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Finite element solution of Poisson Equation over Polygonal Domains using a novel auto mesh generation technique and an explicit integration scheme for nine node linear convex quadrilateral of Lagrange family.

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Abstract :

This paper presents an explicit integration scheme to compute the stiffness matrix of a nine node linear convex quadrilateral element of Lagrange family using symbolic mathematics and discretisation of polygonal domain by such finite elements using a novel auto mesh generation technique , In finite element analysis, the boundary value problems governed by second order linear partial differential equations, the element stiffness matrices are expressed as integrals of the product of global derivatives over the linear convex quadrilateral region. These matrices can be shown to depend on the material properties and the matrix of integrals with integrands as rational functions with polynomial numerator and the linear denominator $(4+\xi+\eta)$ in the bivariates ξ and η over a nine node 2-square $(-1\leq \xi,\eta \leq 1)$ with nodes at the points {(-1,-1),(1,-1),(1,1),(-1,1),(0,-1),(1,0).(0,1),(-1,0),(0,0)} in local parametric space. In this paper, we have computed these integrals in exact forms using the symbolic mathematics capabilities of MATLAB. The proposed explicit finite element integration scheme can be applied to solve boundary value problems in continuum mechanics over convex polygonal domains. We have also developed a novel auto mesh generation technique of all nine node linear convex quadrilaterals for a polygonal domain $\Omega \subset \mathcal{R}^2$ which provides the nodal coordinates and element connectivity. We have used the explicit integration scheme and this novel auto mesh generation technique to solve the Poisson equation $-\nabla^2 u= f$,where u is unknown physical variable and f *is a known smooth function* in $\Omega \subset \mathcal{R}^2$ with given Dirichlet boundary conditions over the given convex polygonal domain.

Key words: Explicit Integration, Finite Element Method,Quadratic 2-D finite element of Lagrange family, Matlab Symbolic Mathematics, All Quadrilateral Mesh Generation Technique,Poisson Equation,Dirichlet Boundary Conditions ,Polygonal Domain, Gauss Legendre Quadrature Rules

1. **Introduction :**

In recent years, the finite element method (FEM) has emerged as a powerful tool for the approximate solution of differential equations governing diverse physical phenomena. Today, finite element analysis is an integral and major component in many fields of engineering design and manufacturing. Its use in industry and research is extensive, and indeed without it many practical problem in science, engineering and emerging technologies such as nanotechnology, biotechnology, aerospace, chemical. etc would be incapable of solution [1,2,3]. In FEM, various integrals are to be determined numerically in the evaluation of stiffness matrix, mass matrix, body force vector, etc. The algebraic integration needed to derive explicit finite element relations for second order continuum mechanics problems generally defies our analytic skill and in most cases, it appears to be a prohibitive task. Hence, from a practical point of view, numerical integration scheme is not only necessary but very important as well. Among various numerical integration schemes, Gauss Legendre quadrature, which can evaluate exactly the (2n-1)th degree polynomial with 'n' Gaussian integration points, is mostly used in view of the accuracy and efficiency of calculation. However, the integrands of global derivative products in stiffness matrix computations of practical applications are not always simple polynomials but rational expressions which the Gaussian quadrature cannot evaluate exactly [7-15]. The integration points have to be increased in order improve the integration accuracy but it is also desirable to make these evaluations by using as few Gaussian points as possible, from the point of view of the computational efficiency. Thus it is an important task to strike a proper balance between accuracy and economy in computation. Therefore analytical integration is essential to generate a smaller error as well as to save the computational costs of Gaussian quadrature commonly applied for science, engineering and technical problems. In explicit integration of stiffness matrix, complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric or equivalently the subparametric finite elements, the presence of determinant of the Jacobian matrix (we refer this as Jacobian here after) in the denominator of the element matrix integrands. This problem is considered in the recent work [16] for the four node linear convex quadrilateral which proposes a new discretisation method and use of pre computed universal numeric arrays which do not depend on element size and shape. In this method a linear polygon is discretized into a set of linear triangles and then each of these triangles is further discretised into three linear four node convex quadrilateral elements by joining the centroid to the mid-point of sides. These quadrilateral elements are then mapped into 2-squares ($-1\leq \xi,\eta \leq 1$) in the natural space (ξ,η) to obtain the same expression of the Jacobian, namely $c(4+\xi+\eta)$ where c is some appropriate constant which depends on the geometric data for the triangle.

Many important problems in engineering,science and applied mathematics are formulated by appropriate differential equations with some boundary conditions imposed on the desired unknown function or the set of functions. There exists a large literature which demonstrates numerical accuracy of the finite element method to deal with such issues [1]. Clough seems to be the first who introduced the fnite elements to standard computational procedures [2]. A further historical development and present day concepts of finite element analysis are widely described in references [1, 3]. In this paper the well-known Laplace and Poisson equations will be examined by means of the finite element method applied to an appropriate 'mesh'. The class of physical situations in which we meet these equations is really broad. Let's recall such problems like heat conduction, seepage through porous media, irrotational flow of ideal fluids, distribution of electrical or magnetic potential, torsion of prismatic shafts, lubrication of pad bearings and others [4]. Therefore, in physics and engineering arises a need of some computational methods that allow to solve accurately such a large variety of physical situations.The considered method completes the above-mentioned task. Particularly, it refers to a standard discrete pattern allowing to find an approximate solution to continuum problem. At the beginning, the continuum domain is discretized by dividing it into a finite number of elements which properties must be determined from an analysis of the physical problem (e. g. as a result of experiments). These studies on particular problem allow to construct so{called the stiffness matrix for each element that, for instance, in elasticity comprising material properties like stress strain relationships [2, 5]. Then the corresponding nodal loads associated with elements must be found. The construction of accurate elements constitutes the subject of a mesh generation recipe proposed by the author within the presented article. In many realistic situations, mesh generation is a time consuming and error prone process because of various levels of geometrical complexity. Over the years, there were developed both semi automatic and fully automatic mesh generators obtained, respectively, by using the mapping methods or, on the contrary, algorithms based on the Delaunay triangulation method [6], the advancing front method [7] and tree methods [8]. It is worth mentioning that the first attempt to create fully automatic mesh generator capable to produce valid finite element meshes over arbitrary domains has been made by Zienkiewicz and Phillips [9].

In the present paper, we propose a similar discretisation method for linear polygon in Cartesian two space (x,y). This discrtisation is carried in two steps, We first discretise the linear polygon into a set of linear triangles in the Cartesian space (x,y) and these linear triangles are then mapped into a standard triangle in a local space (u,v). We further discretise the standard triangles into three linear quadrilaterals by joining the centroid to the midpoints of triangles in (u,v) space which are finally mapped into 2-square in the local (ξ,η) space. We then establish a derivative product relation between the linear convex quadrilaterals in the Cartesian space, (x,y) which are interior to an arbitrary triangle and the linear quadrilaterals in the local space (u,v) interior to the standard triangle. In this procedure, all computations in the local space (u,v) for product of global derivative integrals are free from geometric properties and hence they are pure numbers. We then propose a numerical scheme to integrate the products of global derivatives. We have shown that the matrix product of global derivative integrals is expressible as matrix triple product comprising of geometric properties matrices and the product of local derivative integrals matrix. We have obtained explicit integration of the product of local derivatives which is now possible by use of symbolic integration commands available in leading mathematical softwares MATLAB, MAPLE, MATHEMATIKA etc. In this paper, we have used the MATLAB symbolic mathematics to compute the integrals of the products of local derivatives in (u, v) space .The proposed explicit integration scheme is shown as a useful technique in the formation of element stiffness matrices for second order boundary problems governed by partial differential equations.

This paper presents an explicit integration scheme to compute the stiffness matrix of a nine node linear convex quadrilateral element of Lagrange family using symbolic mathematics and discretisation of polygonal domain by such finite elements using a novel auto mesh generation technique , In finite element analysis, the boundary value problems governed by second order linear partial differential equations, the element stiffness matrices are expressed as integrals of the product of global derivatives over the linear convex quadrilateral region. These matrices can be shown to depend on the material properties and the matrix of integrals with integrands as rational functions with polynomial numerator and the linear denominator $(4+\xi+\eta)$ in the bivariates ξ and η over a nine node 2-square ($-1\leq \xi,\eta \leq 1$) with nodes at the points {(-1,-1),(1,-1),(1,1),(-1,1),(0,-1),(1,0).(0,1),(-1,0),(0,0)}. In this paper, we have computed these integrals in exact forms using the symbolic mathematics capabilities of MATLAB. The proposed explicit finite element integration scheme can be applied to solve boundary value problems in continuum mechanics over convex polygonal domains. We have also developed a novel auto mesh generation technique of all nine node linear convex quadrilaterals for a polygonal domain $\Omega \subset \mathcal{R}^2$ which provides the nodal coordinates and element connectivity. We have used the explicit integration scheme and this novel auto mesh generation technique to solve the Poisson equation $-\nabla^2 u= f$,where u is unknown physical variable and f *is a known smooth function* in $\Omega \subset \mathcal{R}^2$with given Dirichlet boundary conditions over convex polygonal domains.We need a small amount of numerical integration to complete the solution of the Poisson boundary value problem when f *is a known smooth function other than a constant.*

2. POISSON EQUATION

2.1 Statement of the Problem

The Poisson equation

$$-\nabla^2 u = f$$

....(1) is the simplest and most famous elliptic partial differential equations.The source (or load) function is given on some two or three dimensional domain $\Omega \subset \mathcal{R}^2$ or $\mathcal{R}^3$. A solution u satisfying (1.1) will also satisfy boundary conditions on the boundary $\partial\Omega$ of Ω ;for example

$$\alpha u + \beta \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega$$

....(2)

where $\partial u / \partial n$ denotes directional derivative in the direction normal to the boundary $\partial\Omega$ (conveniently pointing outwards) and α and β are constants, although variable coefficients are also possible.The combination of (1.1) and (1.2) together is referred to as boundary value problem. If the constant β in (1.2) is zero,then the boundary condition is known as the Dirichlet type, and the boundary value problem is referred as the Dirichlet problem for the Poisson equation. Alternatively, if the constant α in (1.2) is zero,then we correspondingly have a Neumann boundary value problem. A third possibility is that Dirichlet conditions hold on part of the boundary $\partial\Omega_D$ and Neumann conditions(or indeed mixed conditions where α and β are both nonzero) hold on remainder $\partial\Omega \backslash \partial\Omega_D$. The case $\alpha = 0, \beta = 1$ in (1.2) demands special attention.First, since u=constant satisfies the homogeneous problem with $f = 0, g = 0$,it is clear that a solution to a Neumann problem can only be unique up to an additive constant.Second,integrating (1.1) over Ω using Gauss's theorem gives

$$-\int_{\partial\Omega} \frac{\partial u}{\partial n} = --\int_{\Omega} \nabla^2 u = \int_{\Omega} f$$

....(3)

thus a necessary condition for the existence of a solution to the Neumann problem is that the source and boundary data satisfy the compatibility condition:

$$\int_{\partial\Omega} g + \int_{\Omega} f = 0$$

---(4)

2.2 Weak Formulation of the Poisson Boundary Value Problem

A sufficiently smooth function u satisfying both eqns(1) and (2) is known as classical solution to the Poisson boundary value problem. For a Dirichlet problem, u is a classical solution only if it has continuousecond derivatives in Ω (i.e. u is $C^2(\Omega)$ and is continuous up to the boundary i.e.u is in $C^0(\overline{\Omega})$). In case of nonsmooth domains or discontinuous source functions,the function u satisfying eqns(1) and (2) may not be smooth (or regular) enough to be regarded as classical solution. For problems which arise from , perfectly reasonable mathematical models an alternative description of the boundary value problem is required. Since this alternative description is less restrictive in terms of admissible data it is called weak formulation.

To derive a weak formulation of a Poisson problem,we require that for an appropriate set of test functions v,

$$\int_{\Omega} (\nabla^2 u + f)\, v = 0 \quad .$$

....(5)

This formulation exists provided that the integrals are well defined. If u is a classical solution then it must also satisfy eqn (5). If v is sufficiently smooth however,then the smoothness required of u can be reduced by using the derivative of a product rule and the divergence theorem

$$-\int_{\Omega} v\nabla^2 u = \int_{\Omega} \nabla u \,.\nabla v - \int_{\Omega} \nabla.(v\nabla u)$$

$$= \int_{\Omega} \nabla u \,.\nabla v - \int_{\partial\Omega} v \frac{\partial u}{\partial n},$$

so that

$$\int_{\Omega} \nabla u \,.\nabla v = \int_{\Omega} vf + \int_{\partial\Omega} v \frac{\partial u}{\partial n}$$

....(6a)

The point here is that the problem posed byeqn(6) may have a solution u called a weak solution,that is not smooth enough to be a classical solution. If a classical solution does exist then eqn(6)is equivalent to eqns (1) and (2) and the weak solution is classical. The case of Neumann problem ($\alpha = 0$ $, \beta = 1$) in eqn(2) is particularly straight forward. Substituting from eqn(2) into eqn(6)gives us the following formulation: find u defined on Ω such that

$$\int_{\Omega} \nabla u \,.\nabla v = \int_{\Omega} vf + \int_{\partial\Omega} vg$$

....(6b)

for all suitable test functions v .

2.3 Finite Elements for Poisson's Equation with Dirichlet conditions: Implementation and Review Of Theory

2.3.1 Weak Form

Given Poisson Equation:

$-\Delta u$(x)=f(x) for all x∈ Ω
............................(7a)

$u = g(x)$ on $\partial\Omega$
...............................(7b)

We have already obtained in eqn(6) with ($\alpha = 1$,$\beta = 0$) the weak form of the equation by multiplying both sides by a test function v (i.e a function which is infinitely differentiable and has compact support,integrating over the domain Ω and performing integration by parts or by application of Divergence(GREEN) theorem. The result is

$\int_\Omega \nabla u \,.\nabla v \, d\, x = \int_\Omega vf \, dx$
..............................(7c)

$u = g(x)$ on $\partial\Omega$
...............................(7d)

For all test functions v .

2.3.2 Finite Elements

To find an approximation to the solution u , we choose a finite dimensional space V_h and ask that eqn(7a-b) is satisfied only for $v \; in \; V_h$ rather than for all test functions v. Then we look for a function $u_h \in V_h$ which satisfies

$\int_\Omega \nabla u_h \,.\nabla v \, d\, x = \int_\Omega vf \, dx$ for all $v \in V_h$ ....(8)

u_h is called the finite element solution and functions in V_h are called finite elements.

Note that it is also common for the triangles or quadrilaterals in the mesh to be called elements.

If a basis for V_h is $\{\varphi_j\}_{j=1}^{j=N}$ then we can write $u_h = \sum_{\mathrm{j=1}}^{\mathrm{j=N}} \alpha_\mathrm{j}\varphi_\mathrm{j}$. Substituting this in eqn(8) and choosing v to be a basis function φ_i gives the following set of equations

$\sum_{j=1}^{N} \alpha_j \int_\Omega \nabla\varphi_{\,i} \,.\nabla\varphi_j \, d\, x = \int_\Omega f\varphi_i \, dx$,i=1,2,3,...., N ....(9)

This is really a linear system of the form

$Ku{=}f$ ....(10)

Where, $u = (\alpha_1 \,, \alpha_2 \,, \alpha_3 \,, \ldots \ldots \ldots \alpha_N)^T$ and

$K_{i,j} = \int_\Omega \nabla\varphi_{\,i} \,.\nabla\varphi_j \, d\, x$ ....(11a)

$f_i = \int_\Omega f\varphi_i \, dx$ ....(11b)

and K is called stiffness matrix because the linear system looks like Hookes law if f represents forces and u represents displacemewnts.

In general,$\Omega = \sum_{e=1}^{N_e} \Omega^e$, where N_e is the number of elements discritised in the domain Ω. In two dimensions the mesh elements are triangles or quadrilaterals.The choice of finite element spaces are usually piecewise polynomials.

2.3.3 Overview on the implementation of Finite Element Method

Once we have choosen the finite element space (and the element type),then we can implement the finite element method.The implementation is divided into three steps:

1. Mesh Generation:how does one perform a triangulation or quadrangulation of the domain Ω ?
2. Assembling the Stiffness Matrix:how does one compute the entries in the stiffness matrix in an efficient way?
3. Solving the linear System:What kind of methodse suited for solving the linear system?

In this paper,we present new approach to mesh generation [ ] and explicit computations for the entries in the stiffness matrix [ ] which is vital in Assembling the Stiffness Matrix,since we believe that the methods of solving linear system are well researched and standardised.

We shall first take up the derivations regarding the topic on Assembling the Stiffness Matrix. The Mesh Generation topic will be discussed immediately there after.

2.3.4 Assembling the Stiffness Matrix

In order to assemble the stiffness matrix,we need to compute integrals of the form(see eqn(11) in section (2.3.2)

$K_{i,j} = \int_\Omega \nabla\varphi_{\,i} \,.\nabla\varphi_j \, d\, x$
....(11a)

The most obvious way to assemble the stiffness matrix is to compute the integrals $K_{i,j}$ for the nodal pairs i and j ; this is a node oriented computation and we need to know the common support of basis functions $\varphi_{\,i} \; and \; \varphi_j$.This means we need to know which elements contain both i and j.The mesh generator provides us with the information regarding the nodes on a particular element so we would need to do some extra processing to find the elements that contain a particular node.This is an issue which is very complicated.Hence,in practice assembling is focussed on elements rather than on nodes.We note that on a particular element,the basis functions have a simple expression and the elements themselves are very simple domains like triangles and quadrilaterals. It is very easy to make a change of variables for integrals over triangles and quadrilaterals to standard triangles and squares. In the element oriented computation,we rewrite or interpret the integral in eqn(11) as

$K_{i,j} = \sum_{\Omega^e \;\varepsilon \Omega_h^e} \quad K_{i,j}^e \quad =$
....(12a)

where

$K_{i,j}^{e}$ $=\int_{\Omega^e} \nabla\varphi_{\,i}\,.\nabla\varphi_j\,d\,x$ ....(12b)

and Ω_h^e is the set of (mesh) elements in Ω contributing to $K_{i,j}$ and $\Omega=\sum_{e=1}^{N_e}\Omega^e$, $\Omega^e$is an element contained in the set Ω_h^e .This says us that we can compute $K_{i,j}$ by computing the integrals over each element Ω^e and then summing up over all elements Ω_h^e .

