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On Multi-granulation Rough Sets

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Abstract: ***In this paper, we investigate some properties of the approximations determined by a class of equivalence relations, in Pawlak's single granulation point of view. A comparison of these approximations with the optimistic and the pessimistic multi-granular approximations is also presented. It has been observed that the accuracy measure and the precision of these approximations are greater than those of the two multi-granular approximations. The topology determined by them is found to be stronger than the topology determined by the pessimistic multi-granular approximations. Finally the results are verified through an example in the context of an information system.***

Keywords: Approximations, Rough Sets, Optimistic and Pessimistic Multi-granulation Rough Sets, Accuracy Measure, Definable Sets.

1. Introduction

The rough set theory, introduced by Zdzislaw Pawlak in 1982 [12], has been proved to be a potential tool to handle imperfect and incomplete information. Since its inception, the theory has found applications in various fields like decision analysis, engineering, image processing, pattern recognition, medicine, information theory etc [13]-[14]. In rough set theory, the indiscernibility relation or the equivalence relation enables us to divide the objects in the universe X into three disjoint sets with respect to any subset A ⊆ X as (i) the objects which are surely in A, (ii) the objects which are surely not in A and (iii) the objects which are possibly in A [16]. The objects in class (i) form the lower approximation, the objects in class (iii) form the boundary region and the objects in class (i) and (iii) together form the upper approximation.

In recent years, several extensions of the rough set model have been proposed according to various requirements [2]-[6], [11], [15], [24], [25]. In the general method of granular computing, a concept is described by a set characterized by the lower and upper approximations under a single granulation, ie; using only a single relation on the universe. A simple multi-granulation rough set model was proposed by Y. H. Qian and J. Y. Liang [17]. A study of multi-granulation rough set model in the context of an incomplete information system can be found in [18]. In [19], we find an extension of Pawlak’s single granulation rough set model into an optimistic multi-granulation rough set model (MGRS), where the set approximations are defined by using multi equivalence relations on the universe. The concept of pessimistic multi-granulation rough set was introduced by Y. H. Qian et al. [20]. Further studies on multi-granulation rough sets are done by many researchers [1], [8], [9], [21], [22], [23].

The main objective of this paper is to study the properties of approximations determined by a class of equivalence relations in Pawlak's single granulation point of view and compare them with those of multi-granular approximations. Section 2 recalls some preliminary notions of rough set theory. In section 3, the approximations determined by the intersection relation R are compared with the multi-granular approximations. In section 4, the classification of rough sets determined by R are discussed. In section 5, the results are verified through an example in the context of an information system and section 6 concludes the findings.

2. Basic Concepts

In this section, we review some basic concepts of rough sets and multi-granulation rough sets. Throughout this paper, we assume that the universe of discourse X is a finite nonempty set.

2.1. Rough sets in an approximation space [16]:

Let X be a finite non-empty set of objects and R be an equivalence relation defined on X. Then (X, R) is called an approximation space. The lower and upper approximations of $A \subseteq X$ are defined respectively as

$$\underline{R}(A) = \{x \in X : [x]_R \subseteq A\} \quad (1)$$

$$\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \phi\} \quad (2)$$

where $[x]_R$ represents the equivalence class of R containing x. A subset A of X is called a rough set if $\underline{R}(A) \neq \overline{R}(A)$. Otherwise the set is called exact. The set $\underline{R}(A)$ is called the positive region, $U - \overline{R}(A)$ is called the negative region and the set $\overline{R}(A) - \underline{R}(A)$ is called the boundary region.

The accuracy measure of a non-empty subset A of X is defined as

$$\alpha(R, A) = \frac{|\underline{R}(A)|}{|\overline{R}(A)|} \quad (3)$$

Also, the precision of approximations is defined as

$$\pi(R, A) = \frac{|\underline{R}(A)|}{|X|} \quad (4)$$

2.2. Properties of Approximations [16]:

Let A and B be two subsets of X. Then, the rough set approximations satisfy the following properties.

1) $\underline{R}(\emptyset) = \overline{R}(\emptyset) = \emptyset$
2) $\underline{R}(X) = \overline{R}(X) = X$
3) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$
4) $\underline{R}\left(\underline{R}(A)\right) = \underline{R}\,(A) = \overline{R}\left(\underline{R}(A)\right)$

5) $\overline{R}\left(\overline{R}(A)\right) = \overline{R}(A) = \underline{R}\left(\overline{R}(A)\right)$
6) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$
7) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$
8) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$
9) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$
10) $\underline{R}(A) = (\overline{R}(A^C))^C$
11) $\overline{R}(A) = (\underline{R}(A)^C)^C$
12) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$
13) $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$
14) $\underline{R}([x]_R) = [x]_R = \overline{R}([x]_R)$ for all $x \in X$

2.3. Topological Classification of Rough Sets [16]:

A rough set $A \subseteq X$ can be classified into the following types;

I. If $\underline{R}(A) \neq \emptyset$ and $\overline{R}(A) \neq X$, then A is roughly R-definable.
II. If $\underline{R}(A) = \emptyset$ and $\overline{R}(A) \neq X$, then A is internally R-indefinable.
III. If $\underline{R}(A) \neq \emptyset$ and $\overline{R}(A) = X$, then A is externally R-indefinable.
IV. If $\underline{R}(A) = \emptyset$ and $\overline{R}(A) = X$, then A is totally R-indefinable.

