www.ijecs.in
International Journal Of Engineering And Computer Science ISSN: 2319-7242
Volume 5 Issue 10 Oct. 2016, Page No. 18314-18320

Fixed Point Theorem in Weakly Compatible Self Mappings on Complete Metric Space

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Abstact: *In this article we proved a generalized common fixed point theorem in Weakly Compatible Self Mappings on Complete Metric Space*

Keywords*: Weakly compatible, point of coincidence, complete metric space*

I.Introduction

In 1972 a new geometrically concept introduced which is different from Banach [1] and Kannan [2] for contraction type mapping was by Chatterjee [3] which gives a new direction to the study of the fixed point theory. Chatarjee [3] gives contraction principle,

there exists a number α where $0 < \alpha < 1$ such that for each $x, y \in X$

$$d(Tx, Ty) \leq \alpha\, [\, d(x, Ty) + d(y, Tx)]$$

In 1978, Fisher B. [4] generalized the result of Kannan by choosing α which as follows:

$$d(Tx, Ty) \leq \alpha\, [d(x, Ty) + d(Tx, y)]$$

For all $x, y \in X$ and $0 \leq \alpha \leq \frac{1}{2}$ then T has unique fixed point in X.

Beside this in 1977 Jaggi [5] introduced the rational expression first time which is as follows:

$$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta\, d(x, y)$$

For all $x, y \in X$, $x \neq y$, $0 \leq \alpha + \beta \leq 1$ then T has unique fixed point in X.

Further in 1980 Jaggi and Das [6] obtained fixed point theorem with the mapping satisfying:

$$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} + \beta\, d(x, y) \quad 1.1.f$$

For all $x, y \in X$, $x \neq y$, $0 \leq \alpha + \beta \leq 1$ then T has unique fixed point in X.

Above this results is also valid for $x = y$.

The aim of this chapter is to obtain some fixed point theorem involving occasionally weakly compatible maps in the setting of symmetric space satisfying a rational contractive condition. Our results complement, extend and unify several well known comparable results.

II.Preliminaries

Definition 2.1 Let S and T are self maps of a metric space X. If $w = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T, and w is called a point of coincidence of S and T.

Definition 2.2 Let S and T are self maps of a metric space X, then S and T are said to be weakly compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $x \in X$.

Definition 2.3 Let S and T are self maps of a metric space X, then S and T are said to be weakly compatible if they commute at their coincidence points; i.e. if $Tx = Sx$ for some $x \in X$ then $TSx = STx$.

Definition 2.4 Let Φ be the set of real functions

$$\phi(t_1, t_2, t_3, t_4, t_5): [0,\infty)^5 \to [0,\infty)$$

satisfying the following conditions:

- **i.** ϕ is non increasing in variables t_4 and t_5.
- **ii.** There is an $h_1 > 0$ and $h_2 > 0$ such that $h = h_1 h_2 < 1$ and if $u \geq 0$ and $v \geq 0$ satisfying

  a. $u \leq \phi(v, v, u, u+v, 0)$ or $u \leq \phi(v, u, v, u+v, 0)$

  Then we have $u \leq h_1 v$.

  And if $u \geq 0, v \geq 0$ satisfy

  b. $u \leq \phi(v, v, u, 0, u+v)$ or $u \leq \phi(v, u, v, 0, u+v)$

  Then we have $u \leq h_2 v$.

If $u \geq 0$ is such that

$$u \leq \phi(u,0,0,u,u) \text{ or } u \leq \phi(0,u,0,0,u) \text{ or } u \leq \phi(0,0,u,u,0)$$

Then $u = 0$.

Before giving our second result of this section we Let R^{+} denote the set of non negative real numbers and F a family of all mappings $\phi : (R^{+})^{5} \rightarrow R^{+}$ such that ϕ is upper semi continuous, non decreasing in each coordinate variable and, for any $\phi(t) < kt$.