Notice that the integrals

$K_{i,j}^e=\int_{\Omega^e}\nabla\varphi_{\,i}\,.\nabla\varphi_j\,d\,$x look like the entries $K_{i,j}=\int_{\Omega}\nabla\varphi_{\,i}\,.\nabla\varphi_j\,d\,x=\sum_{\Omega^e\ \varepsilon\Omega_h^e}\ \int_{\Omega^e}\nabla\varphi_{\,i}\,.\nabla\varphi_j\,d\,x$ except the domain of integration is an element Ω^e. So,we only need to save all entries of $K^e=[K_{i,j}^e\ ]$ which corresponds to nodes on Ω^e. Then if Ω^e has d nodes , we can think of K^e as a dxd matrix. In view of the above,the procedure for computing the stiffness maerix is done on an element by element basis.

We must also compute the integrals

$f_i=\int_{\Omega} f\varphi_i\,dx=\sum_{e=1}^{N_e}f_i^e$ ....(12c)

where

f_i^e $=$ $\int_{\Omega^e} f\varphi_i d\,$x ....(12d)

Now further assume that on an element Ω^{e} , $u_h=\mathrm{u^e}=\sum_{\mathrm{j=1}}^{\mathrm{j=d}}\mathrm{u_j^e}\varphi_{\mathrm{j}}$

From eqn(9) and eqns(12a-d) it follows that $Ku=f$ is equivalent to

$\sum_{e=1}^{N_e}K^e\,\mathrm{u^e}=\sum_{e=1}^{N_e}f^{\,e}$ ....(12e)

Where

$\mathrm{u^e}\ =(\mathrm{u_1^e}\ ,\mathrm{u_2^e},\ \mathrm{u_3^e},$ .... $\mathrm{u_d^e}\ )^T$, $f^{\,\mathrm{e}}\ =(f_1^{\mathrm{e}}\,,f_2^{\mathrm{e}},\ f_3^{\mathrm{e}},$ .... $f_{\mathrm{d}}^{\mathrm{e}}\ )^T$ ....(12f)

d referes to number of nodes per element, N_e referes to the total number of elements in the domain Ω

2.3.5 Computing the Integrals $K_{i,j}^e$ and f_i^e

In order to compute the local/element stiffness matrices,we need to compute the integrals $K_{i,j}^e=\int_{\Omega^e}\nabla\varphi_{\,i}\,.\nabla\varphi_j\,d\,$x .These integrals are computed by making a change of variables to a reference element. We now outline a brief procedure for element oriented computation

(1)For each element Ω^e , compute it's local stiffness matrix K^e. This requires computing the integrals $K_{i,j}^e=\int_{\Omega^e}\nabla\varphi_{\,i}\,.\nabla\varphi_j\,d\,$x which we compute by transforming to a reference element. In two dimensions Ω^e is an arbitrary linear triangle and each triangle will be further discritised three convex quadrilaterals Q_{3e-2} , Q_{3e-1} and Q_{3e} Each triangle will be transformed to the corresponding reference elements:the standard triangle(a right isosceles triangle) and further Each triangle will be transformed to the corresponding reference elements:the standard triangle(a right isosceles triangle) and further Each triangle will be transformed to the corresponding reference elements:the standard triangle(a right isosceles triangle) and further each quadrilateral will be transformed into a standard square(1-square or a 2-square).Since in two dimensional space x= (x,y) the explicit form of $K_{i,j}^e=\int_{\Omega^e}\nabla\varphi_{\,i}\,.\nabla\varphi_j\,d\,$x is given by

$K_{i,j}^e=\int_{\Omega^e}\nabla\varphi_{\,i}\,.\nabla\varphi_j\,$dx $=\int_{\Omega^e}\{\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}+\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\}\,dxdy=\sum_{e=1}^{N_e}\sum_{n=0}^{2}\int_{Q_E}\{\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}+\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\}dxdy$ $=\sum_{e=1}^{N_e}\sum_{n=0}^{2}S_{i,j}^E$ ....(12g)

Where $S_{i,j}^E=\int_{Q_E}\{\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}+\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\}dxdy$ and E=3e+n-2,e=1,2,... N_e andn=0,1,2

and hence we must be careful about the derivatives when we perform the change of variables. These bring extra factors involving the affine transformations (when Ω^e is an arbitrary linear triangle) and bilinear transformations(when Ω^e is an arbitrary linear convex quadrilateral)

f_i^e = $\int_{\Omega^e} f\varphi_i$ dxdy can be computed in a straight forward manner if f is a simple function otherwise we have to apply numerical integration

(2) For each element Ω^e, first compute the local stiffness matrices $S^E=[S_{i,j}^E\ ]$ and then add contribution of $K^e=S^{3e-2}$ $+S^{3e-1}+S^{3e}$, to the global stiffness matrix K. We repeat this procedure for all elements i.e for e=1,2,......, N_e; where N_e is the number of elements Ω^e $which$ are discritised in the domain Ω ,in fact we have $\Omega=\sum_{e=1}^{N_e}\Omega^e$ $=\sum_{e=1}^{N_e}\sum_{n=0}^{2}Q_E$, E=3e+n-2

2.4 Finite Element Types

2.4.1 Linear Convex Quadrilateral Elements :

Let us first consider an arbitrary four noded linear convex quadrilateral in the global (Cartesian) coordinate system (u, v) as in Fig 1a, which mapped into a 2-square in the local(natural) parametric coordinate (ξ, η) as in Fig 1b.

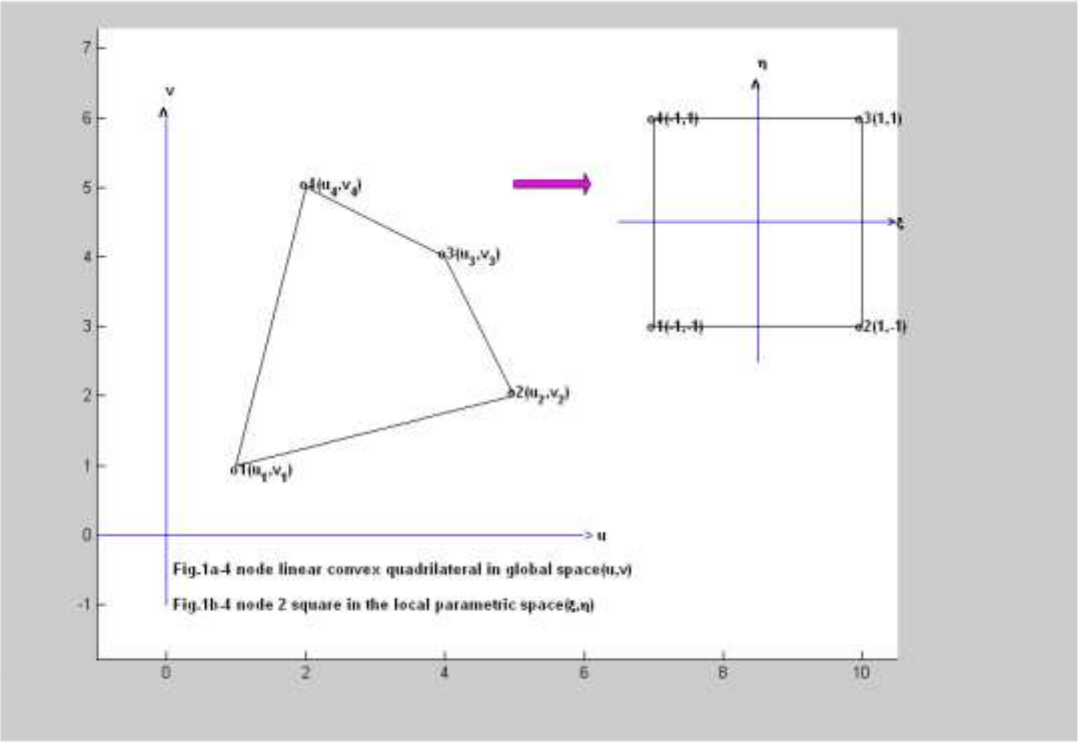


Fig.1a-4 node linear convex quadrilateral in global space(u,v)

Fig.1b-4 node 2 square in the local parametric space(ξ,η)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k=1}^{4} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \ M_k(\xi, \eta) \qquad \text{------------------------ (13)}$$

Where (u_k, v_k) , (k=1,2,3,4) are the vertices of the original arbitrary linear convex quadrilateral in (u, v) plane and $M_k(\xi, \eta)$ denote the well known bilinear basis functions [1-3] in the local parametric space (ξ, η) and they are given by

$$M_k(\xi, \eta) = \frac{1}{4}\ (1 + \xi\xi_k)(1 + \eta\eta_k)\ , \quad k = 1, 2,3,4 \qquad \text{----------------------- (14a)}$$

Where $\{\ (\xi_k, \eta_k), k = 1,2,3,4\}$ = {(-1,-1),(1,-1),(1,1),(-1,1)} ----------------------- (14b)

describes a geometric transformation over a linear convex quadrilateral element from the original global space into the local parametric space.

2.4.2 Isoparametric Transformation :

For the isoparametric coordinate transformation over the linear convex quadrilateral element as shown in Fig 1, we select the field variables, say ϕ, ψ , etc governing the physical problem as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=1}^{4} \begin{pmatrix} \phi_k \\ \psi_k \end{pmatrix} {N_k}^e(\xi, \eta) \qquad \text{------------------------ (15)}$$

Where ϕ_k , ψ_k refer to unknowns at node k and the shape functions ${N_k}^e = M_k$, and M_k are defined as in Eqn.(14a-b)

We have considered the application of explicit stiffness matrix integration scheme and automesh generation technique to find FEM solution of Poisson equation boundary value problems over polygonal domains using linear convex quadrilateral elements under isoparametric transformations[ ].

2.4.3 Subparmetric Transformation :

For the subparametric transformation over the nde – noded element we define the field variables ϕ, ψ (say) governing the physical problem as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=1}^{nde} \begin{pmatrix} {\phi_k}^e \\ {\psi_k}^e \end{pmatrix} {N_k}^e(\xi, \eta) \qquad \text{------------------------ (16)}$$

Where ϕ_k , ψ_k refer to unknowns at node k and nde >4

In our recent paper, the explicit finite element integration scheme is presented by using the isoparametric transformation over the 4 node linear convex quadrilateral element for which we set nde=4

In the present paper, we consider the subparametric transformation for a linear convex quadrilateral element for which nde = 9 , a nine noded 2 square of Lagrange type

3. Nine Node Linear Convex Quadrilateral Element :

In this section, we give a brief description of the 9- node quadrilateral element under subparametric transformation as shown in Fig 1c , Fig 1d.

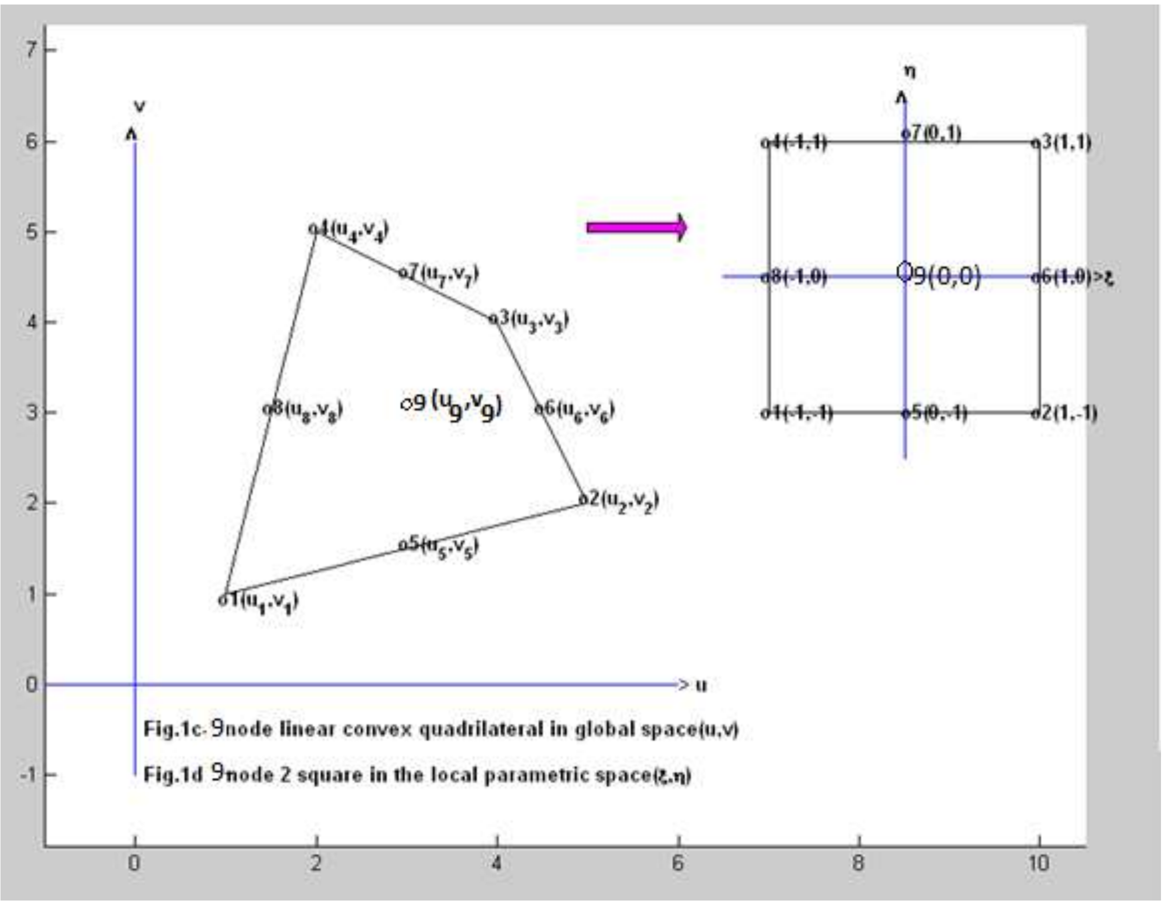


Fig.1c- 9node linear convex quadrilateral in global space(u,v)

Fig.1d 9node 2 square in the local parametric space(ξ,η)

We use the transform of Eqns.(13-14) to define the element geometry i.e.

$\begin{pmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k=1}^{4} \begin{pmatrix} u_k \\ v_k \end{pmatrix} M_k(\xi,\eta)$ ------------------------- (13)

Where $M_k(\xi,\eta) = \frac{1}{4}\,(1+\xi\xi_k)(1+\eta\eta_k)$, $(k = 1,2,3,4\,)$ ----------------------- (14a)

With $(u(\xi_k\,,\eta_k)\,, v(\,\xi_k\,,\eta_k))$, $k = 1,2,3,4$) are the vertices of the linear convex quadrilateral in global (u, v) space .

