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Fixed points of λ-generalized contraction selfmapings in D^*-metric space

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ABSTRACT: The main of this paper is to prove the existence of fixed point on λ-generalized contraction of self mapping functions on D^*-metric space.

Key Words: D*-metric space, K-contraction**,** λ-generalized contraction,

I. INTRODUCTION:

Huang and Zhang [11] generalized the notion of metric spaces, replacing the real numbers by an ordered Banach space and defined cone metric spaces. They have proved Banach contraction mapping theorem and some other fixed point theorems of contractive type mappings in cone metric spaces. Subsequently, Rezapour and Hamlbarani [17], Ilic and Rakocevic [9], contributed some fixed point theorems for contractive type mappings in cone metric spaces.

Gahler [7, 8] introduced the notions of 2-metric space and Dhage [5, 6] defned D-metric spaces as a generalization of metric spaces. In 2003, Zead Mustafa and Brailey Sims[13] introduced a new structure of generalized metric spaces, which are called G-metric spaces. Recently Aage and Salunke[2] generalized G -metric space by replacing R by real Banach space in G -metric spaces. In 2007 Shaban Sedghi et al [18] modify the D-metric space and defined D §-metric spaces. Now in this paper I Generalized D-metric spaces by introducing generalized D -metric space by replacing R by a real Banach space in D-metric spaces.

II. PRELIMINARY NOTES:

Recall that a selfmap f of a D^*-metric space (X, D^*) is called a ***contraction***, if there is a q with $0 \le q < 1$ such that

(2.1) $$D^*(fx, fy, fy) \le q.D^*(x, y, y) \text{ for all } x, y \in X$$

In a different way R. Kannan [9] has defined a contraction for metric spaces which we shall call a K-contraction. Analogously we define the K-contractions for D^*-metric spaces as follows:

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2.2 Definition: A selfmap f of a D^*-metric space (X, D^*) is called a ***K-contraction***, if there is a q with $0 \le q < \frac{1}{2}$ such that

$$D^*(fx, fy, fz) \le q.\max\{D^*(x, fx, fx) + D^*(y, fy, fy)\} \text{ for all } x, y \in X \tag{2.3}$$

The notions of contraction and K-contraction are independent. In this thesis we define a special type of contractions called λ-generalized contractions for D^*-metric spaces as follows:

2.4 Definition: A selfmap f of a D^*-metric space (X, D^*) is called a λ***-generalized contraction***, if for every $x, y \in X$, there exist non-negative numbers q, r, s and t (all depending on x and y) such that

$$\sup_{x, y \in X} \{q + r + s + 2t\} = \lambda < 1 \text{ and} \tag{2.5}$$

$$\begin{aligned} D^*(fx, fy, fz) \le q.D^*(x, y, y) + r.D^*(x, fx, fx) + s.D^*(y, fy, fy) \\ t.\{D^*(x, fy, fy) + D^*(y, fx, fx)\} \end{aligned} \tag{2.6}$$

for all $x, y \in X$

As already noted in the Remark 1.14.3, every contraction and every K-contraction is a λ-generalized contraction. However the following examples show that there are some λ-generalized contraction f on a D^*-metric spaces (X, D^*) which are not contractions and /or K-contractions.

The following is an example of a λ-generalized contraction which is not a contraction.

III. Main Result:

Theorem: suppose f is a selfmap of a D^*-metric space (X, D^*) and X be f-orbitally complete. If f is a λ-generalized contraction, then it has a unique fixed point $u \in X$. In fact,

$$u = \lim_{n \to \infty} f^n x \text{ for any } x \in X \tag{3.1}$$

and

$$D^*(f^n x, u, u) \le \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx) \text{ for all } x \in X \text{ and } n \ge 1. \tag{3.2}$$

Proof: $x \in X$ be arbitrary and define the sequence $\{x_n\}$ by

$x_0 = x,\ x_1 = fx_0,\ x_2 = fx_1 = f^2x, \ldots, x_n = fx_{n-1} = f^n x, \ldots$. Note that the orbit of *x* under *f*,

$O_f(x:\infty) = \{x_n : n = 0, 1, 2, 3, \ldots\}$

Consider

$$D*(x_n, x_{n+1}, x_{n+1}) = D*(fx_{n-1}, fx_n, fx_n)$$

$$\leq q(x_{n-1}, x_n)D*(x_{n-1}, x_n, x_n) + r(x_{n-1}, x_n)D*(x_{n-1}, fx_{n-1}, fx_{n-1})$$

$$+ s(x_{n-1}, x_n)D*(x_n, fx_n, fx_n)$$

$$+ t(x_{n-1}, x_n)\{D*(x_{n-1}, fx_n, fx_n) + D*(x_n, fx_{n-1}, fx_{n-1})\}$$

$$\leq q(x_{n-1}, x_n)D*(x_{n-1}, x_n, x_n) + r(x_{n-1}, x_n)D*(x_{n-1}, x_n, x_n)$$