2.4. Topology of rough set approximations [12]:

The lower approximation operator satisfies the properties of an interior operator and hence it induces a topology

$$\mathcal{T} = \{A : A \subseteq X, \underline{R}(A) = A\} \quad (5)$$

on X in which all open subsets are closed. The upper approximation operator is the closure operator in this topology. The family of all equivalence classes forms a basis for it. For the classical theory of topological spaces, the readers may refer to [7].

2.5. The Optimistic Multi-granulation Rough Set [17]:

Let $R_1, R_2, \ldots, R_m$ be equivalence relations on a finite nonempty set X. The optimistic lower and upper approximations of $A \subseteq X$ with respect to $R_1, R_2, \ldots R_m$ are defined as

$$\underline{\sum_{i=1}^m} Ri^O(A) = \{x \in X: [x]_{Ri} \subseteq A, for\ some\ i\} \quad (6)$$

$$\overline{\sum_{i=1}^m} Ri^O(A) = \left(\underline{\sum_{i=1}^m} Ri^O(A^C)\right)^C \quad (7)$$

respectively.

The boundary region of A is defined as $N(A) = \underline{\sum_{i=1}^m} Ri^O(A) / \overline{\sum_{i=1}^m} Ri^O(A)$. The accuracy measure of the approximations is given by

$$\alpha\left(\sum R_i{}^O, A\right) = \frac{\left|\underline{\sum_{i=1}^m} Ri^O(A)\right|}{\left|\overline{\sum_{i=1}^m} Ri^O(A)\right|} \quad (8)$$

2.6. Properties of optimistic approximations [19]:

Let A and B be two subsets of X. Then,

1) $\underline{\sum_{i=1}^m} Ri^O(\emptyset) = \overline{\sum_{i=1}^m} Ri^O(\emptyset) = \emptyset$
2) $\underline{\sum_{i=1}^m} Ri^O(X) = \overline{\sum_{i=1}^m} Ri^O(X) = X$
3) $\underline{\sum_{i=1}^m} Ri^O(A) \subseteq A \subseteq \overline{\sum_{i=1}^m} Ri^O(A)$
4) $\underline{\sum_{i=1}^m}\, \underline{\sum_{i=1}^m} Ri^O(A) = \underline{\sum_{i=1}^m} Ri^O(A)$
5) $\overline{\sum_{i=1}^m}\, \overline{\sum_{i=1}^m} Ri^O(A) = \overline{\sum_{i=1}^m} Ri^O(A)$
6) $\underline{\sum_{i=1}^m} Ri^O(A) = (\overline{\sum_{i=1}^m} Ri^O(A^C))^C$
7) $\overline{\sum_{i=1}^m} Ri^O(A) = (\underline{\sum_{i=1}^m} Ri^O(A^C))^C$
8) $A \subseteq B \Rightarrow \underline{\sum_{i=1}^m} Ri^O(A) \subseteq \underline{\sum_{i=1}^m} Ri^O(B)$
9) $A \subseteq B \Rightarrow \overline{\sum_{i=1}^m} Ri^O(A) \subseteq \overline{\sum_{i=1}^m} Ri^O(B)$
10) $\underline{\sum_{i=1}^m} Ri^O(A) = \cup_{i=1}^m \underline{Ri}(A)$
11) $\overline{\sum_{i=1}^m} Ri^O(A) = \cap_{i=1}^m \overline{Ri}(A)$
12) $\underline{\sum_{i=1}^m} Ri^O(A \cap B) \subseteq \underline{\sum_{i=1}^m} Ri^O(A) \cap \underline{\sum_{i=1}^m} Ri^O(B)$
13) $\overline{\sum_{i=1}^m} Ri^O(A \cap B) \subseteq \overline{\sum_{i=1}^m} Ri^O(A) \cap \overline{\sum_{i=1}^m} Ri^O(B)$
14) $\underline{\sum_{i=1}^m} Ri^O(A \cup B) \supseteq \underline{\sum_{i=1}^m} Ri^O(A) \cup \underline{\sum_{i=1}^m} Ri^O(A)$
15) $\overline{\sum_{i=1}^m} Ri^O(A \cup \mathrm{B}) \supseteq \overline{\sum_{i=1}^m} Ri^O(A) \cup \overline{\sum_{i=1}^m} Ri^O(A)$

2.7. The Pessimistic Multi-granulation Rough Set [20]:

Let $R_1, R_2, \ldots, R_m$ be equivalence relations on a finite nonempty set X. The pessimistic lower and upper approximations of $A \subseteq X$ with respect to $R_1, R_2, \ldots, R_m$ are defined as

$$\underline{\sum_{i=1}^m} Ri^P(A) = \{x \in X: [x]_{Ri} \subseteq A\ for\ all\ i\} \quad (9)$$

$$\overline{\sum_{i=1}^m} Ri^P(A) = \left(\underline{\sum_{i=1}^m} Ri^P(A^C)\right)^C \quad (10)$$

respectively.