$\{\, (\xi_k,\eta_k), k = 1,2,3,4\} = \{(-1,-1),(1,-1),(1,1),(-1,1)\}$ ----------------------- (14b)

Using the transformation of Eqns.(13-14) and from Fig 1c , Fig 1d we see that there is a one to one correspondence between $((\xi_k,\eta_k),\ k = 5,6,7,8,9) = ((0,1),(1,0),(0,1),(-1,0),(0,0))$

and $((u_k,v_k) = \big(u(\xi_k\,,\eta_k), v(\,\xi_k\,,\eta_k)\big)$, $k = 5,6,7,8,9)$, where

$(u_5,\ v_5) = (\,(u_1+u_2)/2\ \ ,(v_1+v_2)/2\ \,)$

$(u_6,\ v_6) = (\,(u_2+u_3)/2\ \ ,(v_2+v_3)/2\ \,)$

$(u_7, v_7) = ((u_3 + u_4)/2 \ , (v_3 + v_4)/2 \ )$

$(u_8, v_8) = ((u_1 + u_4)/2 \ , (v_1 + v_4)/2 \ )$

$(u_9, v_9) = ((u_1 + u_2 + u_3 + u_4)/4 \ , (v_1 + v_2 + v_3 + v_4)/4 \ )$ -------------------------(14c)

We then define the variation of physical variables ϕ^e, ψ^e (say) over 9- node element of Fig 1c , 1d by Eqn.(16) with nde = 9

$$\begin{pmatrix} \phi^e \\ \psi^e \end{pmatrix} = \sum_{k=1}^{8} N_k{}^e(\xi,\eta) \begin{pmatrix} \phi_k{}^e \\ \psi_k{}^e \end{pmatrix} \qquad \text{------------------------ (16)}$$

Where $\phi_k{}^e$, $\psi_k{}^e$ are the nodal values at node k

The shape functions $N_i{}^e$ of the 9- node element shown in Fig 1c , Fig 1d are given by

$N_1{}^e(\xi,\eta) = (\xi^2 - \xi)(\eta^2 - \eta)/4$
$N_2{}^e(\xi,\eta) = (\xi^2 + \xi)(\eta^2 - \eta)/4$
$N_3{}^e(\xi,\eta) = (\xi^2 + \xi)(\eta^2 + \eta)/4$
$N_4{}^e(\xi,\eta) = (\xi^2 - \xi)(\eta^2 + \eta)/4$
$N_5{}^e(\xi,\eta) = (1 - \xi^2)(\eta^2 - \eta)/2$
$N_6{}^e(\xi,\eta) = (\xi^2 + \xi)(1 - \eta^2)/2$
$N_7{}^e(\xi,\eta) = (1 - \xi^2)(\eta^2 + \eta)/2$
$N_8{}^e(\xi,\eta) = (\xi^2 - \xi)(1 - \eta^2)/2$
$N_9{}^e(\xi,\eta) = (1 - \xi^2)(1 - \eta^2)$
-------------------------(17a)

and we may check that

$N_k{}^e(\xi_k,\eta_k)=1$, $N_k{}^e(\xi_j,\eta_j)=0$,when $j \neq k$

$\{ (\xi_k,\eta_k), k = 1(1)9\} = \{(-1,-1), (1,-1), (1,1), (-1,1), (0,-1), (1,0), (0,1), (-1,0), (0,0)\}$ ----------------------(17b)

4. Explicit Form of the Jacobian and Global Derivatives :

4.1 Jacobian

Let us consider an arbitrary linear convex quadrilateral in the global Cartesian space (u, v) as in Fig 1a , c which is mapped into a 8- node 2- square in the local parametric space (ξ,η) as in Fig 1b, d

From the Eq.(1) and Eq.(2), we have

$$\frac{\partial u}{\partial \xi} = \sum_{k=1}^{4} u_k \frac{\partial M_k}{\partial \xi} = \frac{1}{4}[(-u_1 + u_2 + u_3 - u_4) + (u_1 - u_2 + u_3 - u_4)\,\eta] \qquad \text{------------ (18a)}$$

$$\frac{\partial u}{\partial \eta} = \sum_{k=1}^{4} u_k \frac{\partial M_k}{\partial \eta} = \frac{1}{4}[(-u_1 - u_2 + u_3 + u_4) + (u_1 - u_2 + u_3 - u_4)\,\xi] \qquad \text{------------ (18b)}$$

$$\frac{\partial v}{\partial \xi} = \frac{1}{4}[(-v_1 + v_2 + v_3 - v_4) + (v_1 - v_2 + v_3 - v_4)\,\eta] \qquad \text{------------ (18c)}$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}[(-v_1 - v_2 + v_3 + v_4) + (v_1 - v_2 + v_3 - v_4)\,\xi] \qquad \text{------------ (18d)}$$

Hence the Jacobian, J can be expressed as [1, 2, 3]

$$J = \frac{\partial(u,v)}{\partial(\xi,\eta)} = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \alpha + \beta\xi + \gamma\,\eta \qquad \text{-------------- (19a)}$$

Where

$$\alpha = \frac{1}{8}[(u_4 - u_2)(v_1 - v_3) + (u_3 - u_1)(v_4 - v_2)]$$

$$\beta = \frac{1}{8}[(u_4 - u_3)(v_2 - v_1) + (u_1 - u_2)(v_4 - v_3)]$$

$\gamma = \frac{1}{8}[\,(u_4 - u_1)(v_2 - v_3) + (u_3 - u_2)(v_4 - v_1)\,]$ ---------------- (19b)

4.2 Global Derivatives:

If N_i^e denotes the basis functions of node i of any order of the element e, then the chain rule of differentiation from Eq.(1) we can write the global derivative as in [1, 2, 3]

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix} = \frac{1}{J}\begin{bmatrix} \frac{\partial v}{\partial \eta} & -\frac{\partial v}{\partial \xi} \\ -\frac{\partial u}{\partial \eta} & \frac{\partial u}{\partial \xi} \end{bmatrix}\begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} \qquad \text{---------------------------- (20)}$$

Where $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \xi}$ and $\frac{\partial v}{\partial \eta}$ are defined as in Eqs.(18a)–(18d) while J is defined in Eq.(19a-b) , ($i, j = 1,2,3, -----,$ nde) , nde = the number of nodes per element. We may recall that the explicit integration for linear convex quadrilateral with nde = 4 is already presented by the authors in their recent paper [18]. We take nde = 8 for the present study.

5. Discretisation of an Arbitrary Triangle :

A linear convex polygon in the physical plane (x, y) can be always discretised into a finite number of linear triangles. However, we would like to study a particular discretization of these triangles further into linear convex quadrilaterals. This is stated in the following Lemma [6].

Lemma 1. Let Δ PQR be an arbitrary triangle with the vertices $P(x_p, y_p)$, $Q(x_q, y_q)$ and $R(x_r, y_r)$ and S, T, U be the midpoints of sides PQ, QR and RP respectively, let a, b, c, d, e, f, g, h, I be the midpoints of sides ZS, ZT, ZU, PU, PS, QS, QT, RT, RU and let Z be the centroid of the triangle Δ PQR. We can obtain three linear convex 9- node quadrilaterals (Q_e, e=1,2,3),where Q_1 =ZcUdPeS , Q_2 =ZaSfQgT and Q_3= ZbThRiU from triangle Δ PQR as shown in Fig 2a,b.Further each of these Q1,Q2 and Q3 have centrodal points j,k and l. If we map each of these 9- node linear convex quadrilaterals into 9- node 2- squares in which the nodes are oriented in counter clockwise from Z, then the Jacobian J^e for each element Q_e, (e=1,2,3) is given by

$J = J^e = \frac{1}{48}\Delta\, pqr\, (4 + \xi + \eta), \qquad e = 1,2,3$ --------------- (21)

Where Δ pqr is the area of the triangle Δ PQR

$$2\Delta\, pqr = \begin{vmatrix} 1 & x_p & y_p \\ 1 & x_q & y_q \\ 1 & x_r & y_r \end{vmatrix} = [\,(x_p - x_r)(y_q - y_r) - (x_q - x_r)(y_p - y_r)\,] \qquad \text{------------ (22)}$$

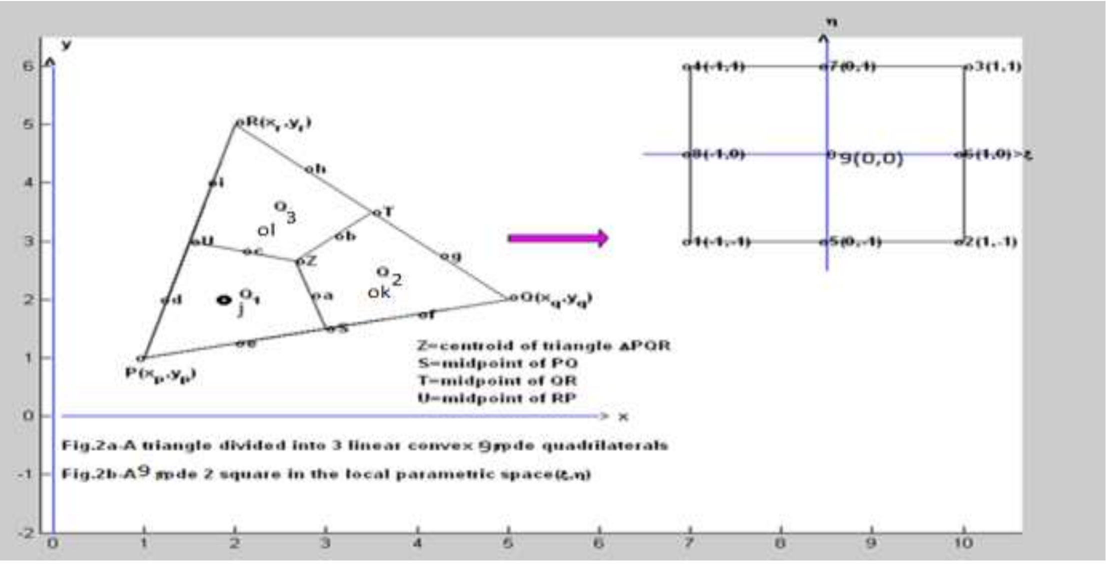


Fig.2a-A triangle divided into 3 linear convex 9 node quadrilaterals

Fig.2b-A 9 node 2 square in the local parametric space(ξ,η)

Proof : Proof is straight forward and it can be elaborated on the lines of proof given in [17].

Lemma 2. Let Δ PQR be the arbitrary linear triangle with the vertices P(x_p, y_p) , Q(x_q, y_q) and R(x_r, y_r) and S, T, U be the midpoints of sides PQ, QR, and RP respectively Further, let a, b, c, d, e, f, g, h, i be the midpoints of sides ZS, ZT, ZU, PU, PS, QS, QT, RT, RU and Z be the centroid of the Δ PQR . We further let j,k,l be the centre point of the quadrilaterals Q_e (e=1,2,3), Then we obtain three linear convex 9- node quadrilaterals Q_e (e=1,2,3),Q_1 =<ZcUdPeSaj> , Q_2 =<ZaSfQgTbk > and Q_3= <ZbThRiUcl> , these quadrilaterals can mapped into the linear conv ex 9- node quadrilateral spanning the vertices GHEICJF with G(1/3, 1/3), H(1/6, 5/12), E(0, ½), I(0, ¼), C(0, 0), J(1/4, 0), F(1/2, 0), K(5/12, 1/6) in the interior of the right isosceles triangle Δ ABC with vertices A(1, 0), B(0, 1) and C(0, 0) in the (u, v) space as shown in Fig3a and Fig3b.

Proof : The sum of the three quadrilaterals Q_1, Q_2, Q_3 is Q_1+ Q_2+ Q_3 = Δ PQR as shown in Fig 2a & Fig 3a.

We know that the linear transformations

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} w + \begin{pmatrix} x_q \\ y_q \end{pmatrix} u + \begin{pmatrix} x_r \\ y_r \end{pmatrix} v \qquad \text{-------------------------} \ (23)$$

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \end{pmatrix} w + \begin{pmatrix} x_r \\ y_r \end{pmatrix} u + \begin{pmatrix} x_p \\ y_p \end{pmatrix} v \qquad \text{--------------------------} \ (24)$$

$$\begin{pmatrix} x^{(3)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \end{pmatrix} w + \begin{pmatrix} x_p \\ y_p \end{pmatrix} u + \begin{pmatrix} x_q \\ y_q \end{pmatrix} v \qquad \text{------------------------} \ (25)$$

with $w = 1 - u - v$

map the arbitrary triangle Δ PQR into a linear right isosceles triangle A(1, 0) , B(0, 1) and C(0, 0) in the uv–plane. We can now verify that the vertices Z, c, U, d, P, e, S in xy palne is mapped into the linear convex 9- node quadrilateral spanning the vertices G, H, E, I, C, J, F, K,L by use of the transformation given in Eqn.(23).

Similarly, we see that the linear convex 8- node quadrilateral Q_2 spanned by vertices Z, a, S, f, Q, g, T, b is mapped into the linear convex 8- node quadrilateral spanned by the vertices G, H, E, I, C, J, F, K ,Lby use of the transformation of Eqn.(24). Finally the quadrilateral Q_3 in xy plane is mapped into the quadrilateral GHEICJFKL in uv- plane by use of the linear transformation of Eqn.(25),
This completes the proof.

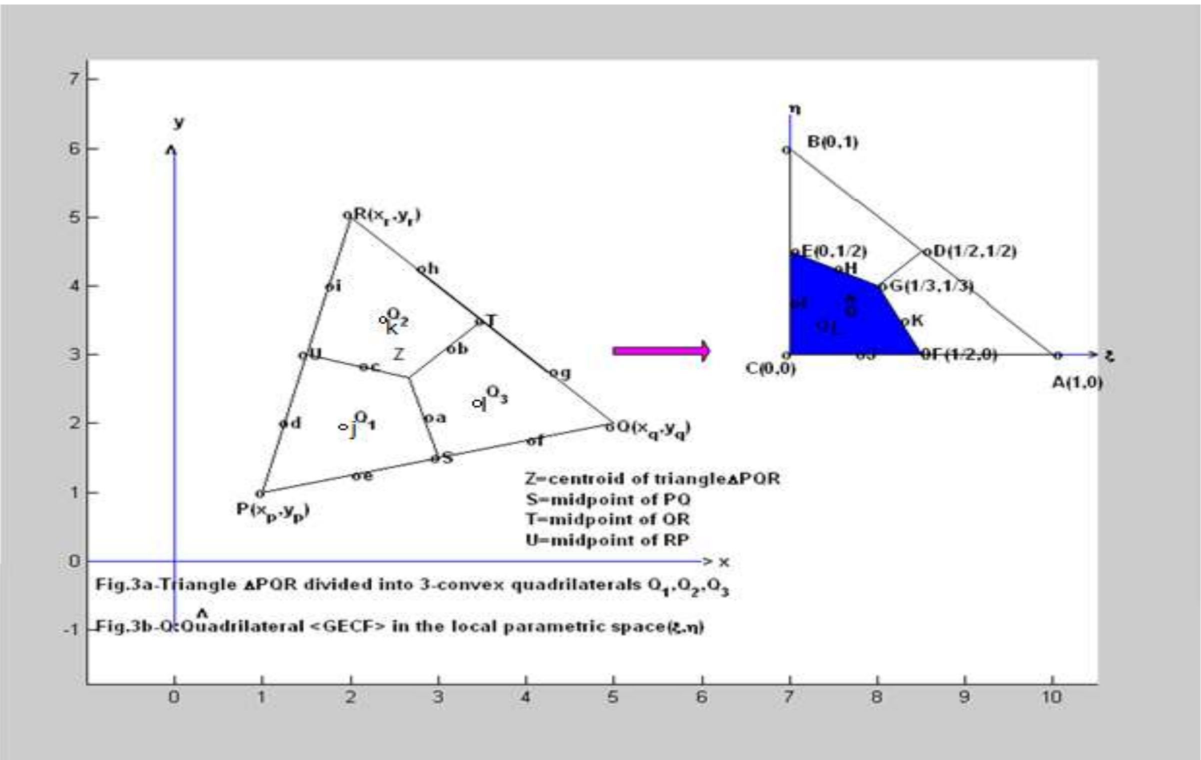


Fig.3a-Triangle ΔPQR divided into 3-convex quadrilaterals Q_1,Q_2,Q_3

Fig.3b-$\hat{Q}$:Quadrilateral <GECF> in the local parametric space(ξ,η)

We have shown in the foregoing Lemma that an arbitrary linear triangle can be discretised into three linear convex 9- node quadrilaterals. Further, each of these quadrilaterals in xy plane can be mapped into a unique linear convex 9- node quadrilateral spanned by the vertices G(1/3, 1/3), H(1/6, 5/12), E(0, ½), I(0, ¼), C(0, 0), J(1/4, 0), F(1/2, 0) , K(5/12, 1/6) and L(5/24,5/24) (see Fig 3a, Fig 3b) using a proper linear transformation as given Eqn.(23) – (25).

6. Integration over a Triangular Region :

6.1 Composite Integration

We shall now establish a composite integration formula for an arbitrary triangular region Δ PQR shown in Fig 2a or Fig 3a. Let $\phi(x, y)$ be an arbitrary and smooth function defined over the region Δ PQR . We now consider

$II_{\Delta PQR} = \iint_{\Delta PQR} \phi(x,y)\, dxdy = \sum_{e=1}^{3} \iint_{Q_e} \phi(x,y)\, dxdy$ --------------------- (26)

$= \iint_{\hat{Q}} \sum_{e=1}^{3} [\phi\,(x^{(e)}(u,v),\ y^{(e)}(u,v)) \frac{\partial\left(x^{(e)}(u,v),\ y^{(e)}(u,v)\right)}{\partial(u,v)}]\ dudv$

$= (2\,\Delta_{pqr})\ \iint_{\hat{Q}} \{ \sum_{e=1}^{3} [\phi\,(x^{(e)}(u,v),\ \ y^{(e)}(u,v))\,]\}\ dudv$ -------------------- (27)

Where $(x^{(e)}(u,v),\ \ y^{(e)}(u,v)), e = 1,2,3)$ are the linear transformations of Eqs.(23)–(25) and $\hat{Q}$ is the linear convex 9- node quadrilateral GHEICJFK spanning the vertices G(1/3, 1/3), H(1/6, 5/12), E(0, ½), I(0, ¼), C(0, 0), J(1/4, 0), F(1/2, 0) , K(5/12, 1/6),L(5/24,5/24) and Δ_{pqr} is the area of triangle Δ PQR, Now, we further use the bilinear transformation of Eqns.(1)–(2) in Eqn.(15) and obtain.

$II_{\Delta PQR} = (2\,\Delta_{pqr})\ \int_{-1}^{1}\int_{-1}^{1}\{ \sum_{e=1}^{3} [\phi\,(x^{(e)}(u,v),\ \ y^{(e)}(u,v)) \frac{\partial(u,v)}{\partial(\xi,\eta)}\}\, d\xi\, d\eta$ -------------- (28)

In Eq.(16) we have used the bilinear transformation given in Eqns.(13)- (14)

$u = u(\xi,\ \eta) = \frac{1}{3} M_1(\xi,\ \eta) + \frac{1}{2} M_4(\xi,\ \eta)$

$v = v(\xi,\ \eta) = \frac{1}{3} M_1(\xi,\ \eta) + \frac{1}{2} M_2(\xi,\ \eta)$ ---------------------- (29)

to map the arbitrary linear convex 9- noded quadrilateral into a 2 – square in (ξ, η) – plane. Thus on using Eqn.(29), the integral of Eqn.(28) simplifies to the following.

$II_{\Delta PQR} = (2\,\Delta_{pqr}) \int_{-1}^{1}\int_{-1}^{1}\ [\sum_{e=1}^{3} \left(\frac{4+\xi+\eta}{96}\right) \phi(\,x^{(e)}(u,v),\ \ y^{(e)}(u,v))\ ]d\xi\, d\eta$ ------------- (30)

We can evaluate Eqn.(30) either analytically or numerically depending on the form of the integrand.