$$+ s(x_{n-1}, x_n)D*(x_n, x_{n+1}, x_{n+1})$$

$$+ t(x_{n-1}, x_n)\{D*(x_{n-1}, x_{n+1}, x_{n+1}) + D*(x_n, x_n, x_n)\}$$

Writing $q_{n-1} = q(x_{n-1}, x_n)$, $r_{n-1} = r(x_{n-1}, x_n)$, $s_{n-1} = s(x_{n-1}, x_n)$ and $t_{n-1} = t(x_{n-1}, x_n)$, we get

$$D*(x_n, x_{n+1}, x_{n+1}) \leq q_{n-1}D*(x_{n-1}, x_n, x_n) + r_{n-1}D*(x_{n-1}, x_n, x_n)$$

$$+ s_{n-1}D*(x_n, x_{n+1}, x_{n+1}) + t_{n-1}\{\ D*(x_{n-1}, x_{n+1}, x_{n+1})$$

$$+ D*(x_n, x_n, x_n)\ \}$$

$$\leq (q_{n-1} + r_{n-1})D*(x_{n-1}, x_n, x_n) + s_{n-1}D*(x_n, x_{n+1}, x_{n+1})$$

$$+ t_{n-1}\{\ D*(x_{n-1}, x_{n+1}, x_{n+1})\ \}$$

$$\leq (q_{n-1} + r_{n-1})D*(x_{n-1}, x_n, x_n) + s_{n-1}D*(x_n, x_{n+1}, x_{n+1})$$

$$+ t_{n-1}D*(x_{n-1}, x_n, x_n) + t_{n-1}D*(x_n, x_{n+1}, x_{n+1})$$

$$\leq (q_{n-1} + r_{n-1} + t_{n-1})D*(x_{n-1}, x_n, x_n) + (s_{n-1} + t_{n-1})D*(x_n, x_{n+1}, x_{n+1})$$

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which implies that

$$(1-s_{n-1}-t_{n-1})D^*(x_n, x_{n+1}, x_{n+1}) \le (q_{n-1}+r_{n-1}+t_{n-1})D^*(x_{n-1}, x_n, x_n)$$

And hence

$$D^*(x_n, x_{n+1}, x_{n+1}) \le \frac{(q_{n-1}+r_{n-1}+t_{n-1})}{(1-s_{n-1}-t_{n-1})} D^*(x_{n-1}, x_n, x_n)$$

$$\le \frac{(\lambda-s_{n-1}-t_{n-1})}{(1-s_{n-1}-t_{n-1})} D^*(x_{n-1}, x_n, x_n)$$

That is,

$$D^*(x_n, x_{n+1}, x_{n+1}) \le \lambda\ D^*(x_{n-1}, x_n, x_n), \text{ where } \lambda = \sup_{x, y \in X} \{q+r+s+2t\}$$

Thus by iteration, we get

$$D^*(x_n, x_{n+1}, x_{n+1}) \le \lambda^n\ D^*(x_0, x_1, x_1) = \lambda^n\ D^*(x_0, fx_0, fx_0) \text{ -------- (A)}$$

Therefore

$$D^*(x_n, x_{n+p}, x_{n+p}) \le D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n+1}, x_{n+2}, x_{n+2})$$

$$+ D^*(x_{n+2}, x_{n+3}, x_{n+3}) + \ . \ . \ . \ + D^*(x_{n+p-1}, x_{n+p}, x_{n+p})$$

$$\le \lambda^n\ D^*(x_0, x_1, x_1) + \lambda^{n+1}\ D^*(x_0, x_1, x_1)$$
$$+ \lambda^{n+2}\ D^*(x_0, x_1, x_1) + \ . \ . \ . \ + \lambda^{n+p-1}\ D^*(x_0, x_1, x_1)$$

$$\le (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ . \ . \ . \ + \lambda^{n+p-1} + \ . \ . \ .\ ) D^*(x_0, x_1, x_1)$$

$$= \frac{\lambda^n}{1-\lambda} D^*(x_0, x_1, x_1) \text{ ------------------ (B)}$$

Hence $D^*(x_n, x_{n+p}, x_{n+p}) \le \frac{\lambda^n}{1-\lambda} D^*(x_0, x_1, x_1) \to 0$ as $n \to \infty$, since $0 \le \lambda < 1$. Thus $\{x_n\}$ is a sequence in $O_f(x:\infty)$ and since X is *f*-orbitally complete, there exists $u \in X$ such that

$$u = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f^n x_0 = \lim_{n \to \infty} f^n x$$