2.8. Properties of pessimistic approximations :

All the above properties except 10, 11, 13 and 16 holds the same. The changes are

10) $\underline{\sum_{i=1}^m} Ri^P(A) = \cap_{i=1}^m \underline{Ri}(A)$
11) $\overline{\sum_{i=1}^m} Ri^P(A) = \cup_{i=1}^m \overline{Ri}(A)$
13) $\underline{\sum_{i=1}^m} Ri^P(A \cap B) = \underline{\sum_{i=1}^m} Ri^P(A) \cap \underline{\sum_{i=1}^m} Ri^P(B)$
15) $\overline{\sum_{i=1}^m} Ri^O(A \cup \mathrm{B}) = \overline{\sum_{i=1}^m} Ri^O(A) \cup \overline{\sum_{i=1}^m} Ri^O(A)$

2.9. Multi-granular approximations in an information system [19]

Let (X, A) be an information system where U is a nonempty set of objects called the universe and $A = \{P_1, P_2, \ldots, P_n\}$ is the set of attributes. The optimistic lower and upper approximations of Y are defined as

$$\underline{\sum_{i=1}^m} Pi^O(Y) = \{x \in X: [x]_{Pi} \subseteq Y\ for\ some\ i\} \quad (11)$$

$$\overline{\sum_{i=1}^m} Pi^O(A) = \left(\underline{\sum_{i=1}^m} Pi^O(A^C)\right)^C \quad (12)$$

respectively.

The pessimistic lower and upper approximations of Y are defined as

$$\underline{\sum_{i=1}^m} Pi^P(Y) = \{x \in X: [x]_{Pi} \subseteq Y\ for\ all\ i\} \quad (13)$$

$$\overline{\sum_{i=1}^m} Pi^P(A) = \left(\underline{\sum_{i=1}^m} Pi^P(A^C)\right)^C \quad (14)$$

respectively.

The quality of the approximations or the degree of dependency of $Q \subseteq A$ is defined as

$$\gamma\left(\sum P_i, Q\right) = \frac{\sum\left\{\left|\underline{\sum_{i=1}^m} P_i(Y)\right| : Y \in X/Q\right\}}{|X|} \quad (15)$$

Also the precision of approximations is defined as

$$\pi\left(\sum P_i, Y\right) = \frac{\left|\underline{\sum_{i=1}^m} P_i(Y)\right|}{|X|} \quad (16)$$

3. Some Properties of the Approximations Determined by the Intersection Relation

Let X be a non-empty set and $R_1, R_2, \ldots, R_m$ be equivalence relations on X. Then, the intersection relation R, defined by $(x, y) \in R\ iff\ (x, y) \in R_i,\ \forall\, i = 1, 2, \ldots, m$ is an equivalence relation on X [15]. Further, $\forall\, x \in X,\ [x]_R = \cap_{i=1}^m [x]_{Ri}$.

The Pawlak's lower and upper approximations of a subset $A \subseteq X$, with respect to R are $\underline{R}(A) = \{x \in X : [x]_R \subseteq A\}$ and $\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \emptyset\}$ respectively.

The following lemma shows that the lower approximation of A with respect to R_i is contained in the lower approximation of A with respect to R and the upper approximation of A with respect to R_i contains the upper approximation of A with respect to R, for each $i = 1,2, \dots, m$.

3.1. Lemma:

For $i = 1, 2, \dots m$, $\underline{R_i}(A) \subseteq \underline{R}(A)$ and $\overline{R}(A) \subseteq \overline{R_i}(A)$

Proof:

Using eqn (1) we get,

$x \in \underline{R_i}(A) \implies [x]_{Ri} \subseteq A \implies \cap_{i=1}^{m}[x]_{Ri} \subseteq A \implies [x]_R \subseteq A \implies x \in \underline{R}(A)$. Therefore, $\underline{R_i}(A) \subseteq \underline{R}(A)$, $\forall\, i$.

Similarly, using eqn (2), $x \in \overline{R}(A) \implies [x]_R \cap A \neq \emptyset \implies \cap_{i=1}^{m}[x]_{Ri} \cap A \neq \emptyset \implies [x]_{Ri} \cap A \neq \varphi, \forall\, i \implies x \in \overline{R_i}(A)$, $\forall i$. Thus, $\overline{R}(A) \subseteq \overline{R_i}(A), \forall\, i$ ■

The following theorem is an extension of proposition 3.1 of [19].

3.2. Theorem:

$\underline{\sum_{i=1}^{m} Ri^O}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \overline{\sum_{i=1}^{m} Ri^O}(A)$, for all $A \subseteq X$.

Proof:

For $A \subseteq X$, $x \in \underline{\sum_{i=1}^{m} Ri^O}(A) \Rightarrow [x]_{Ri} \subseteq A$ for some $i \leq m$. Therefore, $x \in \underline{R_i}(A)$ for some $i \leq m$ and Using lemma 3.1, $x \in \underline{R}(A)$. Hence,

$$\underline{\sum_{i=1}^{m} Ri^O}(A) \subseteq \underline{R}(A) \tag{17}$$

Also, $x \in \overline{R}(A) \Rightarrow x \in \overline{R_i}(A), \forall\, i \implies [x]_{Ri} \cap A \neq \emptyset, \forall\, i \implies [x]_{Ri} \nsubseteq A^C, \forall\, i \implies x \notin \underline{\sum_{i=1}^{m} Ri^O}(A^C)$. Therefore, $x \in \left(\underline{\sum_{i=1}^{m} Ri^O}(A^C)\right)^C$. Thus, $x \in \overline{\sum_{i=1}^{m} Ri^O}(A)$. Hence,

$$\overline{R}(A) \subseteq \overline{\sum_{i=1}^{m} Ri^O}(A) \tag{18}$$

From the properties of approximations 2.2,

$$\underline{R}(A) \subseteq A \subseteq \overline{R}(A) \tag{19}$$

Combining (17), (18) and (19),

$\underline{\sum_{i=1}^{m} Ri^O}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \overline{\sum_{i=1}^{m} Ri^O}(A)$ ■

3.3. Theorem:

$\underline{\sum_{i=1}^{m} Ri^P}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \overline{\sum_{i=1}^{m} Ri^P}(A)$, for all $A \subseteq X$.