Using Numerical Integration , we have from Eqn.(30)

$$II_{\Delta PQR} = 2\Delta_{pqr} \sum_{i=1}^{N}\sum_{j=1}^{N} \left(\frac{W_i^{(N)} W_j^{(N)} (4+\xi_i^{(N)}+\eta_j^{(N)})}{96}\right) \sum_{e=1}^{3} \phi(x^{(e)}\left(u_{i,j}^{(N)}, v_{i,j}^{(N)}\right), y^{(e)}\left(u_{i,j}^{(N)}, v_{i,j}^{(N)}\right))$$

----------------------- (31)

Where from Eqn.(29), we write

$u_{i,j}^{(N)} = u(\xi_i^{(N)},\ \eta_j^{(N)})$

$v_{i,j}^{(N)} = v(\xi_i^{(N)},\ \eta_j^{(N)})$ ---------------------- (32)

and $(W_i^{(N)},\ \xi_i^{(N)})$, $(W_j^{(N)},\ \xi_j^{(N)})$ are the weight coefficients and sampling points along ξ, η directions of the Nth order Gauss Legendre quadrature rules. We could also use Gauss Labatto quadrature rules as well to evaluate the integral of Eqn.(18).

The above composite rule is applied to numerical Integration over polygonal domains using convex quadrangulation and Gauss Legendre Quadrature Rules[27].

In the next section 6.2, we shall apply the above derivations and compute the integral of eqn.(26) by assuming the integrand $\phi(x, y)$ as the product of global derivatives, which are not explicit function of global variates (x, y)

6.2 Global Derivative Integrals :

If $N_i^{(e)}$ (i =1(1)9) denotes the basis functions for node i of a linear convex 9- node linear convex quadrilateral element e , then by use of chain rule of partial differentiation

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{bmatrix} \qquad \text{-------------------------- (33)}$$

We note that to transform 8- node linear convex quadrilateral $Q_e(e = 1,2,3)$ of ΔPQR in Cartesian space (x,y) into $\widehat{Q}$, the 8- node linear convex quadrilateral spanned by vertices (1/3, 1/3), (1/6, 5/12), (0, ½), (0, ¼), (0, 0), (1/4, 0), (1/2, 0) and (5/12, 1/6) in uv-plane.

We must now use the earlier transformations.

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} + \begin{pmatrix} x_q - x_p \\ y_q - y_p \end{pmatrix} u + \begin{pmatrix} x_r - x_p \\ y_r - y_p \end{pmatrix} v \quad \text{for } Q_1 \text{ in } \Delta PQR \qquad \text{--------------------------- (23)}$$

$$\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \end{pmatrix} + \begin{pmatrix} x_r - x_q \\ y_r - y_q \end{pmatrix} u + \begin{pmatrix} x_p - x_q \\ y_p - y_q \end{pmatrix} v \quad \text{for } Q_2 \text{ in } \Delta PQR \qquad \text{-------------------------- (24)}$$

$$\begin{pmatrix} x^3 \\ y^3 \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \end{pmatrix} + \begin{pmatrix} x_p - x_r \\ y_p - y_r \end{pmatrix} u + \begin{pmatrix} x_q - x_r \\ y_q - y_r \end{pmatrix} v \quad \text{for } Q_3 \text{ in } \Delta PQR \qquad \text{------------------------ (25)}$$

And we note that the above transformations viz Eqns.(23)-(25) are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x_a - x_c \\ y_a - y_c \end{pmatrix} u + \begin{pmatrix} x_b - x_c \\ y_b - y_c \end{pmatrix} v \qquad \text{------------------------ (34)}$$

which can map an arbitrary triangle ΔABC , A(x_a ,y_a) ,B(x_b ,y_b) , C(x_c , y_c) in xy – plane into a right isosceles triangle in the uv – plane

Hence, we have from Eqn.(34)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (x_a - x_c) & (x_b - x_c) \\ (y_a - y_c) & (y_b - y_c) \end{pmatrix}^{-1} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \qquad \text{-------------------------- (35)}$$

This gives

$$u = (\propto_a + \beta_a x + \gamma_a y)/(2\,\Delta_{abc})$$

$$v = (\propto_b + \beta_b x + \gamma_b y)/(2\,\Delta_{abc}) \qquad \text{------------------------ (36)}$$

where

$\propto_a = (x_b y_c - x_c y_b)$, $\qquad \propto_b = (x_c y_a - x_a y_c)$,

$\beta_a = (y_b - y_c)$, $\qquad \beta_b = (y_c - y_a)$,

$\gamma_a = (x_c - x_b)$, $\qquad \gamma_b = (x_a - x_c)$, $\qquad$ -----------------------(37a)

and

$$\frac{\partial(x,y)}{\partial(u,v)} = 2\Delta_{abc} = \begin{vmatrix} 1 & x_a & y_a \\ 1 & x_b & y_b \\ 1 & x_c & y_c \end{vmatrix} = 2 * \text{area of the triangle } \Delta ABC$$

$$= (\gamma_b \beta_a - \gamma_a\,\beta_b) \qquad \text{------------------- (37b)}$$

From Eqn.(33) and Eqn.(36), we obtain

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} = \begin{pmatrix} \beta_a{}^* & \beta_b{}^* \\ \gamma_a{}^* & \gamma_b{}^* \end{pmatrix} \begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix} \qquad \text{---------------------------- (38a)}$$

where $\beta_a{}^* = \frac{\beta_a}{(2\Delta_{abc})}$, $\beta_b{}^* = \frac{\beta_b}{(2\Delta_{abc})}$

$\gamma_a{}^* = \frac{\gamma_a}{(2\Delta_{abc})}$, $\gamma_b{}^* = \frac{\gamma_b}{(2\Delta_{abc})}$ --------------------------- (38b)

Letting,

$$D_{x,y}^{i,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} , \; P = \begin{pmatrix} \beta_a{}^* & \beta_b{}^* \\ \gamma_a{}^* & \gamma_b{}^* \end{pmatrix} , \; D_{u,v}^{i,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix} \qquad \text{---------------------------- (39)}$$

We obtain from Eqn.(38) and Eqn.(39)

$$D_{x,y}^{i,e} = P\, D_{u,v}^{i,e} \qquad \text{----------------------------- (40)}$$

Hence from Eqn.(39) and Eqn.(40)

$$G_{x,y}^{i,j,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} \left(\frac{\partial N_j^e}{\partial x} \;\; \frac{\partial N_j^e}{\partial y} \right) = (D_{x,y}^{i,e})\,(D_{x,y}^{j,e})^T$$

$$= \begin{pmatrix} \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} & \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial y} \\ \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial x} & \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y} \end{pmatrix} \qquad \text{----------------------------- (41a)}$$

$$G_{u,v}^{i,j,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix} \left(\frac{\partial N_j^e}{\partial u} \;\; \frac{\partial N_j^e}{\partial v} \right) = (D_{u,v}^{i,e})\,(D_{u,v}^{j,e})^T$$

$$= \begin{pmatrix} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial u} & \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial v} \\ \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial u} & \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial v} \end{pmatrix} \qquad \text{----------------------------- (41b)}$$

We have now from Eqn.(40) and Eqn.(41a- b)

$$G_{x,y}^{i,j,e} = (P\, D_{u,v}^{i,e}) \; \left(D_{u,v}^{j,e}\, P\right)^T$$

$$= P\,(D_{u,v}^{i,e}) \; \left(D_{u,v}^{j,e}\right)^T P^T$$

$$= P\, G_{u,v}^{i,j,e}\, P^T \qquad \text{--------------------------- (41c)}$$

We now define the submatrices of global derivative integrals in (x,y) and (u,v) space associated with the nodes i and j $(i, j = 1, 2, 3, 4, 5, 6, 7, 8, 9)$ as

$$S^{i,j,e} = \iint_{Q_e} G_{x,y}^{i,j,e} \; dx\, dy , \qquad \text{--------------------------- (42)}$$

$$K^{i,j,e} = \iint_{\hat{Q}} G_{u,v}^{i,j,e} \; du\, dv \qquad \text{--------------------------- (43)}$$

where, we have already defined the 8- node linear convex quadrilaterals Q_e (e=1,2,3) in (x,y) space and $\widehat{Q}$ in (u,v) space in Fig 3a- 3b. From Eqns.(41)-(43) , we obtain the following relations connecting the submatrices $S^{i,j,e}$ and $K^{i,j,e}$

We now obtain the submatrices $S^{i,j,e}$ and $K^{i,j,e}$ in an explicit form from Eqs.(41a)- (41b)

$$S^{i,j,e} = \iint_{Q_e} G_{x,y}^{i,j,e} \; dx\, dy = \begin{pmatrix} \iint_{Q_e} \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x}\, dxdy & \iint_{Q_e} \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial y}\, dxdy \\ \iint_{Q_e} \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial x}\, dxdy & \iint_{Q_e} \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}\, dxdy \end{pmatrix}$$

$$= \begin{pmatrix} S^{e}_{2i-1,2j-1} & S^{e}_{2i-1,2j} \\ S^{e}_{2i,2j-1} & S^{e}_{2i,2j} \end{pmatrix} \quad \text{(say)} \quad \text{------------------(44)}$$

and in similar manner

$$K^{i,j,e} = \iint_{\widehat{Q}} G^{i,j,e}_{u,v} \; du\,dv = \begin{pmatrix} \iint_{\widehat{Q}} \frac{\partial N^e_i}{\partial u}\frac{\partial N^e_j}{\partial u}\,dudv & \iint_{\widehat{Q}} \frac{\partial N^e_i}{\partial u}\frac{\partial N^e_j}{\partial v}\,dudv \\ \iint_{\widehat{Q}} \frac{\partial N^e_i}{\partial v}\frac{\partial N^e_j}{\partial u}\,dudv & \iint_{\widehat{Q}} \frac{\partial N^e_i}{\partial v}\frac{\partial N^e_j}{\partial v}\,dudv \end{pmatrix}$$

$$= \begin{pmatrix} K^{e}_{2i-1,2j-1} & K^{e}_{2i-1,2j} \\ K^{e}_{2i,2j-1} & K^{e}_{2i,2j} \end{pmatrix} \quad \text{(say)} \quad \text{------------------(45)}$$

We have now from the above Eqns.(41)-(45)

$$S^{i,j,e} = \iint_{Q_e} G^{i,j,e}_{x,y} \; dx\,dy = \iint_{\widehat{Q}} (P\,G^{i,j,e}_{u,v}P^{T}) \,\frac{\partial(x,y)}{\partial(u,v)}\; du\,dv$$

$$= 2\Delta_{abc} \iint_{\widehat{Q}} (P\,G^{i,j,e}_{u,v}P^{T}) \; du\,dv$$

$$= 2\Delta_{abc}\, P\, (\iint_{\widehat{Q}} G^{i,j,e}_{u,v}\, du\,dv)\; P^{T}$$

$$= 2\Delta_{abc}\; P\,(K^{i,j,e})\; P^{T} \quad , \; (i,j = 1,2,3,4,5,6,7,8,9) \quad \text{----------------(46)}$$

We can thus obtain the global derivative integrals in the physical space or Cartesian space (x,y) by using the matrix triple product established in Eqn.(46).

We note that $\widehat{Q}$ is the 8- node linear convex quadrilateral in (u, v) space spanned by the vertices (1/3, 1/3), (1/6, 5/12), (0, ½), (0, ¼), (0, 0), (1/4, 0), (1/2, 0) , (5/12, 1/6) and (5/24,5/24) in uv- plane hence from Eqn.(45)

$$K^{i,j,e} = \iint_{\widehat{Q}} G^{i,j,e}_{u,v} \; du\,dv \quad \text{-------------------------(47)}$$

$$= \int_{-1}^{1}\int_{-1}^{1} G^{i,j,e}_{u,v}\, \frac{\partial(u,v)}{\partial(\xi,\eta)}\; d\xi\, d\eta \quad \text{------------------------(48)}$$

We now refer to section 6.1 of this paper, in this section, we have derived the necessary relations to integrate Eq.(47). As in Eqns.(27)-(28), we use the transformation of Eqn.(29) to map the 9- node quadrilateral $\widehat{Q}$ to the 9- node 2-square $-1 \le \xi,\eta \le 1$ Using Eqn.(29) in Eqn.(48), we obtain

$$K^{i,j,e} = \iint_{\widehat{Q}} G^{i,j,e}_{u,v} \; (\tfrac{4+\xi+\eta}{96})\; d\xi d\eta \quad \text{----------------------- (49)}$$

Thus, we have from Eq.(46)

$$S^{i,j,e} = \; (2\Delta_{abc})\; P\,(K^{i,j,e})\; P^{T} \quad \text{---------------------- (50)}$$

Where $K^{i,j,e}$ is given in Eqn.(49)

In Eqn.(50) , $2\Delta_{abc}$= 2 * area of the triangle spanning vertices A(x_a ,y_a) ,B(x_b ,y_b) , C(x_c , y_c) is a scalar.

The matrices P, P^T depend purely on the nodel coordinates (x_a ,y_a) ,(x_b ,y_b) , (x_c , y_c) the matrix $K^{i,j,e}$ can be explicity computed by the relations obtained in section 2 – 6. We find that $K^{i,j,e}$ is a (2X2) matrix of integrals whose integrands are rational functions with polynomial numerator and the linear denominator $(4 + \xi + \eta)$. Hence these integrals can be explicity computed. The explicit values of these integrals are expressible in terms of logarithmic constants. We have used symbolic mathematics software of MATLAB to compute the explicit values and their conversion to any number of digits can be obtained by using variable precision arithmetic (vpa) command. The matrix K^e as noted in Eqn.(45) is of order $(2xn_{de})$ x $(2xn_{de})$, n_{de} = 8 = for 8-node convex quadrilateral element.

We have computed K^e for the four node element n_{de}= 4 in our resent paper [18]. In the present paper, we have computed K^e for the 8- node linear convex quadrilateral $\widehat{Q}$ in uv – space. This is listed in Table 1A and Table 1B.

We may note that In order to compute the local/element stiffness matrices for the Poisson Boundary Value problem, we need to compute the integrals Eqns(12a-b)

$$K^e_{i,j} = \int_{\Omega^e} \nabla\varphi_i \,.\, \nabla\varphi_j \, d\mathrm{x} = \int_{\Omega^e} \{ \frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y} \} \, dxdy \, ,$$

,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,(51a)

from the above derivations, we can rewrite $K^e_{i,j}$ in the notations of this sections by taking $\varphi_i = \mathrm{N}_i$ and $\varphi_j = N_j$ and $\Omega^e = Q_e$ so that

$$K^e_{i,j} = \int_{Q_e} \nabla \mathrm{N}_i \,.\, \nabla N_j \, d\mathrm{x} = \int_{Q_e} \{ \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \} \, dxdy = \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} + \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}}$$

....................................(51b)

6.3 Computation of $K^e_{i,j}$

The explicit integration scheme explained above compute four derivative product integrals as given in eqn(44) and they are necessary to compute the stiffness matrix entries of plane stress/plane strain problems in elasticity and sevral other applications in continuum mechanics.But this computation requires matrix triple product as given in eqn (50).Since,we only need the sum of two of these integrals viz : $\mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} + \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}}$ .We now present an efficient method to compute this sum by using matrix product.

Let $F^{i,j}_{p.q} = \frac{\partial N_i}{\partial p}\frac{\partial N_j}{\partial q}$, $I^{i,j}_{p.q} = \int_{Q_e} F^{i,j}_{p.q} \, dpdq$, then we have from eqns(44-45) :

$$\mathrm{S}^{\mathrm{i,j,e}} = \iint_{\mathrm{Q_e}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{x\,,y}} \;\; \mathrm{dx\,dy} = \begin{pmatrix} \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{x}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{x}} \, \mathrm{dxdy} & \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{x}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{y}} \, \mathrm{dxdy} \\ \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{y}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{x}} \, \mathrm{dxdy} & \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{y}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{y}} \, \mathrm{dxdy} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} & \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}} \\ \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}-1} & \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}} \end{pmatrix} \;\; \text{(say)}$$

$$= \begin{pmatrix} I^{i,j}_{x,x} & I^{i,j}_{x,y} \\ I^{i,j}_{y,x} & I^{i,j}_{y,y} \end{pmatrix}$$

..(52a)

$$\mathrm{K}^{\mathrm{i,j,e}} = \iint_{\hat{\mathrm{Q}}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{u\,,v}} \;\; \mathrm{du\,dv} = \begin{pmatrix} \iint_{\hat{\mathrm{Q}}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{u}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{u}} \, \mathrm{dudv} & \iint_{\hat{\mathrm{Q}}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{u}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{v}} \, \mathrm{dudv} \\ \iint_{\hat{\mathrm{Q}}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{v}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{u}} \, \mathrm{dudv} & \iint_{\hat{\mathrm{Q}}} \frac{\partial \mathrm{N^e_i}}{\partial \mathrm{v}}\frac{\partial \mathrm{N^e_j}}{\partial \mathrm{v}} \, \mathrm{dudv} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} & \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}} \\ \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}-1} & \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}} \end{pmatrix} \;\; \text{(say)}$$

$$= \begin{pmatrix} I^{i,j}_{u,u} & I^{i,j}_{u,v} \\ I^{i,j}_{v,u} & I^{i,j}_{v,v} \end{pmatrix}$$

...(52b)

Let $\mathrm{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \; P^T = \begin{pmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{pmatrix}$

...(53)