III. Main Result

Theorem 3.1 Let A, B, S, T be continuous self mappings defined on the complete metric space X into itself satisfies the following conditions:

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$

(ii) The pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

(iii) $$d(Ax, By) \leq \phi\begin{pmatrix} \frac{(d(Ax,Sx))^2 + (d(By,Ty))^2}{d(Ax,Sx)+d(By,Ty)}, \\ \frac{(d(Ax,Ty))^2 + (d(By,Sx))^2}{d(Ax,Ty)+d(By,Sx)}, \\ \frac{(d(Ax,Sx))^2 + (d(Ax,Ty))^2}{d(Ax,Sx)+d(Ax,Ty)}, \\ \frac{(d(By,Sx))^2 + (d(By,Ty))^2}{d(By,Sx)+d(By,Ty)}, \\ d(Sx, Ty) \end{pmatrix}$$

For all $x, y \in X$, $(x \neq y)$ and $\phi \in \Phi$. Then A, B, S, T have unique common fixed point in X.

Proof For any arbitrary x_0 in X we define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n} \text{ and } Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$$

for all n= 0, 1, 2, …..

On taking $y_{2n} \neq y_{2n+1}$

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$

From (iii) we have

$$d(Ax_{2n}, Bx_{2n+1}) \leq \phi\begin{pmatrix} \frac{(d(Ax_{2n},Sx_{2n}))^2+(d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Ax_{2n},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{(d(Ax_{2n},Tx_{2n+1}))^2+(d(Bx_{2n+1},Sx_{2n}))^2}{d(Ax_{2n},Tx_{2n+1})+d(Bx_{2n+1},Sx_{2n})}, \\ \frac{(d(Ax_{2n},Sx_{2n}))^2+(d(Ax_{2n},Tx_{2n+1}))^2}{d(Ax_{2n},Sx_{2n})+d(Ax_{2n},Tx_{2n+1})}, \\ \frac{(d(Bx_{2n+1},Sx_{2n}))^2+(d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Bx_{2n+1},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})}, \\ d(Sx_{2n}, Tx_{2n+1}) \end{pmatrix}$$

$$d(y_{2n}, y_{2n+1}) \leq \phi\begin{pmatrix} \frac{(d(y_{2n},y_{2n-1}))^2+(d(y_{2n+1},y_{2n}))^2}{d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n})}, \\ \frac{(d(y_{2n},y_{2n}))^2+(d(y_{2n+1},y_{2n-1}))^2}{d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})}, \\ \frac{(d(y_{2n},y_{2n-1}))^2+(d(y_{2n},y_{2n}))^2}{d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n})}, \\ \frac{(d(y_{2n+1},y_{2n-1}))^2+(d(y_{2n+1},y_{2n}))^2}{d(y_{2n+1},y_{2n-1})+d(y_{2n+1},y_{2n})}, \\ d(y_{2n-1}, y_{2n}) \end{pmatrix}$$

$$d(y_{2n}, y_{2n+1}) \leq \phi\begin{pmatrix} (d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})), \\ (d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})), \\ d(y_{2n}, y_{2n-1}) \\ , d(y_{2n}, y_{2n-1}) + 2d(y_{2n+1}, y_{2n}), \\ d(y_{2n-1}, y_{2n}) \end{pmatrix}$$

from the property of ϕ we have

$$d(y_{2n}, y_{2n+1}) \leq k\, d(y_{2n}, y_{2n-1})$$

similarly we can show that

$$d(y_{2n}, y_{2n-1}) \leq k^n\, d(y_{2n-2}, y_{2n-1})$$

processing the same way we can write,

for any integer m we have

$$d(y_{2n}, y_{2n+m}) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + \dots\dots\dots\dots\dots + d(y_{2n+m-1}, y_{2n+m})$$

$$d(y_{2n}, y_{2n+m}) \leq k^n . d(y_0, y_1) + k^{n+1} . d(y_0, y_1) + \dots\dots\dots\dots + k^{n+m} . d(y_0, y_1)$$

$$d(y_{2n}, y_{2n+m}) \leq k^n [1 + k + k^2 + \cdots \dots\dots\dots + k^m] . d(y_0, y_1)$$

$$d(y_{2n}, y_{2n+m}) \leq \frac{k^n}{1-k} . d(y_0, y_1)$$

as $n \to \infty$ gives that

$$d(y_{2n}, y_{2n+m}) \to 0 .$$

Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since T(X) is complete subspace of X then the subsequence $y_{2n} = Tx_{2n+1}$ is Cauchy sequence in T(X) which converges to the some point say u in X. Let $v \in T^{-1}u$ then $Tv = u$. Since $\{y_{2n}\}$ is converges to u and hence $\{y_{2n+1}\}$ also converges to same point u .