Proof:

Let, $A \subseteq X$. Then, $\in \underline{\sum_{i=1}^{m} Ri^P}(A) \Rightarrow [x]_{Ri} \subseteq A$, $\forall\, i \leq m \implies x \in \underline{R_i}(A)$, $\forall\, i \implies x \in \underline{R}(A)$. Therefore,

$$\underline{\sum_{i=1}^{m} Ri^P}(A) \subseteq \underline{R}(A) \tag{20}$$

Also, $x \in \overline{R}(A) \Rightarrow x \in \overline{R_i}(A), \forall\, i \Rightarrow [x]_{Ri} \cap A \neq \emptyset, \forall\, i \implies [x]_{Ri} \nsubseteq A^C, \forall\, i \implies x \notin \underline{\sum_{i=1}^{m} Ri^P}(A^C)$. Therefore, $x \in \left(\underline{\sum_{i=1}^{m} Ri^P}(A^C)\right)^C$. So, $x \in \overline{\sum_{i=1}^{m} Ri^P}(A)$. Therefore,

$$\overline{R}(A) \subseteq \overline{\sum_{i=1}^{m} Ri^P}(A) \tag{21}$$

From the properties of approximations 2.2,

$$\underline{R}(A) \subseteq A \subseteq \overline{R}(A) \tag{22}$$

Combining (20), (21) and (22),

$\underline{\sum_{i=1}^{m} Ri^P}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \overline{\sum_{i=1}^{m} Ri^P}(A)$ ■

For each subset $A \subseteq X$, the best approximations are given by the largest set contained in A and the smallest set containing A. Theorems 3.2 and 3.3 show that the lower and upper approximations defined by R are closer to the set than the optimistic or pessimistic multi-granular approximations. The following example illustrates this fact.

3.4. Example:

Let $X = \{a, b, c, d, e, f\}$. Consider two equivalence relations R_1 and R_2 on X given by $X/R_1 = \{\{a, b\}, \{c, d\}, \{e, f\}\}$ and $X/R_2 = \{\{a, c, d\}, \{b, e\}, \{f\}\}$. Then the intersection relation is $X/R = \{\{a\}, \{b\}, \{c, d\}, \{e\}, \{f\}\}$. Take $A = \{b, d, f\}$. Then, $\underline{R}(A) = \{b, f\}$ and $\overline{R}(A) = \{b, c, d, f\}$. Further we obtain, $\underline{\sum_{i=1}^{m} Ri^O}(A) = \{f\}$ and $\overline{\sum_{i=1}^{m} Ri^O}(A) = \{a, b, c, d, e, f\}$. Also, $\underline{\sum_{i=1}^{m} Ri^P}(A) = \emptyset$ and $\overline{\sum_{i=1}^{m} Ri^P}(A) = \{a, b, c, d, e, f\}$. Clearly $(\{b, f\}, \{b, c, d, f\})$ is a better approximation for the set $A = \{b, d, f\}$ than $(\{f\}, \{a, b, c, d, e, f\})$ or $(\emptyset, \{a, b, c, d, e, f\})$. Also from the example it follows that in general, $\underline{R}(A) \neq \underline{\sum_{i=1}^{m} Ri^O}(A)$ and $\overline{R}(A) \neq \overline{\sum_{i=1}^{m} Ri^O}(A)$. Further, $\underline{R}(A) \neq \underline{\sum_{i=1}^{m} Ri^P}(A)$ and $\overline{R}(A) \neq \overline{\sum_{i=1}^{m} Ri^P}(A)$ ■

Another comparison of the approximations in terms of accuracy and precision of approximations are given in the following propositions. The accuracy measure and the precision of the approximations with respect to R are found to be greater than those of the two multi-granular approximations.

3.5. Proposition:

The accuracy measure,

$$\alpha(R, A) \geq \alpha(\Sigma Ri^O, A) \text{ and } \alpha(R, A) \geq \alpha(\Sigma Ri^P, A)$$

Proof:

We have $(R, A) = \frac{|\underline{R}(A)|}{|\overline{R}(A)|}$, $\alpha(\Sigma Ri^O, A) = \frac{\left|\underline{\sum_{i=1}^{m} Ri^O}(A)\right|}{\left|\overline{\sum_{i=1}^{m} Ri^O}(A)\right|}$ and $\alpha(\Sigma Ri^P, A) = \frac{\left|\underline{\sum_{i=1}^{m} Ri^P}(A)\right|}{\left|\overline{\sum_{i=1}^{m} Ri^P}(A)\right|}$. Also, $\underline{\sum_{i=1}^{m} Ri^O}(A) \subseteq \underline{R}(A)$ and $\overline{R}(A) \subseteq \overline{\sum_{i=1}^{m} Ri^O}(A)$. Hence, $\left|\underline{\sum_{i=1}^{m} Ri^O}(A)\right| \leq \left|\underline{R}(A)\right|$ and $\left|\overline{R}(A)\right| \leq \left|\overline{\sum_{i=1}^{m} Ri^O}(A)\right|$. Therefore, $\frac{\left|\underline{\sum_{i=1}^{m} Ri^O}(A)\right|}{\left|\overline{\sum_{i=1}^{m} Ri^O}(A)\right|} \leq \frac{|\underline{R}(A)|}{|\overline{R}(A)|}$ which gives, $\alpha(R, A) \geq \alpha(\Sigma Ri^O, A)$.

