P_- + P_+)T(0)$. Then, $U = R - T$ is a solution of the system (3.2) with $U(0) = R(0) - (P_- + P_+)T(0)$. Thus, $U \in D$and $RT = F$. Consequently, the mapping R is a bounded, one-to-one linear operator from one Banach space $B$to another Banach space B . Hence, R^{-1}exists and bounded, where

$\left\|R^{-1}F\right\|_B \leq \left\|R^{-1}\right\|\|F\|_B$ for all $F \in$ B

It follows that

$$\left\|R^{-1}F\right\| = \left(\left\|R^{-1}\right\| - 1\right)\|F\| \leq K_0\|F\|_B$$

where$K_0 = \left\|R^{-1}\right\| - 1$, which is equivalent to (3.3).

Let $\theta_1$and $\theta_2$be any fixed real numbers such that $\theta_1 < 0 < \theta_2$ and $F: \mathbb{R} \to \mathbb{R}^{n^2}$be a function in B which vanishes on $(-\infty, \theta_1] \cup [\theta_2, \infty)$. Then it is easy to see that the function $T: \mathbb{R} \to \mathbb{R}^{n^2}$ defined as

$$T(t)=\begin{cases} -\int_{\theta_1}^{0} \phi(t)P_0\phi^{-1}(\sigma(s))f(s)ds-\int_{\theta_1}^{\theta_2} \phi(t)P_+\phi^{-1}(\sigma(s))f(s)\Delta s, t<\theta_1 \\ -\int_{\theta_1}^{t} \phi(t)P_-\phi^{-1}(\sigma(s))f(s)\Delta s+\int_{0}^{t} \phi(t)P_0\phi^{-1}(\sigma(s))f(s)\Delta s \\ -\int_{\theta_1}^{\theta_2} \phi(t)P_-\phi^{-1}(\sigma(s))f(s)\Delta s,,\theta_1\le t\le\theta_2 \\ \int_{0}^{\theta_2} \phi(t)P_+\phi^{-1}(\sigma(s))f(s)\Delta s, t>\theta_2 \end{cases}$$

is the solution in D of the system (1.1). Now if we put

$$G(t,s)=\begin{cases} \Phi(t)\Psi^*(t)P_-{\Psi^*}^{-1}(s)\Phi^{-1}(s), & s\le 0<t \\ \Phi(t)\Psi^*(t)(P_0+P_+){\Psi^*}^{-1}(s)\Phi^{-1}(s), & 0<s<t \\ -\Phi(t)\Psi^*(t)P_+{\Psi^*}^{-1}(s)\Phi^{-1}(s), & 0<t\le s \\ \Phi(t)\Psi^*(t)P_-{\Psi^*}^{-1}(s)\Phi^{-1}(s), & s<t\le 0 \\ -\Phi(t)\Psi^*(t)(P_0+P_+){\Psi^*}^{-1}(s)\Phi^{-1}(s), & t\le s<0 \\ -\Phi(t)\Psi^*(t)P_+{\Psi^*}^{-1}(s)\Phi^{-1}(s), & t\le 0\le s \end{cases}.$$

Then, G is continuous on $\mathbb{R}^{n^2}$ at all points except at $t=s$, and at $t=sG$ has a jump discontinuity of unit-magnitude (I_n). Then, we have

$T(t)=\int_{\theta_1}^{\theta_2} G(t,s)F(s)ds$ for $t\in\mathbb{R}$.

Indeed, for $\theta_1>t$, we have

$$\int_{\theta_1}^{\theta_2} G(t,s)F(s)ds=-\int_{\theta_1}^{0}\Phi(t)\Psi^*(t)(P_0+P_+){\Psi^*}^{-1}(s)\Phi^{-1}(s)F(s)ds-$$

$\int_0^{\theta_2}\Phi(t)\Psi^*(t)P_+{\Psi^*}^{-1}(s)\Phi^{-1}(s)F(s)ds.$

Rewrite the second integral as $\int_0^{\theta_1}\dots ds+\int_{\theta_1}^{\theta_2}\dots ds$, we get

$$\int_{\theta_1}^{\theta_2} G(t,s)F(s)ds=-\int_{\theta_1}^{0}\Phi(t)\Psi^*(t)P_0{\Psi^*}^{-1}(s)\Phi^{-1}(s)F(s)ds-$$

$\int_{\theta_1}^{\theta_2}\Phi(t)\Psi^*(t)P_+{\Psi^*}^{-1}(s)\Phi^{-1}(s)F(s)ds=T(t).$

For $t\in[\theta_1,0)$, we have

$$\int_{\theta_1}^{\theta_2} G(t,s)F(s)ds=-\int_{\theta_1}^{t}\Phi(t)\Psi^*(t)P_-{\Psi^*}^{-1}(s)\Phi^{-1}(s)F(s)ds-$$

$$\int_t^0 \Phi(t)\Psi^*(t)(P_0+P_+)\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds -$$

$\int_0^{\theta_2} \Phi(t)\Psi^*(t)P_+\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds=$

$\int_{\theta_1}^t \Phi(t)\Psi^*(t)P_-\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds+$

$$\int_0^t \Phi(t)\Psi^*(t)P_0\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds -$$

$\int_t^{\theta_2} \Phi(t)\Psi^*(t)P_1\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds = T(t).$