We set $x = x_{2n}$ and $y = v$ in 6.2.1(iv)

$$d(Ax_{2n}, Bv) \leq \phi\left(\begin{array}{c} \frac{(d(Ax_{2n},Sx_{2n}))^2 + (d(Bv,Tv))^2}{d(Ax_{2n},Sx_{2n})+d(Bv,Tv)}, \\ \frac{(d(Ax_{2n},Tv))^2 + (d(Bv,Sx_{2n}))^2}{d(Ax_{2n},Tv)+d(Bv,Sx_{2n})}, \\ \frac{(d(Ax_{2n},Sx_{2n}))^2 + (d(Ax_{2n},Tv))^2}{d(Ax_{2n},Sx_{2n})+d(Ax_{2n},Tv)}, \\ \frac{(d(Bv,Sx_{2n}))^2 + (d(Bx_{2n+1},Tv))^2}{d(Bv,Sx_{2n})+d(Bx_{2n+1},Tv)}, \\ d(Sx_{2n}, Tv) \end{array}\right)$$

as $n \to \infty$

$$d(u, Bv) \leq d(u, Bv)$$

which contradiction

implies that $Bv = u$ also $B(X) \subset S(X)$ so $Bv = u$ implies that $u \in S(X)$.

Let $w \in S^{-1}(X)$ then $w = u$ setting $x = w$ and $y = x_{2n+1}$ in 2.2.2(iii) we get

$$d(Ax_{2n}, Bv) \leq \phi\left(\begin{array}{c} \frac{(d(Aw,Sw))^2 + (d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Aw,Sw)+d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{(d(Aw,Tx_{2n+1}))^2 + (d(Bx_{2n+1},Sw))^2}{d(Aw,Tx_{2n+1})+d(Bx_{2n+1},Sw)}, \\ \frac{(d(Aw,Sw))^2 + (d(Aw,Tx_{2n+1}))^2}{d(Aw,Sw)+d(Aw,Tx_{2n+1})}, \\ \frac{(d(Bx_{2n+1},Sw))^2 + (d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Bx_{2n+1},Sw)+d(Bx_{2n+1},Tx_{2n+1})}, \\ d(Sw, Tx_{2n+1}) \end{array}\right)$$

as $n \to \infty$, $d(Aw, u) \leq d(Aw, u)$

which contradiction

implies that, $Aw = u$ this means $Aw = Sw = Bv = Tv = u$.

since $Bv = Tv = u$ so by weak compatibility of (B, T) it follows that, $BTv = TBv$ and so we get

$$Bu = BTv = TBv = Tu .$$

Since $Aw = Sw = u$ so by weak compatibility of (A, S) it follows that $SAw = ASw$ and So we get

$$Au = ASw = SAw = Su$$

Thus from (iii) we have

$$d(Ax_{2n}, Bv) \leq \phi\left(\begin{array}{c}\frac{(d(Aw,Sw))^2+(d(Bu,Tu))^2}{d(Aw,Sw)+d(Bu,Tu)}, \frac{(d(Aw,Tu))^2+(d(Bu,Sw))^2}{d(Aw,Tu)+d(Bu,Sw)}, \\ \frac{(d(Aw,Sw))^2+(d(Aw,Tu))^2}{d(Aw,Sw)+d(Aw,Tu)}, \frac{(d(Bu,Sw))^2+(d(Bu,Tu))^2}{d(Bu,Sw)+d(Bu,Tu)}, \\ d(Sw, Tu)\end{array}\right)$$

$$d(u, Bu) \leq d(u, Bu)$$

which contradiction

implies that $Bu = u$.

Similarly we can show $Au = u$ by using (iii) . Therefore

$$u = Au = Bu = Su = Tu.$$

Hence the point u is common fixed point of A, B, S, T.

If we assume that $S(X)$ is complete then the argument analogue to the previous completeness argument proves the theorem. If $A(X)$ is complete then $u \in A(X) \subset T(X)$. similarly if $B(X)$ is complete then $u \in B(X) \subset S(X)$. This complete prove of the theorem.

Uniqueness Let us assume that z is another fixed point of A, B, S, T in X different from u. i.e. $u \neq z$ then

$$d(u, z) = d(Au, Bz)$$

from (iii) we get

$$d(u, z) \leq d(u, z)$$

which contradiction the hypothesis . Hence u is unique common fixed point of A, B, S, T in X.

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