From eqns(44) , (46) and (52a-b)

$$\mathrm{S}^{\mathrm{i,j,e}} = \iint_{\mathrm{Q_e}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{x\,,y}} \;\; \mathrm{dx\,dy} \; = 2\Delta_{\mathrm{abc}} \, \mathrm{P} \, (\iint_{\hat{\mathrm{Q}}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{u\,,v}} \, \mathrm{du\,dv}) \;\; \mathrm{P^T}$$

$$= 2\Delta_{\mathrm{abc}} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} I^{i,j}_{u,u} & I^{i,j}_{u,v} \\ I^{i,j}_{v,u} & I^{i,j}_{v,v} \end{pmatrix} \begin{pmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{pmatrix}$$

$$= 2\Delta_{\mathrm{abc}} \begin{pmatrix} \{ \mathrm{P_{11}}(\mathrm{P_{11}}\mathrm{I^{i,j}_{u.u}} + \mathrm{P_{12}}\mathrm{I^{i,j}_{u,v}}) + \mathrm{P_{12}}(\mathrm{P_{11}}\mathrm{I^{i,j}_{v,u}} + \mathrm{P_{12}}\mathrm{I^{i,j}_{v,v}}) \; \} & \{ \mathrm{P_{11}}(\mathrm{P_{21}}\mathrm{I^{i,j}_{u,u}} + \mathrm{P_{22}}\mathrm{I^{i,j}_{u.v}}) + \mathrm{P_{12}}(\mathrm{P_{21}}\mathrm{I^{i.j}_{v,u}} + \mathrm{P_{22}}\mathrm{I^{i,j}_{v.v}}) \} \\ \{ \mathrm{P_{21}}(\mathrm{P_{11}}\mathrm{I^{i,j}_{u,u}} + \mathrm{P_{12}}\mathrm{I^{i,j}_{u,v}}) + \mathrm{P_{22}}(\mathrm{P_{11}}\mathrm{I^{i,j}_{u.u}} + \mathrm{P_{12}}\mathrm{I^{i.j}_{v,v}}) \; \} & \{ \mathrm{P_{21}}(\mathrm{P_{21}}\mathrm{I^{i,j}_{u,u}} + \mathrm{P_{22}}\mathrm{I^{i.j}_{u,v}}) + \mathrm{P_{22}}(\mathrm{P_{21}}\mathrm{I^{i,j}_{v,u}} + \mathrm{P_{22}}\mathrm{I^{i,j}_{v,v}}) \} \end{pmatrix}$$

……………………………………………………………………………………………………(54)

From eqn(51a-b) and eqn(46) , we find

$$\text{trace}\ (S^{i,j,e}) = \text{trace}(\iint_{Q_e} G^{i,j,e}_{x,y}\ dxdy\ ) = (S^e_{2i-1,2j-1} + S^e_{2i,2j}) = K^e_{i,j} = \int_{Q_e} \nabla N_i \,.\, \nabla N_j\ dx = \int_{Q_e}\{\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\}\, dxdy$$

$$= (P_{11}^2 + P_{21}^2)I^{i,j}_{u,u} + (P_{11}\,P_{12} + P_{21}\,P_{22})\left(I^{i,j}_{u,v} + I^{i,j}_{v,u}\right) + (P_{12}^2 + P_{22}^2)I^{i,j}_{v,v} \qquad \text{..........(55)}$$

W e can obtain the above integraql $\int_{Q_e}\{\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\}\, dxdy$ by use of matrix operations which doesnot need the computation matrix triple product.This procedure is presented below

From eqn (44b) and eqn(45),let us do the following:

$$(P^T\,P)\,.* \begin{pmatrix} I^{i,j}_{u,u} & I^{i,j}_{u,v} \\ I^{i,j}_{v,u} & I^{i,j}_{v,v} \end{pmatrix} = \begin{bmatrix} (P_{11}^2 + P_{21}^2)\, I^{i,j}_{u,u} & (\,P_{11}\,P_{12} + P_{21}\,P_{22})I^{i,j}_{u,v} \\ (\,P_{11}\,P_{12} + P_{22}\,P_{21})I^{i,j}_{v,u} & (P_{12}^2 + P_{22}^2)I^{i,j}_{v,v} \end{bmatrix} \text{...................................(56)}$$

We observe from eqn(56) that sum of all the entries gives us the value of the integral i.e

$$\int_{Q_e}\{\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\}\, dxdy = sum(sum\left(\ (P^T\,P).* \begin{pmatrix} I^{i,j}_{u,u} & I^{i,j}_{u,v} \\ I^{i,j}_{v,u} & I^{i,j}_{v,v} \end{pmatrix}\ \right)) \qquad \text{............................(57)}$$

Where,sum is a Matlab function.We note that S=sum(X) gives the sum of the elements of vector X. If X is a matrix then S is a row vector with the sum over each column.It is clear that sum(sum(X)) gives the sum of all the entries in a matrix X.

6.4 Computing of Force Vector Integrals $\int_{\Omega^e} f\varphi_i$ dxdy

We shall now propose numerical integration for the complicated integrands in the force vector integrals over the domain Ω^e which is an arbitrary linear triangle and $\phi(x,y) = f\varphi_i$ .We also refer to the section 2 for the theory necessary to derive the composite numerical integration formla

We shall now establish a composite integration formula for an arbitrary linear triangular region Δ PQR shown in Fig 2a or Fig 3a. We have for an arbitrary smooth function $\phi(x, y)$

$$II_{\Delta PQR} = \iint_{\Delta PQR} \phi(x,y)\ dxdy = \sum_{e=1}^{3} \iint_{Q_e} \phi(x,y)\ dxdy \qquad \text{---------------------- (58)}$$

$$= \iint_{\hat{Q}}\ \sum_{e=1}^{3}[\phi\ (x^{(e)}(u,v), y^{(e)}(u,v))\,\frac{\partial\left(x^{(e)}(u,v), y^{(e)}(u,v)\right)}{\partial(u,v)}]\ dudv$$

$$= (2\ \Delta_{pqr})\ \iint_{\hat{Q}}\ \{\sum_{e=1}^{3}[\phi\ (x^{(e)}(u,v),\ \ y^{(e)}(u,v))\ ]\}\ dudv \qquad \text{------------------- (59)}$$

Where $(x^{(e)}(u,v),\ \ y^{(e)}(u,v)), e = 1, 2, 3)$ are the transformations of Eqs.(8)–(10) and $\hat{Q}$ is the quadrilateral in uv- plane spanned by vertices G(1/3 ,1/3) , E(0, ½) , C(0, 0) and F(1/2,0) , and Δ_{pqr} is the area of triangle Δ PQR, Now using the transformations defined in Eqs.(1)–(2) we obtain

$$II_{\Delta PQR} = (2\ \Delta_{pqr})\ \iint_{\hat{Q}}\ \{\sum_{e=1}^{3}[\phi\ (x^{(e)}(u,v),\ \ y^{(e)}(u,v))\,\frac{\partial(u,v)}{\partial(\xi,\eta)}\}\ d\xi\ d\eta \qquad \text{--------------- (60)}$$

In Eq.(14) we have used the transformation

$$u(\xi,\ \eta) = \frac{1}{3}N_1(\xi,\ \eta) + \frac{1}{2}N_4(\xi,\ \eta)$$

$$v(\xi,\ \eta) = \frac{1}{3}N_1(\xi,\ \eta) + \frac{1}{2}N_2(\xi,\ \eta) \qquad \text{---------------------- (61)}$$

to map the quadrilateral $\hat{Q}$ into a 2 – square in ξη – plane.

We can now obtain from Eqs.(14)–(15)

$$II_{\Delta PQR} = (2\ \Delta_{pqr}) \int_{-1}^{1}\int_{-1}^{1}\ [\sum_{e=1}^{3}\ \left(\frac{4+\xi+\eta}{96}\right)\phi(\ x^{(e)}(u,v),\ \ y^{(e)}(u,v))\ ]d\xi\ d\eta \qquad \text{------------- (62)}$$

We can evaluate Eq.(16) either analytically or numerically depending on the form of the integrand.

Using Numerical Integration ;

$$II_{\Delta PQR} = 2\Delta_{pqr} \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{W_i^{(N)}W_j^{(N)}(4+\xi_i^{(N)}+\eta_j^{(N)})}{96}\right) \sum_{e=1}^{3}\phi(x^{(e)}\left(u_{i,j}^{(N)}, v_{i,j}^{(N)}\right), y^{(e)}\left(u_{i,j}^{(N)}, v_{i,j}^{(N)}\right)\,)$$

------ ----------------- (63)

Where,

$u_{i,j}^{(N)} = u(\xi_i^{(N)},\ \eta_j^{(N)})$ and $v_{i,j}^{(N)} = v(\xi_i^{(N)},\ \eta_j^{(N)})$ --------- ------------- (64)

and $(W_i^{(N)},\ \xi_i^{(N)})$, $(W_j^{(N)},\ \xi_j^{(N)})$ are the weight coefficients and sampling points of Nth order Gauss Legendre Quadrature rules.

The above composite rule is applied to numerical Integration over polygonal domains using convex quadrangulation and Gauss Legendre Quadrature Rules[27].

The above method will help in integrating $\int_{\Omega^e} f\varphi_i$ dxdy,when the intgrand $f\varphi_i$ is complicated

7 A New Approach To Mesh Generation

The first step in implimenting finite element method isto generate a mesh.In a recent work the author and his co-workers have proposed a new approach to mesh generation which can discretise a convex polygon into an all quadrilateral mesh.This will be presented next.This new approach to mesh generation meets the necessary requirements of regularity on the shape of elements.There are two types of them which usually suffice in finite element computations.The first is called shape regularity. It says that the ratio of the diameter of the element to the radius of the inner circle must be less than some constant. For triangles,the diameter of the triangle is related to the smallest circle which contain the triangle.The inner circle refers to the largest circle which fits inside the triangle. Shape regularity focuses on the shape of individual triangles and doesnot refer to how the shapes of different elements relate to each other. So some elements can be large wshile others might be very small. There is a second type of requirement on the shape of elements.This requirement says that ratio of the maximum diameter of elements to the radius of the inner circle of an element must be less than some constant .If a mesh satisfies this requirement,it is called quasiuniform.This requirement is more important when we perform refinements.We must note that a mesh generation gives us the nodes on a particular element as well as the coordinates of the nodes.We now give an account of this novel mesh generation technique with an aim to use it further in the solution of Poisson problem. Stated in eqn(7a-b).

In our recent paper[ ], the explicit finite element integration scheme is presented by using the isoparametric transformation over the 4 node linear convex quadrilateral element which is applied to torison of square shaft, on considering symmetry of the problem domain, mesh generation for 1/8 of the cross section which is a triangle was discritised into an all quadrilateral mesh. In this paper we consider applications to polygonal domains.

7.1 An automatic indirect quadrilateral mesh generator

A wide range of problems in applied science and engineering can be simulated by partial derivative equations(PDE).In the last few decade,one of the most relevant techniques to solve is the Finite Element Method(FEM).It is well known that a good quality mesh is required in order to obtain an accurate solution.Hence the construction of a mesh is on e of the most important steps.

In the next few sections , we present a novel mesh generation scheme of all quadrilateral elements for convex polygonal domains. This scheme converts the elements in background triangular mesh into quadrilaterals through the operation of splitting. We first decompose the convex polygon into simple subregions in the shape of triangles. These simple subregions are then triangulated to generate a fine mesh of triangles. We propose then an automatic triangular to quadrilateral conversion scheme in which each isolated triangle is split into three quadrilaterals according to the usual scheme, adding three vertices in the middle of edges and a vertex at the barrycentre of the triangular element. Further, to preserve the mesh conformity a similar procedure is also applied to every triangle of the domain and this fully discretizes the given convex polygonal domain into all quadrilaterals, thus propogating uniform refinement. In section 4.2, we present a scheme to discretize the arbitrary and standard triangles into a fine mesh of six node triangular elements. In section 4.3, we explain the procedure to split these triangles into quadrilaterals. In section 4.4,we have presented a method of piecing together of all triangular subregions and eventually creating a all quadrilateral mesh for the given convex polygonal domain. In section 4.5,we present several examples to illustrate the simplicity and efficiency of the proposed mesh generation method for standard and arbitrary triangles,rectangles and convex polygonal domains.

7.2 Division of an Arbitrary Triangle

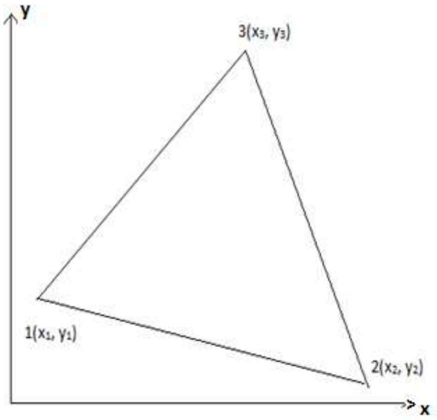
We can map an arbitrary triangle with vertices $(\ (x_i, y_i),\ i = 1, 2, 3)$ into a right isosceles triangle in the (u, v) space as shown in Fig. 4a, b. The necessary transformation is given by the equations.

$$x = x_1 + (x_2 - x_1)u + (x_3 - x_1)v$$

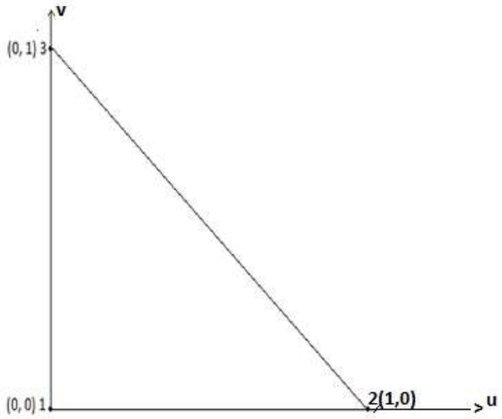
$$y = y_1 + (y_2 - y_1)u + (y_3 - y_1)v$$

(57)

The mapping of eqn.(1) describes a unique relation between the coordinate systems. This is illustrated by using the area coordinates and division of each side into three equal parts in Fig. 5a Fig. 5b. It is clear that all the coordinates of this division can be determined by knowing the coordinates $(\,(x_i, y_i),\ i = 1,2,3)$ of the vertices for the arbitrary triangle. In general , it is well known that by making 'n' equal divisions on all sides and the concept of area coordinates, we can divide an arbitrary triangle into n^2 smaller triangles having the same area which equals Δ/n^2 where Δ is the area of a linear arbitrary triangle with vertices $(\,(x_i, y_i),\ i = 1,2,3)$ in the Cartesian space.



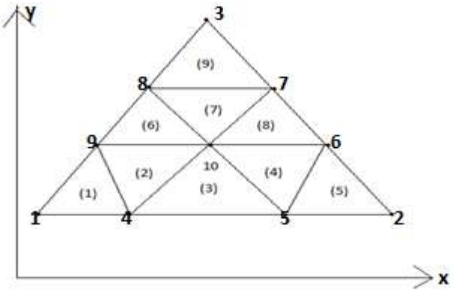
4.a



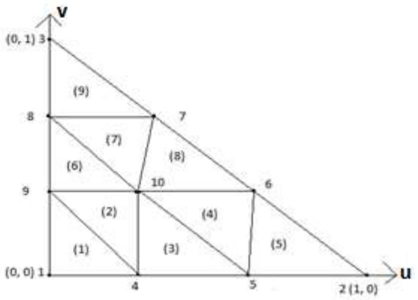
4 b

Fig. 4a An Arbitrary Linear Triangle in the (x, y) space

Fig. 4b A Right Isosceles Triangle in the (u, v) space



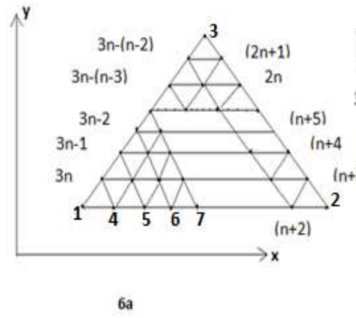
5a



5b

Fig. 5a Division of an arbitrary triangle into Nine triangles in Cartesian space

Fig. 5b Division of a right isosceles triangle into Nine right isosceles triangles in (u, v) space



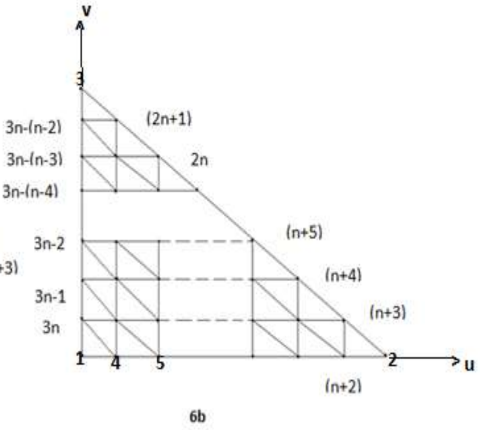


Fig.6a Division of an arbitrary triangle into n^2 triangle in Cartesian space (x, y), where each side is divided into n divisions of equal length

Fig. 6b Division of a right isosceles triangle into n^2 right isosceles triangle in (u, v) space, where each side is divided into n divisions of equal length

We have shown the division of an arbitrary triangle in Fig. 6a , Fig. 6b, We divided each side of the triangles (either in Cartesian space or natural space) into n equal parts and draw lines parallel to the sides of the triangles. This creates (n+1) (n+2) nodes. These nodes are numbered from triangle base line l_{12} (letting l_{ij} as the line joining the vertex (x_i, y_i) and (x_j, y_j)) along the line $v = 0$ and upwards up to the line $v = 1$. The nodes 1, 2, 3 are numbered anticlockwise and then nodes 4, 5, ------, (n+2) are along line $v = 0$ and the nodes (n+3), (n+4), ------, 2n, (2n+1) are numbered along the line l_{23} i.e. $u + v = 1$ and then the node (2n+2), (2n+3), -------, 3n are numbered along the line $u = 0$. Then the interior nodes are numbered in increasing order from left to right along the line $v = \frac{1}{n}, \frac{2}{n}, ---, \frac{n-1}{n}$ bounded on the right by the line $u + v = 1$. Thus the entire triangle is covered by (n+1) (n+2)/2 nodes. This is shown in the $\underline{rr}$ matrix of size $(n+1) \times (n+1)$, only nonzero entries of this matrix refer to the nodes of the triangles