To show that u is a fixed point of *f*, first we prove

$\lim_{n \to \infty} D^*(fu, fx_n, fx_n) = 0$. Since *f* is λ-generalized contraction, we have

$$D^*(fu, fx_n, fx_n) \le q(u, x_n) D^*(u, x_n, x_n) + r(u, x_n) D^*(u, fu, fu)$$

$$+ s(u, x_n) D^*(x_n, fx_n, fx_n) + t(u, x_n)\{D^*(u, fx_n, fx_n)$$

$$+ D^*(x_n, fu, fu)\}$$

That is,

$$D^*(fu, fx_n, fx_n) \le q(u, x_n) D^*(u, x_n, x_n) + r(u, x_n) D^*(u, fu, fu)$$

$$+ s(u, x_n) D^*(x_n, x_{n+1}, x_{n+1}) + t(u, x_n)\{D^*(u, x_{n+1}, x_{n+1})$$

$$+ D^*(x_n, fu, fu)\}$$

$$\le q(u, x_n) D^*(u, x_n, x_n) + r(u, x_n) D^*(u, x_{n+1}, x_{n+1})$$

$$+ r(u, x_n) D^*(x_{n+1}, fu, fu) + s(u, x_n) D^*(x_n, x_{n+1}, x_{n+1})$$

$$+ t(u, x_n) D^*(u, x_{n+1}, x_{n+1}) + t(u, x_n) D^*(x_n, x_{n+1}, x_{n+1})$$

$$+ t(u, x_n) D^*(x_{n+1}, fu, fu)$$

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$$\leq q(u,x_n)D^*(u,x_n,x_n)+\{r(u,x_n)+t(u,x_n)\}D^*(u,x_{n+1},x_{n+1})$$

$$+\{r(u,x_n)+t(u,x_n)\}D^*(x_{n+1},fu,fu)$$

$$+\{s(u,x_n)+t(u,x_n)\}D^*(x_n,x_{n+1},x_{n+1})$$

$$\leq \lambda D^*(u,x_n,x_n)+\lambda D^*(u,x_{n+1},x_{n+1})$$

$$+\lambda D^*(x_{n+1},fu,fu)+\lambda D^*(x_n,x_{n+1},x_{n+1})$$

That is,

$$(1-\lambda)D^*(fu,fx_n,fx_n)\leq \lambda\{D^*(u,x_n,x_n)+D^*(u,x_{n+1},x_{n+1})$$

$$+D^*(x_n,x_{n+1},x_{n+1})\}$$

Therefore

$$D^*(fu,fx_n,fx_n)\leq \frac{\lambda}{(1-\lambda)}\{D^*(u,x_n,x_n)+D^*(u,x_{n+1},x_{n+1})$$

$$+D^*(x_n,x_{n+1},x_{n+1})\}$$

Which implies that $\lim_{n\to\infty} D^*(fu,fx_n,fx_n)=0$, and hence

$fu=\lim_{n\to\infty} fx_n=\lim_{n\to\infty} x_{n+1}=u$, showing u is a fixed point of f.

To prove that f has unique fixed point, suppose that $fx_0=x_0$ and $fy_0=y_0$ for some $x_0,y_0\in X$. Then by the definition of λ-generalized contraction, it follows that

$$D^*(x_0, y_0, y_0) = D^*(fx_0, fy_0, fy_0)$$

$$\leq qD^*(x_0, y_0, y_0) + rD^*(x_0, fx_0, fx_0) + sD^*(y_0, fy_0, fy_0)$$

$$+ t\{D^*(x_0, fy_0, fy_0) + qD^*(y_0, fx_0, fx_0)\}$$

$$= qD^*(x_0, y_0, y_0) + rD^*(x_0, x_0, x_0) + sD^*(y_0, y_0, y_0)$$

$$+ t\{D^*(x_0, y_0, y_0) + qD^*(y_0, x_0, x_0)\}$$

$$= (q + 2t)D^*(x_0, y_0, y_0)$$

$$\leq \lambda D^*(x_0, y_0, y_0)$$

This implies that $D^*(x_0, y_0, y_0) = 0$, since $\lambda < 1$, and hence $x_0 = y_0$. Thus f has unique fixed point.

Since x is arbitrary in the above discussion, it follows that $u = \lim_{n \to \infty} f^n x$ for any $x \in X$ and hence (2.2.2) is proved. Finally, since $D^*(x_n, x_{n+p}, x_{n+p}) = \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx)$ (by (B)), on letting $p \to \infty$, we get

$D^*(x_n, u, u) = \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx)$, proving (2.2.3). This completes the proof of theorem.

(3.4) **Corollary:** Suppose f is a selfmap of a D^*-metric space (X, D^*) and X is f-orbitally complete. If f is a contraction of (X, D^*), then it has a unique fixed point $u \in X$. In fact,

(3.5) $u = \lim_{n \to \infty} f^n x$ for any $x \in X$

and

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(3.6) $D^*(f^n x,u,u) \le \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx)$ for all $x \in X$ and $n \ge 1$.

Proof: In view of the fact that, every contraction is λ-generalized contraction, Corollary follows from Theorem 2.2.1

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