Similarly, $\alpha(R, A) \geq \alpha(\Sigma Ri^P, A)$. ■

3.6. Proposition:

The precision of approximation

$$\pi(R, A) \geq \pi(\Sigma Ri^O, A) \text{ and } \pi(R, A) \geq \pi(\Sigma Ri^P, A).$$

Proof:

We have, $(R, A) = \frac{|\underline{R}(A)|}{|X)|}$, $\pi(\Sigma Ri^O, A) = \frac{\left|\underline{\sum_{i=1}^{m} Ri^O}(A)\right|}{|X|}$ and $\pi(\Sigma Ri^P, A) = \frac{\left|\underline{\sum_{i=1}^{m} Ri^P}(A)\right|}{|X|}$. From $\underline{\sum_{i=1}^{m} Ri^O}(A) \subseteq \underline{R}(A)$ and $\underline{\sum_{i=1}^{m} Ri^P}(A) \subseteq \underline{R}(A)$, we obtain $\pi(R, A) \geq \pi(\Sigma Ri^O, A)$ and $\pi(R, A) \geq \pi(\Sigma Ri^P, A)$ ■

Y. She and X. He [23] proved that the pessimistic multi-granular lower approximation induces a topology on X given by $\{ A \subseteq X : \underline{\sum_{i=1}^{m} Ri^P}(A) = A\}$. The results obtained in theorems 3.2 and 3.3 pave way to the following fact.

3.7. Theorem:

The topology induced by R is stronger than the topologies induced by the pessimistic multi-granular approximations.

Proof:

Let $\mathcal{T}$ and $\mathcal{T}_P$ be the topology induced by R and the pessimistic multi-granular approximations respectively. If A is an open set in $\mathcal{T}_P$, Then, $\underline{\sum_{i=1}^{m} Ri^P}(A) = A$. Also, $\underline{\sum_{i=1}^{m} Ri^P}(A) \subseteq \underline{R}(A)$. It follows that $A \subseteq \underline{R}(A)$ and hence $\underline{R}(A) = A$. Thus, $A \in \mathcal{T}$. Therefore $\mathcal{T}_P \subseteq \mathcal{T}$, which means that $\mathcal{T}$ is stronger ■

3.8. Theorem:

The topology induced by R is stronger than the topologies induced by R_i for each i.

Proof:

Let $\mathcal{T}$ and $\mathcal{T}_{R_i}$ be the topologies induced by R and R_i respectively. Let A be an open set in $\mathcal{T}_{R_i}$. Then $\underline{R_i}(A) = A$. But $\underline{R_i}(A) \subseteq \underline{R}(A)$. Hence we get $A \subseteq \underline{R}(A)$. As $\underline{R}(A) \subseteq A$, it follows that $\underline{R}(A) = A$. Therefore A is an open set in $\mathcal{T}$. So $\mathcal{T}_{R_i} \subseteq \mathcal{T}$ ■

In [10], the authors has defined multi-granulation topological rough space, where the topology induced by the natural mappings with respect to the multi-granulations has been considered. The following proposition shows that this topology is same as the topology induced by R.

3.9. Proposition:

The topology $\mathcal{T}$ induced by the intersection relation R on X is the same as $\mathcal{T}_1 \sqcap \mathcal{T}_2 \sqcap \ldots \sqcap \mathcal{T}_m$, the topology generated by the intersection of the natural mappings with respect to R_1, R_2,…., R_m.

Proof:

Let A be a basic open set of $\mathcal{T}_1 \sqcap \mathcal{T}_2 \sqcap \ldots \sqcap \mathcal{T}_m$. The natural mapping maps an element to its equivalence class. Hence A will be the intersection of the equivalence classes of some element $x \in X$. That is; $A = [x]_{R1} \cap [x]_{R2} \cap \ldots \cap [x]_{Rm}$. Then, $A = [x]_R$ which is a basic element of $\mathcal{T}$.

Similarly the basic elements of $\mathcal{T}$ are the basic elements of $\mathcal{T}_1 \sqcap \mathcal{T}_2 \sqcap \ldots \sqcap \mathcal{T}_m$ ■

4. Topological Classification of Rough Sets Determined by the Intersection Relation

In this section, we investigate the effect of the nature of rough sets determined by the individual equivalence relations on the nature of rough sets determined by R with respect to the definability of rough sets. Also, the relations between the nature of rough sets and the nature of their union, intersection and complement are also discussed.