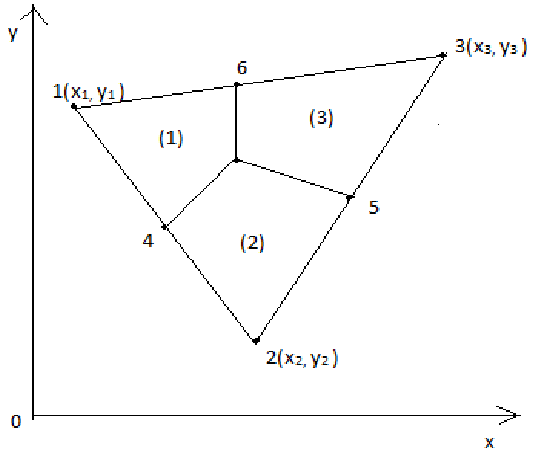
$$\underline{rr} = \begin{bmatrix} 1, & 4, & 5, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & (n+2) & 2 \\ 3n, & (3n+1), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & 3n+(n-2), & (n+3) & 0 \\ 3n-1, & 3n+(n-1) \ldots\ldots\ldots\ldots\ldots & ,3n+(n-2)+(n-3), & (n+4) \quad 0 & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 3n-(n-3), & \frac{(n+1)(n+2)}{2}, & 2n & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 \\ 3n-(n-2), & (2n+1), & 0 & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. & 0 \\ 3 & 0 & 0 & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 \end{bmatrix}$$

..(58)

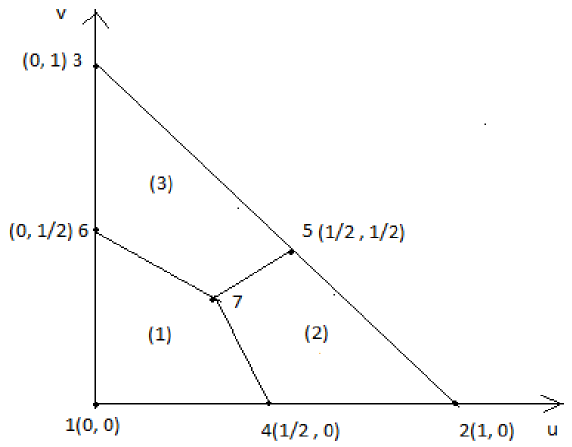
7.3. Quadrangulation of an Arbitrary Triangle

We now consider the quadrangulation of an arbitrary triangle. We first divide the arbitrary triangle into a number of equal size six node triangles. Let us define l_{ij} as the line joining the points (x_i, y_i) and (x_j, y_j) in the Cartesian space (x, y). Then the arbitrary triangle with vertices at $((x_i, y_i), i = 1, 2, 3)$ is bounded by three lines l_{12}, l_{23}, and l_{31} . By dividing the sides l_{12}, l_{23}, l_{31} into $n = 2m$ divisions (m, an integer) creates m^2 six node triangular divisions. Then by joining the centroid of these six node triangles to the midpoints of their sides, we obtain three quadrilaterals for each of these triangle. We have illustrated this process for the two and four divisions of l_{12}, l_{23}, and l_{31} sides of the arbitrary and standard triangles in Figs. 4 and 5

Two Divisions of Each side of an Arbitrary Triangle



7(a)

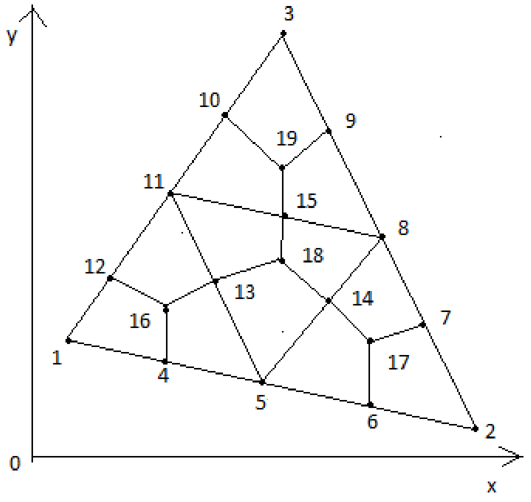


7(b)

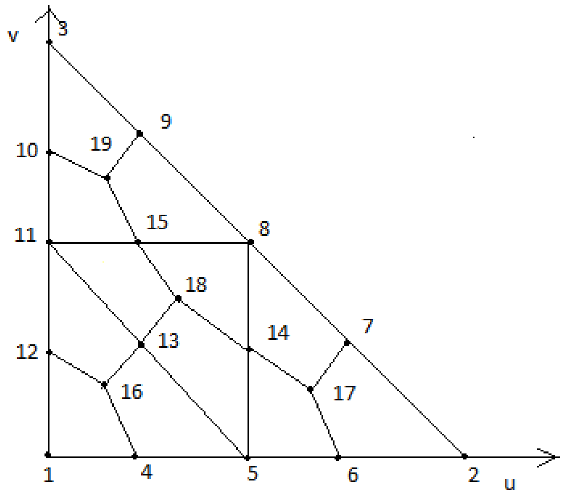
Fig 7(a). Division of an arbitrary triangle into three quadrilaterals

Fig 7(b). Division of a standard triangle into three quadrilaterals

Four Divisions of Each side of an Arbitrary Triangle



8a



8b

Fig 8a. Division of an arbitrary triangle into 4 six node triangles

Fig 8b. Division of a standard triangle into 4 right isosceles triangle

In general, we note that to divide an arbitrary triangle into equal size six node triangle, we must divide each side of the triangle into an even number of divisions and locate points in the interior of triangle at equal spacing. We also do similar divisions and locations of interior points for the standard triangle. Thus n (even) divisions creates ($n/2)^2$ six node triangles in both the spaces. If the entries of the sub matrix $\underline{rr}\,(i\,;\,i+2,\;j\,;\,j+2)$ are nonzero then two six node triangles can be formed. If $\underline{rr}\,(i+1,\;j+2) = \underline{rr}\;(i+2,\;j+1;\;j+2) = 0$ then one six node triangle can be formed. If the sub matrices $\underline{rr}\,(i\,;\,i+2,\;j\,;\,j+2)$ is a (3×3) zero matrix , we cannot form the six node triangles. We now explain the creation of the six node triangles using the $\underline{rr}$ matrix of eqn.( ). We can form six node triangles by using node points of three consecutive rows and columns of $\underline{rr}$ matrix. This procedure is depicted in Fig. 9 for three consecutive rows $i\,,i+1$, $i+2$ and three consecutive columns $j\,,\;j+1,\;j+2$ of the $\underline{rr}$ sub matrix

Formation of six node triangles using sub matrix $\underline{rr}$

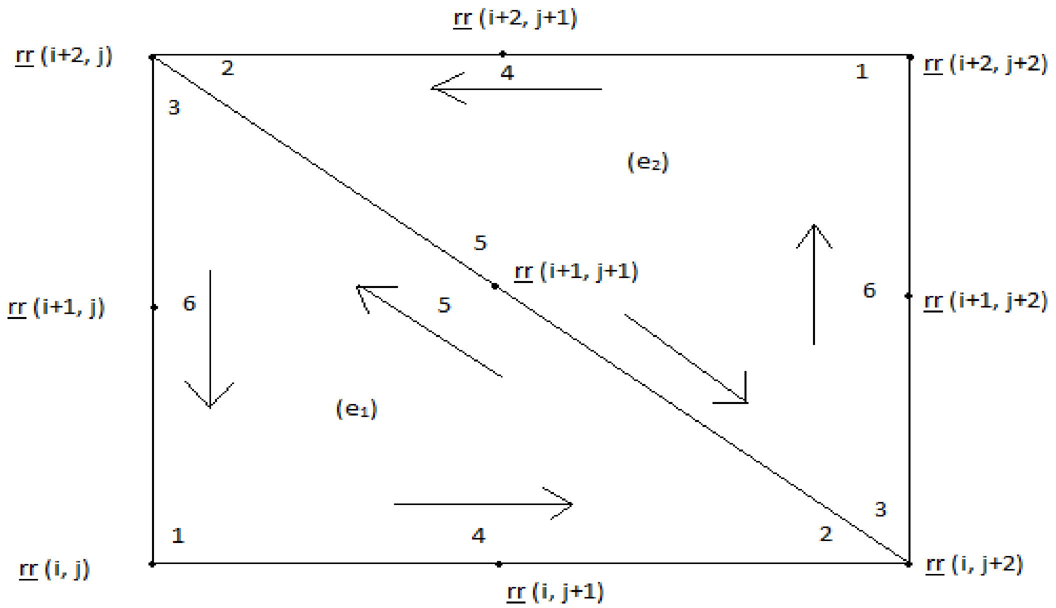


Fig. 9 Six node triangle formation for non zero sub matrix $\underline{rr}$

If the sub matrix ($(\underline{rr}\ (\ k,l), k = i, i+1, i+2), l = j,\ j+1,\ j+2)$ is nonzero, then we can construct two six node triangles. The element nodal connectivity is then given by

(e_1) $< \underline{rr}\ (i,\ j), \underline{rr}\,(i, i+2), \underline{rr}\,(i+2,\ j), \underline{rr}\,(i, j+1), \underline{rr}\,(i+1, j+1), \underline{rr}\,(i+1, j) >$

(e_2) $< \underline{rr}\ (i+2, j+2), \underline{rr}\,(i+2, j), \underline{rr}\,(i, j+2), \underline{rr}\,(i+2, j+1), \underline{rr}\,(i+1, j+1), \underline{rr}\ \ i+1, j+2) >$ ....(59)

If the elements of sub matrix ($(\underline{rr}\ (\ k,l), k = i, i+1, i+2),\ l = j,\ j+1,\ j+2)$ are nonzero, then as standard earlier, we can construct two six node triangles. We can create three quadrilaterals in each of these six node triangles. The nodal connectivity for the 3 quadrilaterals created in (e_1) are given as

$Q_{3n_1-2} < c_1, \underline{rr}\ (i+1,\ j), \underline{rr}\,(i,\ j), \underline{rr}\,(i, j+1) >$

$Q_{3n_1-1} < c_1, \underline{rr}\ (i,\ j+1), \underline{rr}\,(i,\ j+2), \underline{rr}\,(i+1, j+1) >$

$Q_{3n_1} < c_1, \underline{rr}\ (i+1,\ j+1), \underline{rr}\ (i+2,\ j), \underline{rr}\ (i+1, j) >$ ....(60)

and the nodal connectivity for the 3 quadrilaterals created in (e_2) are given as

$Q_{3n_2-2} < c_2, \underline{rr}\ (i+1,\ j+2), \underline{rr}\,(i+2,\ j+2), \underline{rr}\,(i+2,\ j+1) >$

$Q_{3n_2-1} < c_2, \underline{rr}\ (i+2,\ j+1), \underline{rr}\,(i+2,\ j), \underline{rr}\,(i+1,\ j+1) >$

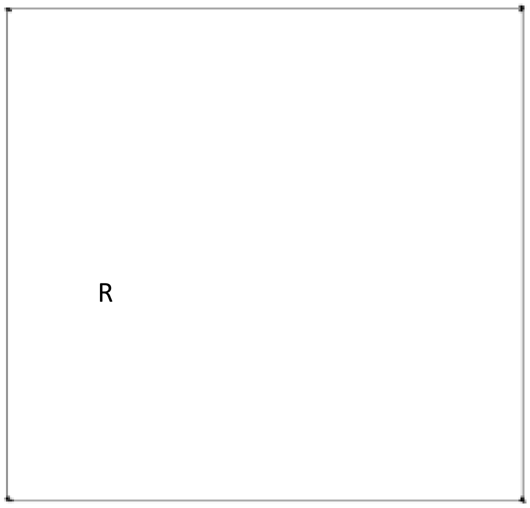
$Q_{3n_1} < c_2, \underline{rr}\ (i+1,\ j+1), \underline{rr}\,(i,\ j+2), \underline{rr}\,(i+1, j+2) >$ -------------------- (61)

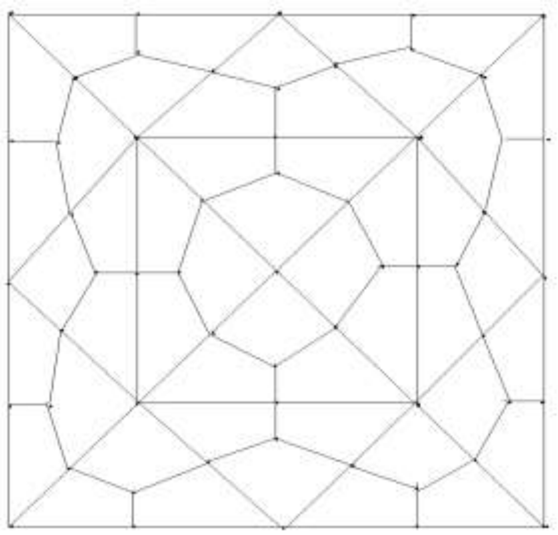
7.4 Quadrangulation of the Polygonal Domain

We can generate polygonal meshes by piecing together triangular with straight sides. Subsection (called LOOPs). The user specifies the shape of these LooPs by designating six coordinates of each LOOP

As an example, consider the geometry shown in Fig. 8(a). This is a a square region which is simply chosen for illustration. We divide this region into four LOOPs as shown in Fig.8(d). These LOOPs 1,2,3 and 4 are triangles each with

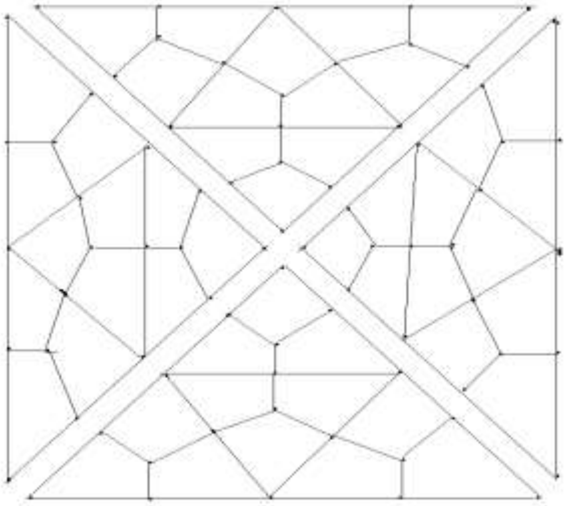
three sides. After the LOOPs are defined, the number of elements for each LOOP is selected to produce the mesh shown in Fig. 8(c).The complete mesh is shown in Fig.8(b)

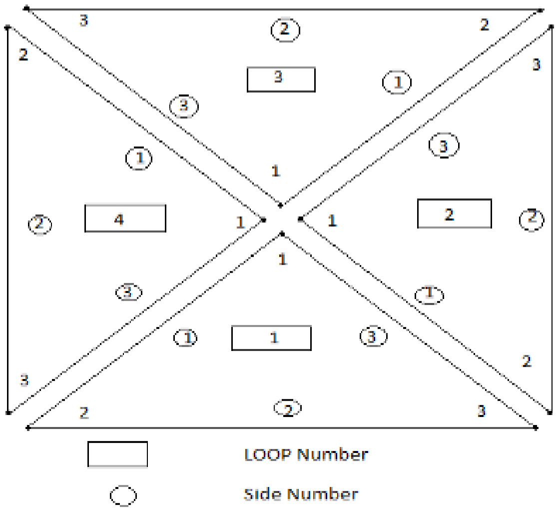




10a **10b**

(i)Fig.10a: Region R to be analyzed **(ii) Fig.10b: Example of completed mesh**





10c **10d**

(iii)Fig.10c:Exploded view showing four loops (iv)Fig.10d:Example of a loop and side numbering scheme

How to define the LOOP geometry, specify the number of elements and piece together the LOOPs will now be explained

Joining LOOPs : A complete mesh is formed by piecing together LOOPs. This piecing is done sequentially thus, the first LOOP formed is the foundation LOOP, with subsequent LOOPs joined either to it or to other LOOPs that have already been defined. As each LOOP is defined, the user must specify for each of the three sides of the current LOOP.

In the present mesh generation code, we aim to create a convex polygon. This requires a simple procedure. We join side 3 0f LOOP 1 to side 1 of LOOP 2, side 3 of LOOP 2 will joined to side 1 of LOOP 3, side 3 of LOOP 3 will be joined to side 1 of LOOP 4. Finally side 3 of LOOP 4 will be joined to side 1 of LOOP 1.

When joining two LOOPs, it is essential that the two sides to be joined have the same number of divisions. Thus the number of divisions remains the same for all the LOOPs. We note that the sides of LOOP (i) and side of LOOP ($i+1$) share the same node numbers. But we have to reverse the sequencing of node numbers of side 3 and assign them as node

numbers for side 1 of LOOP $(i+1)$. This will be required for allowing the anticlockwise numbering for element connectivity.

The auto mesh generation technique discritises a polygonal domain into all four node special quadrilateral elements.We can convert these into nine node special quadrilateral elements by adding one node at the midpoint of each side of the four node special quadrilateral elements,and also the ninth node at the centroid of the quadrilateral elements. We have written codes to carry this conversion schemes in the programs of all four node special quadrilaterals proposed in[ ].We include here some meshes of all nine node special quadrilaterals at initial stagtes of mesh generation which is self explainatory.