For $t \in (0,\theta_2)$, we have

$$\int_{\theta_1}^{\theta_2} G(t,s)F(s)ds = \int_{\theta_1}^0 \Phi(t)\Psi^*(t)P_-\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds +$$

$$\int_0^t \Phi(t)\Psi^*(t)(P_0+P_-)\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds -$$

$$\int_t^{\theta_2} \Phi(t)\Psi^*(t)P_+\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds =$$

$$\int_{\theta_1}^t \Phi(t)\Psi^*(t)P_-\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds +$$

$$\int_0^t \Phi(t)\Psi^*(t)P_0\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds -$$

$\int_t^{\theta_2} \Phi(t)\Psi^*(t)P_+\Psi^{*^{-1}}(s)\Phi^{-1}(s)F(s)ds = T(t).$

For $t > \theta_2$, we can easily show that $\int_{\theta_1}^{\theta_2} G(t,s)F(s)ds = T(t)$. Therefore,

$\sup_{t\in\mathbb{R}} \left\|\Phi(t)\Psi^*(t)\int_{\theta_1}^{\theta_2} G(t,s)F(s)ds\right\| \le K\int_{\theta_1}^{\theta_2}\|\Phi(t)\Psi^*(t)F(t)\|dt$

for all $t \in \mathbb{R}$. Hence,

$\left\|\Phi(t)\Psi^*(t)G(t,s)\Psi^{*^{-1}}(s)\Phi^{-1}(s)\right\| \le K$ for all $t \in \mathbb{R}$.

Now, to prove the converse statement, suppose the fundamental matrices of $Y$and $Z$of (1.2) and (1.3) satisfy the condition (3.1) for some $K > 0$. Let $F:\mathbb{R} \to \mathbb{R}^{n^2}$be a Lebesgue (Φ,Ψ)-delta integrable function on$\mathbb{R}$. We consider the function $U:\mathbb{R} \to \mathbb{R}^{n^2}$defined by

$$U(t) = \int_{-\infty}^{\infty} \Phi(t)\Psi^*(t)\Psi^{*^{-1}}(s)\Phi^{-1}(s)T(s)ds +$$

$$\int_0^t \Phi(t)\Psi^*(t)\Psi^{*^{-1}}(s)\Phi^{-1}(s)T(s)ds -$$

$\int_t^{\infty} \Phi(t)\Psi^*(t)\Psi^{*^{-1}}(s)\Phi^{-1}(s)T(s)ds.$ (3.3)

Then, the function is well defined on$\mathbb{R}$, and

$\|\Phi(t)\Psi^*(t)U(t)\| \le K\int_{-\infty}^{\infty}\|\Phi(s)\Psi^*(s)T(s)\|ds,$

which shows that U is (Φ,Ψ)-bounded on$\mathbb{R}$. Hence the proof is complete. □

Theorem 3.2: If the homogeneous Sylvester system (1.1) has no non-trivial (Φ, Ψ)-bounded solution on$\mathbb{R}$, then (1.1) has a unique (Φ, Ψ)-bounded solution on$\mathbb{R}$ for every Lebesgue (Φ, Ψ)-integrable function$F: \mathbb{R} \to \mathbb{R}^{n^2}$if and only there exists a $K > 0$ such that

$$\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)P_-{\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K\, s \text{ for } -\infty < s < t < \infty$$

and

$$\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)P_+{\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K\, t \text{ for } -\infty < t < s < \infty$$

The proof follows by taking $P_0 = 0$in Theorem 3.1.

Theorem 3.3: Suppose that a fundamental matrix $Y(t)$of $T' = AT$and a fundamental matrix $Z(t)$of $T' = B^*T$satisfy the conditions:

(i) $\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)P_-{\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K \text{ for } t > 0, s \leq 0$

$\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)(P_0 + P_-){\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K \text{ for } t > 0, s > 0, s < t$

$\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)P_+{\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K \text{ for } t > 0, s > 0, s > t$

$\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)P_-{\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K \text{ for } t \leq 0, s < t$

$\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)(P_0 + P_-){\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K \text{ for } t \leq 0, s > t, s < 0$

$\left\| \Phi(t)Y(t)Z^*(t)\Psi^*(t)P_+{\Psi^*}^{-1}(s){Z^*}^{-1}(s)Y^{-1}(s)\Phi^{-1}(s) \right\| \leq K \text{ for } t \leq 0, s > 0, s \geq t$

(ii) $\lim_{t\to\infty} \|\Phi(t)Y(t)Z^*(t)\Psi^*(t)P_0\| = 0$

$\lim_{t\to\infty} \|\Phi(t)Y(t)Z^*(t)\Psi^*(t)P_1\| = 0$

$\lim_{t\to\infty} \|\Phi(t)Y(t)Z^*(t)\Psi^*(t)P_-\| = 0$

and

(iii) the function $F: \mathbb{R} \to \mathbb{R}^{n^2}$is Lebesgue-delta integrable on$\mathbb{R}$.

Then, every (Φ, Ψ)-bounded solution T of (1.1) is such that

$\lim_{t\to\pm\infty} \|\Phi(t)Y(t)Z^*(t)\Psi^*(t)T(t)\| = 0.$

The proof is similar to that of the Theorem 3.3 in [4].

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