4.1. Lemma:

a) If $\underline{R}i(A) \neq \emptyset$ for some i, then $\underline{R}(A) \neq \emptyset$

b) If $\overline{R}i(A) \neq X$ for some i, then $\overline{R}(A) \neq X$

Proof:

Since, $[x]_R = \bigcap_{i=1}^{m} [x]_{Ri}, \forall x \in X$, we get, $[x]_R \subseteq [x]_{Ri}, \forall i$.

a) If $\underline{R}i(A) \neq \emptyset$, there exists an $x \in X$ such that $[x]_{Ri} \subseteq A$. Hence $[x]_R \subseteq A$ and $x \in \underline{R}(A)$. Therefore, $\underline{R}(A) \neq \emptyset$.

b) If $\overline{R}i(A) \neq X$ for some i, there exists an $x \in X$ such that $x \notin \overline{R}i(A)$. Then $[x]_{Ri} \cap A = \emptyset$. Hence, $[x]_R \cap A = \emptyset$ and $\notin \overline{R}(A)$. Therfore, $\overline{R}(A) \neq X$ ■

is expressed in the following theorem.

4.2. Theorem:

a) If A is a type I rough set with respect to R_i for some i, then A is a type I rough set with respect to R.

b) If A is a type II rough set with respect to R_i for some i, then A is a type I or type II rough set with respect to R.

c) If A is a type II rough set with respect to R_i for some i and A is type III rough set with respect to R_j for some j, then A is a type I rough set with respect to R.

d) If A is a type III rough set with respect to R_i for some i, then A is a type III or type I rough set with respect to R.

e) If A is a type IV rough set with respect to R_i for some i, then A can be any of the four types of rough sets with respect to R.

Proof:

a) Since A is a type I rough set with respect to R_i for some i, $\underline{R}i(A) \neq \emptyset$ and $\overline{R}i(A) \neq X$. Then $\underline{R}(A) \neq \emptyset$ and $\overline{R}(A) \neq X$. Hence, A is a type I rough set with respect to R.

b) Since A is a type II rough set with respect to R_i for some i, $\underline{R}i(A) = \emptyset$ and $\overline{R}i(A) \neq X$. Then $\overline{R}(A) \neq X$ and both $\underline{R}(A) \neq \emptyset$ and $\underline{R}(A) = \emptyset$ are possible. Hence, A is either Type I or Type II rough set with respect to R.

c) Since A is a type II rough set with respect to R_i for some i, $\underline{R_i}(A) = \emptyset$ and $\overline{R}_i(A) \neq X$. Since A is a type III rough set with respect to R_j for some j, $\underline{R_j}(A) \neq \emptyset$ and $\overline{R}_j(A) = X$. Then, $\underline{R}(A) \neq \emptyset$ *and* $\overline{R}(A) \neq X$. Hence, A is a Type I rough set with respect to R.

d) Since A is a type III rough set with respect to R_i for some i, $\underline{R}i(A) \neq \emptyset$ and $\overline{Ri}(A) = X$ for some i. Then, $\underline{R}(A) \neq \emptyset$ and both $\overline{R}(A) \neq X$ and $\overline{R}(A) = X$ are possible. Hence, A is either Type III or Type I rough set with respect to R.

e) Since A is a type IV rough set with respect to R_i for some i, $\underline{R}i(A) = \emptyset$ and $\overline{Ri}(A) = X$ for some i. Then both $\underline{R}(A) \neq \emptyset$ and $\underline{R}(A) = \emptyset$ and both $\overline{R}(A) \neq X$ and $\overline{R}(A) = X$ are possible. Hence A can be any of the four types of rough sets with respect to R.

4.3. Table for $R = R_1 \cap R_2$

In the case of two equivalence relations R1 and R2, the above results can be summarized as given in the following table. Each entry gives the type of A with respect to R.

Table 1: Type of A under R

		Type of A with respect to R1			
		Type I	Type II	Type III	Type IV
Type of A with respect to R2	Type I	Type I	Type I	Type I	Type I
	Type II	Type I	Type I/ Type II	Type I	Type I/ Type II
	Type III	Type I	Type I/ Type II	Type I/ Type III	Type I/ Type III
	Type IV	Type I	Type I/ Type II	Type I/ Type III	Any of the four types

4.4. Table for A^C

The types of A^C corresponding to the types of A with respect to R is given in the following table. This result holds for any equivalence relation on X.

Table 2: Type of A^C

Type of A	Type of A^C
Type I	Type I
Type II	Type III
Type III	Type II
Type IV	Type IV

4.5. Table for $A \cup B$:

The types of $A \cup B$ are expressed in the following table.

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Table 3: Type of $A \cup B$

		Type of B			
		Type I	Type II	Type III	Type IV
Type of A	Type I	Type I/ Type III	Type I/ Type III	Type III	Type III
	Type II	Type I/ Type III	Any of the four types	Type III	Type III/ Type IV
	Type III	Type III	Type III	Type III	Type III
	Type IV	Type III	Type III/ Type IV	Type III	Type III/ Type IV

4.6. Table for $A \cap B$

The types of $A \cap B$ are given in the following table.

Table 4: Type of $A \cap B$

		Type of B			
		Type I	Type II	Type III	Type IV
Type of A	Type I	TypeI/ TypeII	Type II	Type I/ Type II	Type II
	Type II	Type II	Type II	Type II	Type II
	Type III	TypeI/ TypeII	Type II	Any of the four types	Type II/ Type IV
	Type IV	Type II	Type II	Type II/ Type IV	Type II/ Type IV

5. Intersection Relation in Information Systems

In [19], the authors proved that the accuracy measure, degree of dependency and the precision of approximation determined by using multi-granulations is greater than those determined by Pawlak's single granulation. But the propositions deal with the case of only one attribute or only one equivalence relation. If we consider the indiscernibility relation determined by the entire attribute set I(A), then this corresponds to the equivalence relation R in section 3. Thus we arrive at the following two propositions.