Example 1: right isosceles triangle
x=sym([0;1/2;1/2])
y=sym([0 0;1/2])
We use this mesh to solve torsion of a square cross section,due to symmetry considerations mesh generation over the above domain.This is a case of Poisson equation with constant right hand side(=-2).Our main aim is to compute torsional constant

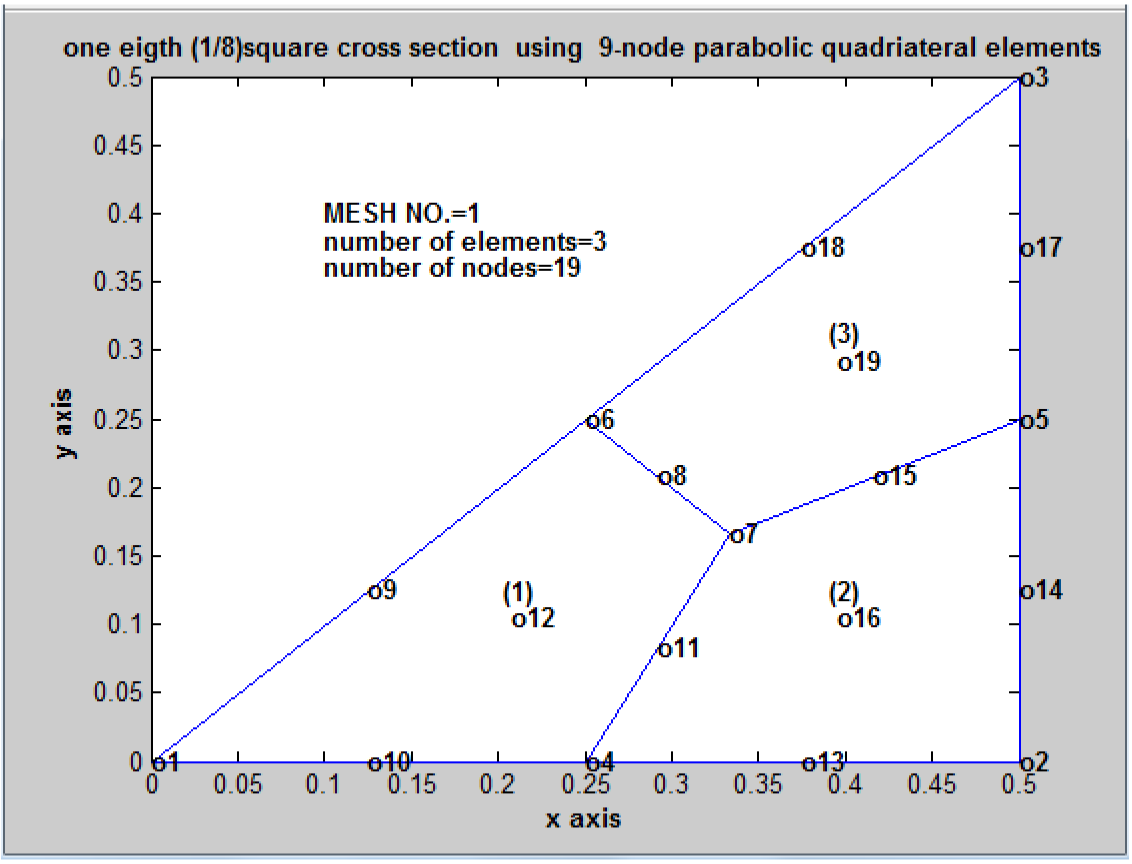


Fig.11 Initial mesh of right isosceles triangle

Example 2: equilateral triangle,each side=2*sqrt(3)
x=sym([-sqrt(3);sqrt(3); 0])
y=sym([-1; -1; 2])
We use this mesh to solve torsion of an arbitrary triangular cross section.This is a case of Poisson equation with constant right hand side(=-1. Our aim are to compute torsional constant and contour lines of Prandtl stress function

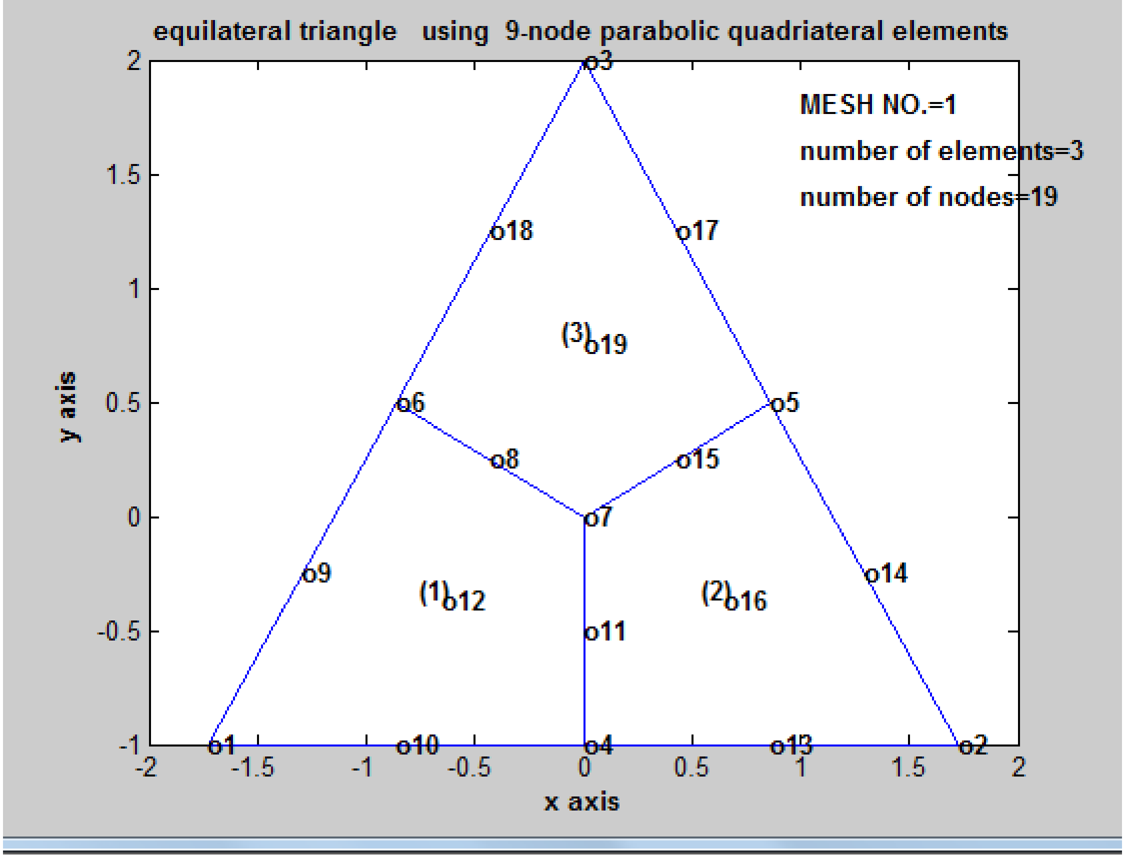


Fig.12 Initial mesh of equilateral triangle,each side=2*sqrt(3)

Example 3 :a square domain with eight triangles(9-boundary nodes)

x=sym([1/2;1/2;1; 1; 1;1/2;0; 0;0])%FOR UNIT SQUARE
y=sym([1/2; 0;0;1/2; 1; 1;1;1/2;0])%FOR UNIT SQUARE
We use this mesh to solve torsion of a square cross section.We would like to draw countour lines of Prandtl stress function over the entire domain.This is a case of Poisson equation with constant right hand side(=-2)

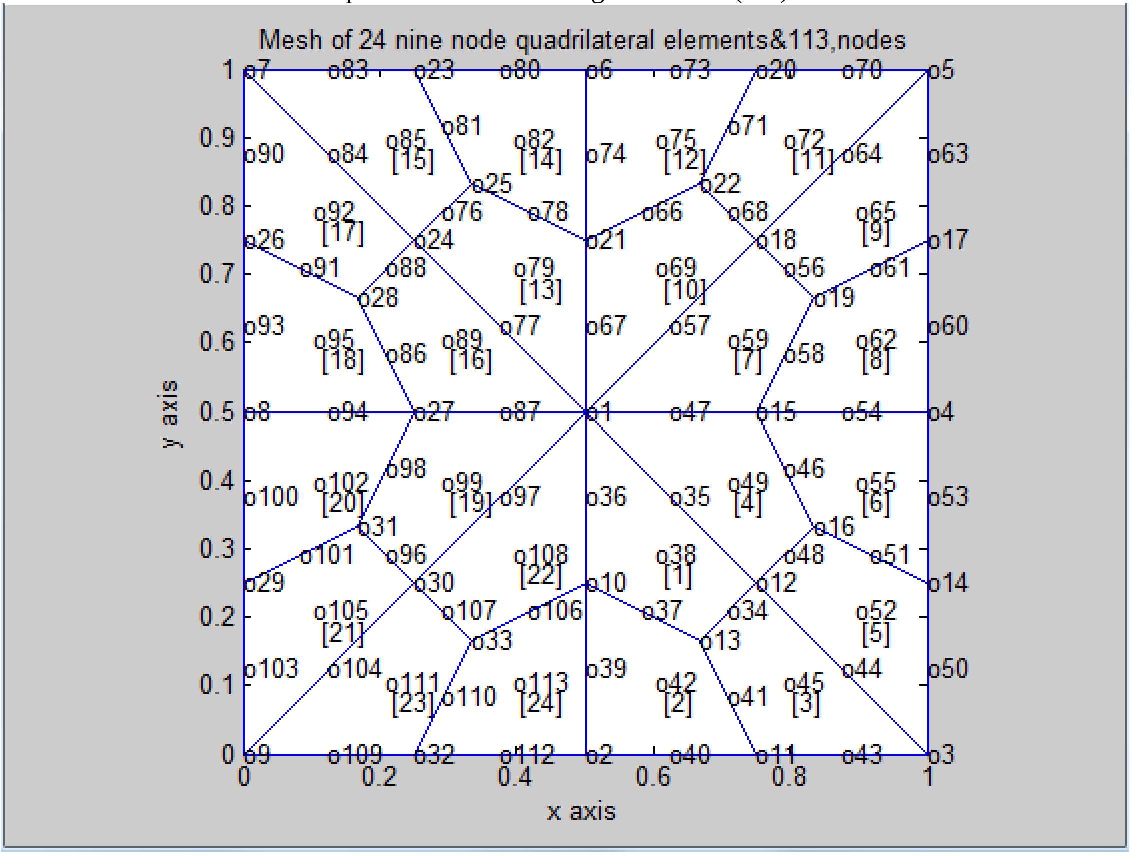


Fig.13 Initial Mesh for a square domain with eight triangles(9-boundary nodes)

Example 4:pentagonal domain with seven triangles(8-boundary nodes)

x=sym([1/2;1/2;1; 1;1/2;0; 0;0])%for MOIN EXAMPLE
y=sym([1/2; 0;0;1/2; 1;1;1/2;0])%for MOIN EXAMPLE
We use this mesh to solve Poisson equation with a nonconstant smooth function on right hand side,with a known analytical solution

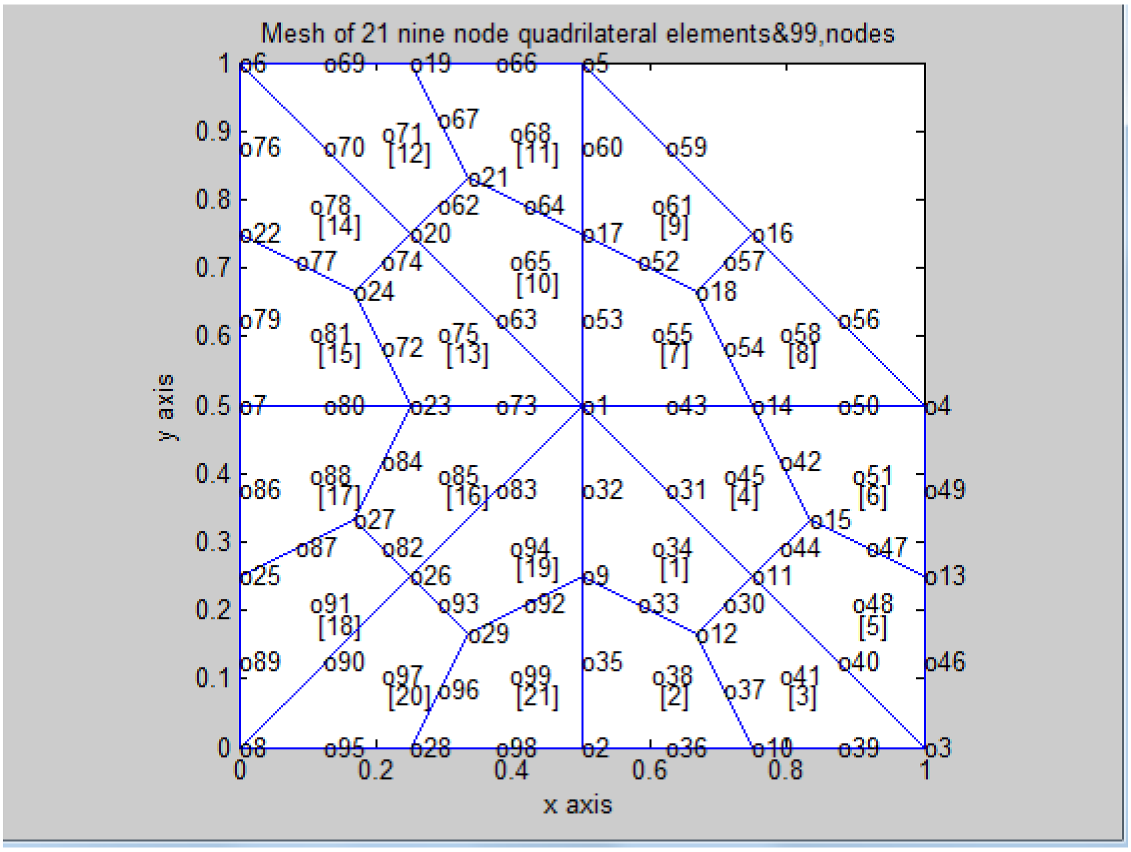


Fig 14:Initial Mesh for a pentagonal domain seven triangles(8-boundary nodes)

Example 5 :a square domain with eight triangles(9-nodes)

x=sym([0;   0; 1/2;1/2;1/2; 0;-1/2;-1/2;-1/2])
y=sym([0;-1/2;-1/2; 0;1/2;1/2; 1/2;   0;-1/2])
We use this mesh to solve torsion of a square cross section.We would like to draw countour lines of Prandtl stress function over the entire domain.This is a case of Poisson equation with constant right hand side(=-2)

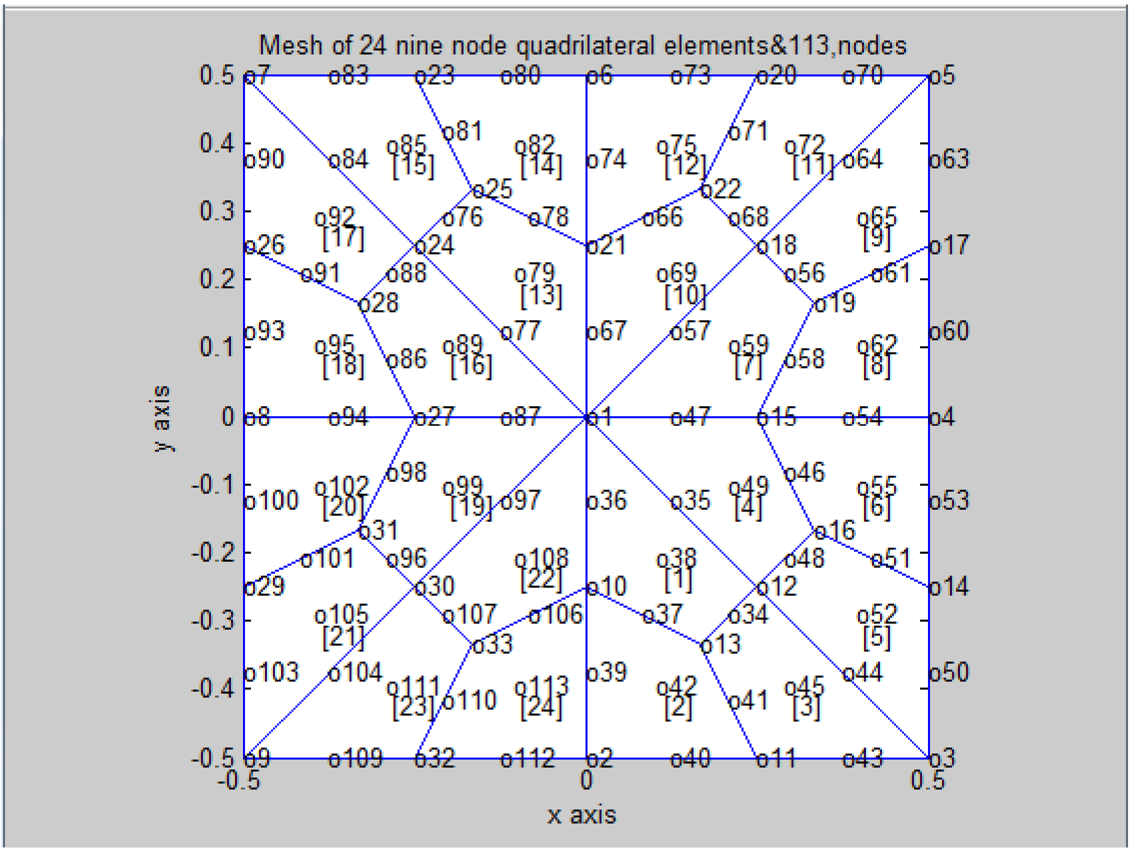


Fig 15: Initial Mesh for a square domain with eight triangles(9-boundary nodes)

7.5 Application Examples

7.5.1 Mesh Generation Over a Standard Triangle & a Square cross sections

Examples 1 &5

Let us use the explicit integration scheme and the auto mesh generation techniques which are developed in the previous sections to solve the Poisson Equation with Dirichlet boundary value problem:

$-\Delta \boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{x} \epsilon \Omega \subset \mathcal{R}^2$ ....(1)

$\boldsymbol{u} = \boldsymbol{g},$ $\boldsymbol{x} \epsilon \partial \Omega$

..(2)

Where Ω is a triangular or polygonal domain and Δ is the standard standard Laplace operator

In this section, we examine the application of the proposed explicit integration scheme to the Saint Venant Torsion problem [24]. Exact solutions of this problem for simple cross sections such as circle, ellipse, equilateral triangle and rectangle have been rigorously derived. These problems are described by the following boundary value problem ;

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta = 0$ in R -------------------- (62a)

$\phi = 0$ on ∂R , the boundary of R --------------------- (62b)

where ϕ(x,y) is known as Prandtl stress function, G is the shear modulus, θ is the angle of twist per unit length, R is the cross sectional region and ∂R is the boundary of R. We choose Gθ = 1 for the sake of simplicity. Then the corresponding torisonal constant is given by the equation

$.t_c = 2 \iint_R \phi(x, y) dx\, dy$ ---------------------------------------(62c)

We take R as the 9-node special quadrilateral meshes described in Examples 1 &5 in previous section.

