Cone metric spaces and fixed point theorems of contractive mappings

C Vijender *
*Dept of Science and Humanities
Sreenidhi Institute of Science and Technology
Hyderabad.

Abstract
In this paper we introduce cone metric spaces, prove some fixed point theorems of contractive mappings on cone metric spaces.
In this paper, we replace the real numbers by ordering Banach space and define cone metric spaces *(X, d)*. We discuss some properties of convergence of sequences. We prove some fixed point theorems for contractive mappings. Our results generalized some fixed point theorems in metric spaces.

Keywords: Cone metric space, Fixed point Contractive mapping, Ordered Banach space.

Introduction:

Cone metric space:

In this section we shall define cone metric spaces and prove some properties.

Let *E* always be a real Banach space and *P* a subset of *E*. *P* is called a cone if and only if:
 (i) *P* is closed, nonempty, and $P = \{0\}$;
 (ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to *P* by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in$ int *P*, int *P* denotes the interior of *P* .

The cone *P* is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying above is called the normal constant of *P* .

The cone *P* is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ *(n → ∞)*. Equivalently the cone *P* is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose *E* is a Banach space, *P* is a cone in *E* with   int $P = \emptyset$ and $\leq$ is partial ordering with respect to *P*.

Definition 1. Let *X* be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies
(d1) $0 < d (x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then *d* is called a cone metric on *X*, and *(X, d)* is called a cone metric space.

It is obvious that cone metric spaces generalize metric spaces.

Example 1. Let $E = R^2$, $P = \{(x, y) \in E \mid x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then *(X, d)* is a cone metric space.

Definition 2. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 << c$ there is N such that for all $n > N$, $d(x_n, x) << c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

Lemma 1. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0 \quad (n \to \infty)$.*

Proof. Suppose that $\{x_n\}$ converges to x. For every real $\varepsilon > 0$, choose $c \in E$ with $0 << c$ and $K\,K\|c\| < \varepsilon$. Then there is N, for all $n > N$, $d(x_n, x) << c$. So that when $n > N$,
$\|d(x_n, x)\| \le K\|c\| < \varepsilon$. This means $d(x_n, x) \to 0$ $(n \to \infty)$.

   Conversely, suppose that $d(x_n, x) \to 0$ $(n \to \infty)$. For $c \in E$ with $0 << c$, there is $\delta > 0$, such that $x < \delta$ implies $c - x \in \text{int } P$. For this δ there is N, such that for all $n > N$, $\|d(x_n, x)\| < \delta$. So $c - d(x_n, x) \in \text{int } P$. This means $d(x_n, x) << c$. Therefore $\{x_n\}$ converges to x.

Lemma 2. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.*

Proof. For any $c \in E$ with $0 << c$, there is N such that for all $n > N$, $d(x_n, x) << c$ and $d(x_n, y) << c$. We have

$$d(x, y) \le d(x_n, x) + d(x_n, y) \le 2c.$$

Hence $\|d(x, y)\| \le 2 K\|c\|$. Since c is arbitrary $d(x, y) = 0$; therefore $x = y$.

Definition 3. Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 << c$, there is N such that for all $n, m > N$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 4. Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Definition 5. Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that $\{x_{ni}\}$ is convergent in X. Then X is called a sequentially compact cone metric space.

Lemma 3. *Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Proof. For any $c \in E$ with $0 << c$, there is N such that for all $n, m > N$, $d(x_n, x) << c/2$ and $d(x_m, x) << c/2$. Hence $d(x_n, x_m) << d(x_n, x) + d(x_m, x) << c$. Therefore $\{x_n\}$ is a Cauchy sequence.

Lemma 4. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.*

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. For every $\varepsilon > 0$, choose $c \in E$ with $0 << c$ and $K\|c\| < \varepsilon$. Then there is N, for all $n, m > N$, $d(x_n, x_m) << c$. So that when $n, m > N$,
$\|d(x, y)\| \le K\|c\| < \varepsilon$. This means $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

   Conversely, suppose that $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. For $c \in E$ with $0 << c$, there is $\delta > 0$, such that $x < \delta$ implies $c - x \in \text{int } P$. For this δ there is N, such that for all $n, m > N$, $\|d(x_n, x_m)\| \le K\|c\| < \varepsilon$. So $c - d(x_n, x_m) \in \text{int } P$. This means $d(x_n, x_m) << c$. Therefore $\{x_n\}$ is a Cauchy sequence.

Lemma 5. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x$, $y_n \to y$ $(n \to \infty)$. Then*

$$d(x_n, y_n) \to d(x, y) \quad (n \to \infty).$$

Proof. For every $\varepsilon > 0$, choose $c \in E$ with $0 << c$ and $\|c\| < \dfrac{\varepsilon}{4K + 2}$ $y_n \to y$, there is N such that for all $n > N$, $d(x_n, x) << c$ and $d(y_n, y) << c$. We have

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y_n, y) \le d(x, y) + 2c,$$
$$d(x, y) \le d(x_n, x) + d(x_n, y_n) + d(y_n, y) \le d(x_n, y_n) + 2c.$$

Therefore $d(x_n, y_n) \to d(x, y)$ $(n \to \infty)$

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