5.1. Proposition:

In an information system, the accuracy measure of a set described by I(A) is greater than the accuracy measure of the set described by both type of multi-granulations. ie; $\alpha(I(A),Y) \geq \alpha\left(\sum P_i^{\,O}, Y\right)$, $\alpha(I(A),Y) \geq \alpha\left(\sum P_i^{\,P}, Y\right)$ and $\pi(I(A),Y) \geq \pi\left(\sum P_i^{\,O}, Y\right)$, $\pi(I(A),Y) \geq \pi\left(\sum P_i^{\,P}, Y\right)$.

Proof:

By proposition 3.2 and 3.3, $\forall\, Y \subseteq X$,
$\underline{\sum_{i=1}^{m}} Pi^{O}(Y) \subseteq \underline{B}(Y) \subseteq Y \subseteq \overline{B}(Y) \subseteq \overline{\sum_{i=1}^{m}} Pi^{O}(Y)$ and $\underline{\sum_{i=1}^{m}} Pi^{P}(Y) \subseteq \underline{B}(Y) \subseteq Y \subseteq \overline{B}(Y) \subseteq \overline{\sum_{i=1}^{m}} Pi^{P}(Y)$. Hence, $\left|\underline{\sum_{i=1}^{m}} Pi^{O}(Y)\right| \leq \left|\underline{B}(Y)\right| \leq \left|\overline{B}(Y)\right| \leq \left|\overline{\sum_{i=1}^{m}} Pi^{O}(Y)\right|$ and $\left|\underline{\sum_{i=1}^{m}} Pi^{P}(Y)\right| \leq \left|\underline{B}(Y)\right| \leq \left|\overline{B}(Y)\right| \leq \left|\overline{\sum_{i=1}^{m}} Pi^{P}(Y)\right|$. Thus, from the definitions of the uncertainty measures, it follows that $\alpha(I(A),Y) \geq \alpha\left(\sum P_i^{\,O}, Y\right)$, $\alpha(I(A),Y) \geq \alpha\left(\sum P_i^{\,P}, Y\right)$ and $\pi(I(A),Y) \geq \pi\left(\sum P_i^{\,O}, Y\right)$ and $\pi(I(A),Y) \geq \pi\left(\sum P_i^{\,P}, Y\right)$ ■

5.2. Proposition:

The degree of dependency of $Q \subseteq A$ with respect to I(A) is greater than that with respect to both type of multi-granulations. That is; $\gamma(I(A),Q) \geq \gamma\left(\sum P_i^{\,O}, Q\right)$ and $\gamma(I(A),Q) \geq \gamma\left(\sum P_i^{\,P}, Q\right)$

Proof:

Since, $\gamma\left(\sum P_i, Q\right) = \frac{\sum\left\{\left|\underline{\sum_{i=1}^{m}} P_i(Y)\right| : Y \in X/Q\right\}}{|X|}$ and using proposition 3.2 and 3.3, $\underline{\sum_{i=1}^{m}} Pi^{O}(Y) \subseteq \underline{B}(Y)$ and $\underline{\sum_{i=1}^{m}} Pi^{P}(Y) \subseteq \underline{B}(Y)$, $\forall\, Y \in X/Q$. Therefore, we get $\left|\underline{\sum_{i=1}^{m}} Pi^{O}(Y)\right| \leq \left|\underline{B}(Y)\right|$ and $\left|\underline{\sum_{i=1}^{m}} Pi^{P}(Y)\right| \leq \left|\underline{B}(Y)\right|$. Hence, $\gamma(I(A),Q) \geq \gamma\left(\sum P_i^{\,O}, Q\right)$ and $\gamma(I(A),Q) \geq \gamma\left(\sum P_i^{\,P}, Q\right)$ ■

These results are verified for the information system presented in table 5 [22], in which six stores are characterized by four attributes, E - empowerment of sales personnel, Q - perceived quality of merchandise, L - high traffic location, P - store profit or loss

Table 5

Store	E	Q	L	P
1	High	Good	No	Profit
2	Medium	Good	No	Loss
3	Medium	Good	No	Profit
4	No	Average	No	Loss
5	Medium	Average	Yes	Loss
6	High	Average	Yes	Profit

Let P_1, P_2, P_3 denote the equivalence relations corresponding to the condition attributes and P_d denote the equivalence relation corresponding to the decision attribute P. Then, we have, $X/P_1 = \{\{1,6\},\{2,3,5\},\{4\}\}$, $X/P_2 = \{\{1,2,3\},\{4,5,6\}\}$, $X/P_3 = \{\{1,2,3,4\},\{5,6\}\}$ and $X/P_d = \{\{1,3,6\},\{2,4,5\}\}$. Let $C = \{E,Q,L\}$. Then, $/I(C) = \{\{1\},\{2,3\},\{4\},\{5\},\{6\}\}$.

Let $= \{2,4,6\}$. Then $\underline{C}(Y) = \{4,6\}$ and $\overline{C}(Y) = \{2,3,4,6\}$. Also, $\underline{\sum_{i=1}^{3}} Pi^{O}(Y) = \{4\}$ and $\overline{\sum_{i=1}^{3}} Pi^{O}(Y) = \{1,2,3,4,5,6\}$. Again, $\underline{\sum_{i=1}^{m}} Pi^{P}(Y) = \emptyset$ and $\overline{\sum_{i=1}^{m}} Pi^{P}(Y) = \{1,2,3,4,5,6\}$. Thus $\underline{C}(Y) \supseteq \underline{\sum_{i=1}^{3}} Pi^{O}(Y)$ and $\overline{C}(Y) \subseteq \overline{\sum_{i=1}^{3}} Pi^{O}(Y)$. Also, $\underline{C}(Y) \supseteq \underline{\sum_{i=1}^{3}} Pi^{P}(Y)$ and $\overline{C}(Y) \subseteq \overline{\sum_{i=1}^{3}} Pi^{P}(Y)$.

Further, $\alpha(I(C),Y) = \frac{|\underline{C}(Y)|}{|\overline{C}(Y)|} = \frac{2}{4} = 0.5$,

$\pi(I(C),Y) = \frac{|\underline{C}(Y)|}{|X|} = \frac{2}{6} = 0.33$,

$\alpha\left(\sum P_i^{\,O}, Y\right) = \frac{\left|\underline{\sum_{i=1}^{3}} Pi^{O}(Y)\right|}{\left|\overline{\sum_{i=1}^{m}} Pi^{O}(Y)\right|} = \frac{1}{6} = 0.167$,

$\pi\left(\sum P_i^{\,O}, Y\right) = \frac{\left|\underline{\sum_{i=1}^{3}} Pi^{O}(Y)\right|}{|X|} = \frac{1}{6} = 0.167$,

$\alpha\left(\sum P_i^{\,P}, Y\right) = \frac{\left|\underline{\sum_{i=1}^{3}} Pi^{P}(Y)\right|}{\left|\overline{\sum_{i=1}^{m}} Pi^{P}(Y)\right|} = \frac{0}{6} = 0$ and

$\pi\left(\sum P_i^{\,P}, Y\right) = \frac{\left|\underline{\sum_{i=1}^{3}} Pi^{O}(Y)\right|}{|X|} = \frac{0}{6} = 0.$

From the above measures it follows that
$\alpha(I(C),Y) \geq \alpha\left(\sum P_i^{\,P}, Y\right)$, $\alpha(I(C),Y) \geq \alpha\left(\sum P_i^{\,P}, Y\right)$,
$\pi(I(C),Y) \geq \pi\left(\sum P_i^{\,P}, Y\right)$ and $\pi(I(C),Y) \geq \pi\left(\sum P_i^{\,P}, Y\right)$.

If $D = \{P\}$, then, $X/D = X/P_d = \{\{1,3,6\},\{2,4,5\}\}$.
So, $\underline{C}(\{1,3,6\},) = \{1,6\}$ and $\underline{C}(\{2,4,5\}) = \{4,5\}$.
Then the degree of dependency of the decision attribute set D on the condition attribute set C with respect to I(C) is given by $(C,D) = \frac{\sum\{|\underline{C}(Y)| : Y \in X/D\}}{|X|} = \frac{2+2}{6} = 0.67$.

But $\underline{\sum_{i=1}^{m}}\ P_i{}^{O}(\{1,3,6\}) = \{1,6\}$ and $\underline{\sum_{i=1}^{m}}\ P_i{}^{O}(\{2,4,5\}) = \{4\}$. Hence, $\gamma\left(\sum_{P_i \in C} P_i{}^{O}, D\right) = \frac{\sum\left\{\left|\underline{\sum_{i=1}^{3}} P_i{}^{O}(Y)\right| : Y \in X/Q\right\}}{|X|} = \frac{2+1}{6} = 0.5$. Also, $\underline{\sum_{i=1}^{m}}\ P_i{}^{P}(\{1,3,6\}) = \emptyset$ and $\underline{\sum_{i=1}^{m}}\ P_i{}^{P}(\{2,4,5\}) = \emptyset$. Hence $\left(\sum_{P_i \in C} P_i{}^{P}, D\right) = \frac{\sum\left\{\left|\underline{\sum_{i=1}^{3}} P_i{}^{P}(Y)\right| : Y \in X/Q\right\}}{|X|} = \frac{0+0}{6} = 0$.
Therefore, $\gamma(C, D) \geq \gamma\left(\sum_{P_i \in C} P_i{}^{O}, D\right)$ and
$\gamma(C, D) \geq \gamma\left(\sum_{P_i \in C} P_i{}^{P}, D\right)$.

6. CONCLUSION

In this paper, we have presented some properties of approximations determined by a class of equivalence relations in Pawlak's single granulation point of view and the results are compared with those of optimistic and the pessimistic multi-granulation rough sets. We have found that the intersection relation R provides approximations which are closer to the set than the optimistic or pessimistic multi-granular approximations. Also, the accuracy measure of the approximations by R is found to be greater than those of the two multi-granular approximations. Further, we have proved that the topology determined by R is stronger than the topology determined by the pessimistic multi-granular approximations. Some topological properties of these approximations are also studied. We have verified the findings through an example in the context of an information system. However, the individuality of the equivalence relations is more dominant in the multi-granular approximations.

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