In a recent paper[26] a new approach to automatic generation of all quadrilateral mesh for finite analysis is proposed and it was applied to discretise the 1/8-th of the square cross section a triangular region into an all quadrilateral mesh. We have demonstrated the proposed explicit integration scheme to solve the St. Venant Torsion problem for a square cross section. Monotonic convergence from below is observed with known analytical solutions for the Prandtl stress function and the torisonal constant which are expessed in terms of infinite series.This triangular domain is a right isosceles triangle and it was discretised by 8-noded special linear convex quadrilaterals of serendipity family.We have considered this problem again and illustrated the application of 9-node quadrilateral of Lagrange family.

7.5.2 Mesh Generation Over an Arbitrary Triangular Domain

Exmple 2

In applications to boundary value problems due to symmetry considerations or otherwise also, we may have to discretize an arbitrary triangle. Our purpose is to have a code which automatically generates convex quadrangulations of the domain by assuming the input as coordinates of the boundary vertices. We use the theory and procedure developed in section 7.2 and section 7.3 for this purpose..

Let us use the explicit integration scheme and the auto mesh generation techniques which are developed in the previous sections to solve the Poisson Equation with Dirichlet boundary value problem:
$\nabla^2 u = -1,\ x\epsilon\Omega \subset \mathcal{R}^2$
..(9)
$u = 0,$ $x\epsilon\partial\Omega$
..(10)
Where Ω is a regular triangular or polygonal domain and ∇^2 is the standard Laplace operator
Example 2
In this example, we would like to consider the linear elastic torsion of an equilateral triangle which is inscribed in a circle of unit radius.
In our recent we considered torsion of an equilateral triangular cross section and it was discretised by 8-noded special linear convex quadrilaterals of serendipity family.Now we would like to illustrate the t St. Venant Torsion problem for an arbitrary triangular cross section by using 9-node special linear convex quadrilaterals of Lagrange family.

The following MATLAB Codes-I are written for the Poisson Equation with Dirichlet boundary value problem, when the force vector is a constant(either=-1 or=-2)

(1)quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen.m
(2)D2LaplaceEquationQ9Ex3automeshgenNew.m
(3)nodaladdresses_special_convex_quadrilaterals_2nd_orderLG.m
(4)coordinate_arbitrarytriangle_2ndorderLAGR.m
(5)coordinate_special_quadrilaterals_in_stdtriangle_2nd_orderLAGR.m
(6)nodaladdresses4Lagrangespecial_convex_quadrilaterals_2nd_order.m
(7)D2LaplaceEquationQ9Ex3automeshgenNewPolygon.m
(8) polygonal_domain_coordinates_2nd_orderLG(see **MATLAB Codes-I I)**
(9) nodaladdresses_special_convex_quadrilaterals_trial_2nd_orderLG.m(see **MATLAB Codes-I I**)

7.5.3 Mesh Generation over a Convex Polygonal Domain

In several physical applications in science and engineering, the boundary value problem require meshes generated over convex polygons. Again our aim is to have a code which automatically generates a mesh of 9 noded convex quadrilaterals of Lagrange family for the complex domains such as those in [27-32]. We use the theory and procedure developed in sections 7.2, 7.3 and 7.4 for this purpose. The following MATLAB codes are written for this purpose.

Example 3(with nonconstant smooth function as right hand side of Poisson equation)

$-\Delta u = 2\pi^2 sin(\pi x)sin(\pi y)\ ,\ (x,y)\epsilon\Omega \subset \mathcal{R}^2$

$u(x,0) = 0, on\ \ y = 0, 0 \leq x \leq 1$

$u(x,1) = 0, on\ \ y = 1, 0 \leq x \leq 1\ ,$

$u(1, y) = 0, on\ x = 1, 0 \leq y \leq 1/2$,

$u(x, y) = sin(\pi x)sin(\pi y)$, **on the line x= 1 - 0.5t ,y=0.5+ 0.5t ,0≤** $t \leq 1$ **......................................(62)**

Where Δ is a standard Laplace operator and Ω is a pentagonal domain joining the vertices {(0,0),(1,0),(1,0.5),(0.5,1),(0,1)}

The exact solution of the above boundary value problem is $u(x, y) = sin(\pi x)sin(\pi y)$.

Example 4(with nonconstant smooth function as right hand side of Poisson equation)

$-\Delta u = 2\pi^2 sin(\pi x)sin(\pi y)$, $(x, y)\epsilon\Omega \subset \mathcal{R}^2$

u =0 , on the boundary $\partial\Omega$ **..(63)**

Where Δ is a standard Laplace operator and Ω is a square domain $[0, 1]^2$ **.**

We have written the following MATLAB Codes-II to solve the Poisson Equations with Dirichlet Boundary Conditions over linear convex polygonal domains when the force vector is a smooth continuous function ($= 2\pi^2 sin(\pi x)sin(\pi y)$)

(1)quadrilateral_mesh4MOINEX_q9LG.m
(2)polygonal_domain_coordinates_2nd_orderLG.m
(3)nodaladdresses_special_convex_quadrilaterals_trial_2nd_orderLG.m
(4)generate_area_coordinate_over_the_standard_triangle.m
(5)glsampleptsweights.m
(6)D2PoissonEquationQ9MoinEx_MeshgridContour.m

8.0 Conclusions

This paper presents the explicit integration scheme for a unique(special) linear convex 9- node quadrilateral which can be obtained from an arbitrary linear triangle by joining the centroid to the midpoints of sides of the triangle. The explicit integration scheme proposed for these unique linear convex 9- node quadrilaterals is derived by using the standard transformations in two steps. We first map an arbitrary linear triangle into a standard right isosceles triangle by using the affine linear transformation from global (x, y) space into a local space (u, v). We then discritise this standard right isosceles triangle in (u, v) space into three unique linear convex 9- node quadrilaterals. We have shown by proving a lemma that any unique linear convex 9-node quadrilateral in (x, y) space can be mapped into one of the unique 9-node quadrilaterals in (u, v) space. We have then mapped these linear convex 9- node quadrilaterals into a 2-square in the local (ξ, η) space by use of the bilinear transformation between (u, v) and (ξ, η) space. Using these two mappings, we have established an integral derivative product relation between the linear convex 9- node quadrilaterals in the global (x ,y) space interior to the arbitrary triangle and the linear convex 9- node quadrilaterals in the local (u, v) space which are interior to the standard right isosceles triangle. We have then shown that the product of global derivative integrals $S^{i,j,e}$ in global (x, y) space can be expressed as a matrix triple product $P * (K^{i,j,e}) * P^T * (2 *$ area of the arbitrary triangle in (x, y) space), in which P is a geometric properties matrix and $K^{i,j,e}$ is the product of global derivative integrals in (u, v) space, and $(i, j = 1, 2, 3, 4, 5, 6, 7, 8, 9)$. We have shown that the explicit integration of the global derivative products in (u, v) space over the unique 8- node quadrilateral spanning vertices {(1/3, 1/3), (1/6, 5/12), (0, ½), (0, ¼), (0, 0), (1/4, 0), (1/2, 0) , (5/12, 1/6)} and (5/24 ,5/24) is now possible by application of symbolic processing capabilities in MATLAB which are based on MAPLE –V mathematical software package. The proposed explicit integration scheme is a useful technique for boundary value problems governed by either a single or a system of partial differential equations. The physical applications of such problems are numerous in science , engineering,medical,business and social sciences. The well known examples are the Laplace and Poisson equations with suitable boundary conditions and the some examples of system of equations are the plane stress, plane strain and axisymmetric stress analysis, flow through porous media, shallow water circulation, dispersion and viscous incompressible flow etc in the areas of solid and fluid mechanics. We have first demonstrated the proposed explicit integration scheme to solve the St. Venant Torsion problem for an equilateral triangular cross section. Monotonic convergence from below is observed with known analytical solutions for the Prandtl stress function and the torisonal constant. We have demonstrated the proposed explicit integration scheme to solve the Poisson Boundary Value Problem for pentagonal and square domains which are to be considered as simple polygonal domains. Monotonic convergence from below is observed with known analytical solutions for the governing unknown function of Poisson Boundary Value Problem.We have shown the solutions in Tables which list both the FEM and exact solutions.The graphical solutions of nine noded quadrilateral meshes and contour level curves for FEM and exact solutions are also displayed. We conclude that efficient scheme on explicit integration of stiffness matrix and a novel automesh generation technique developed in this paper will be useful for the solution of many physical problems governed by second order partial differential equations.

We hope that the scheme developed in this paper will be useful for the solution of boundary value problems governed by second order partial differential equations with fast convergence and economy for the computational problems.

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APPENDIX-1

We now propose to compute the following integrals(see eqns(52a-b)) and they will be listed in Tables 1a-1i

$$\mathbf{K}^{\mathbf{i,j,e}} = \iint_{\hat{Q}} \mathbf{G}^{\mathbf{i,j,e}}_{\mathbf{u\,,v}} \ \mathbf{du\,dv} = \begin{pmatrix} \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial u}\,\mathbf{dudv} & \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial v}\,\mathbf{dudv} \\ \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial u}\,\mathbf{dudv} & \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial v}\,\mathbf{dudv} \end{pmatrix} = \begin{pmatrix} K^e_{2i-1\,,2j-1} & K^e_{2i-1\,,2j} \\ K^e_{2i\,,2j-1} & K^e_{2i\,,2j} \end{pmatrix} \text{(say)} \qquad \text{-----------}$$

(52a-b)

We map $\hat{Q}$ into a 2-square We map $\hat{Q}$ into a $2-$ square, $-1 \le \xi,\eta \le 1$

in the natural parametric space (ξ,η) by using the bilinear transformation from (u,v) to (ξ,η)

This gives(see eqns(18-20))

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix} = \frac{1}{J}\begin{bmatrix} \frac{\partial v}{\partial \eta} & -\frac{\partial v}{\partial \xi} \\ -\frac{\partial u}{\partial \eta} & \frac{\partial u}{\partial \xi} \end{bmatrix}\begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix}$$

Let us replace the Greek letters $\boldsymbol{\xi},\boldsymbol{\eta}$ by English letters **r,s** for computing the integrals by using MATLAB programming.With this assumption,we denote the entries of submatrix $\mathbf{K}^{\mathbf{i,j,e}}$ **= intJdnidnjuvrs ,(i,j=1(1)9)**

and we have from eqn(52a-b):

$$\textbf{intJdnidnjuvrs=}\begin{pmatrix} \textbf{intJdnidnjuvrs}(1,1) & \textbf{intJdnidnjuvrs}(1,2) \\ \textbf{intJdnidnjuvrs}(2,1) & \textbf{intJdnidnjuvrs}(2,2) \end{pmatrix}$$

$$K^e_{2i-1\,,2j-1} = \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial u}\,\mathbf{dudv} = \textbf{intJdnidnjuvrs}(1,1),\ K^e_{2i-1\,,2j} = \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial v}\,\mathbf{dudv} = \textbf{intJdnidnjuvrs}(1,2),$$

$$K^e_{2i\,,2j-1} = \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial u}\,\mathbf{dudv} = \textbf{intJdnidnjuvrs}(2,1),\ K^e_{2i\,,2j} = \iint_{\hat{Q}} \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial v}\,\mathbf{dudv} = \textbf{intJdnidnjuvrs}(2,2)$$

Tables 1a-1i

Table-1a

(i=1,j=1(1)9)

```
intJdn1dn1uvrs =[ 0.68800369699992917040751928886589, 0.63026863732902919654693031749554;...
                  0.63026863732902919654693031749554, 0.68800369699992917040751928886589]
intJdn1dn2uvrs = [ 0.049469766679465431322423128490783, -0.092492499790138905674372588717718;...
                  0.074174166876527760992294077948948, -0.058041502241256898516961218298311]
 intJdn1dn3uvrs = [ -0.027639187071864196478244612691051, -0.036406227705135716084660539278176;...
                  -0.036406227705135716084660539278176, -0.027639187071864196478244612691051]
intJdn1dn4uvrs = [ -0.058041502241256898516961218298311, 0.074174166876527760992294077948948;...
                  -0.092492499790138905674372588717718, 0.049469766679465431322423128490783]
intJdn1dn5uvrs =[ -0.42034020155248898256147959938904, 0.23835534506556861836117847035703;...
                  -0.42831132160109804830548819630964, 0.10203781293970562330870803190571]
intJdn1dn6uvrs =[  0.12626870005164588340746772851 13, 0.16865169113202404137334892155088;...
                  0.16865169113202404137334892155088, 0.15224655673422927054938212412807]
intJdn1dn7uvrs =[ 0.15224655673422927054938212412807, 0.16865169113202404137334892155088;...
                  0.16865169113202404137334892155088,  0.12626870005164588340746772851 13]
intJdn1dn8uvrs =[ 0.10203781293970562330870803190571, -0.42831132160109804830548819630964;...
                  0.23835534506556861836117847035703, -0.42034020155248898256147959938904]
intJdn1dn9uvrs =[ -0.61200564253936530143881487152335, -0.72289148243880098858257938459775;...
                  -0.72289148243880098858257938459775, -0.61200564253936530143881487152335]
```

Table-1b

(i=2,j=1(1)9)

```
intJdn2dn1uvrs =[  0.049469766679465431322423128490783,  0.074174166876527760992294077948948;...
                  -0.092492499790138905674372588717718, -0.058041502241256898516961218298311]
intJdn2dn2uvrs =[  0.24822242723570359859415910167674, -0.13943577781062161645243966285652;...
                 -0.13943577781062161645243966285652,  0.22525137401606776447455758470085]
intJdn2dn3uvrs =[ -0.093683290648218768462051538677453, -0.082852283681797324065968279843 49;...
                   0.083814382984869342600698386823177, 0.037904800898126436475032582647755]
intJdn2dn4uvrs =[ 0.00062925700025302140451751959728441,   0.026100052047122979851449223466987;...
                  0.026100052047122979851449223466987, 0.00062925700025302140451751959728447]
intJdn2dn5uvrs =[ -0.46868216752790132335135919627624,  -0.34646194452109534302782806359 57;...
                  0.32020472214557132363883860307097, 0.085836729209541381287229242816599]
intJdn2dn6uvrs =[  0.20486711030201898764584585313023,  0.25718774442435573105221634721679;...
                   -0.40947892224231093561445031944987, -0.28073862677651621463393803503496]
intJdn2dn7uvrs =[ 0.050885917641966498462888866392175,   -0.10335794828229434487750587372274;...
                  -0.10335794828229434487750587372274, -0.0017609016019421764322513358165054]
intJdn2dn8uvrs =[ -0.041736856619169902011341267601507, -0.094765689307712237483661393080961;...
                   -0.094765689307712237483661393080961,   0.033881015844374607003400451583 06]
```

```
intJdn2dn9uvrs =[ 0.050027835935882456394917533267998,   0.40941168025551439401144362446668;...
                   0.40941168025551439401144362446668, -0.042962146348647921061586792195777]
```

Table-1c

(i=3,j=1(1)9)

```
intJdn3dn1uvrs =[ -0.027639187071864196478244612691051, -0.036406227705135716084660539278176]
                  -0.036406227705135716084660539278176, -0.027639187071864196478244612691051;...
intJdn3dn2uvrs =[ -0.093683290648218768462051538677453, 0.083814382984869342600698386823177]
                   -0.08285228368179732406596827984349, 0.037904800898126436475032582647755;...
intJdn3dn3uvrs =[ 0.33203856905137126709551224250652,  0.29812794600013651335576030277 24;...
                   0.29812794600013651335576030277 24, 0.33203856905137126709551224250652]
intJdn3dn4uvrs =[ 0.037904800898126436475032582647755,  -0.082852283681797324065968279843 49;...
                  0.083814382984869342600698386823177, -0.093683290648218768462051538677453]
intJdn3dn5uvrs =[ 0.15710844821544667058002407563081,  0.12945549547911074908308207666547;...
                   0.12945549547911074908308207666547, 0.043059644229262196507731226976332]
intJdn3dn6uvrs =[ 0.14159217838663875388665351330935, -0.35360191630747700837476451284707;...
                   0.31306475035918965829190215381 96, -0.34086853405320051524855521401992]
intJdn3dn7uvrs =[ -0.34086853405320051524855521401992,  0.31306475035918965829190215381 96;...
                   -0.35360191630747700837476451284707, 0.14159217838663875388665351330935]
intJdn3dn8uvrs =[ 0.043059644229262196507731226976332, 0.12945549547911074908308207666547;...
                    0.12945549547911074908308207666547, 0.15710844821544667058002407563081]
intJdn3dn9uvrs =[ -0.24951262900756184435610227568234, -0.48105764260800696388913166477738;...
                   -0.48105764260800696388913166477738, -0.24951262900756184435610227568234]
```

Table-1d

(i=4,j=1(1)9)

```
intJdn4dn1uvrs =[ -0.058041502241256898516961218298311, -0.092492499790138905674372588717718;...
                    0.074174166876527760992294077948948,  0.049469766679465431322423128490783]
intJdn4dn2uvrs =[ 0.00062925700025302140451751959728441,   0.026100052047122979851449223466987;...
                    0.026100052047122979851449223466987, 0.00062925700025302140451751959728447]
intJdn4dn3uvrs =[ 0.037904800898126436475032582647755,  0.083814382984869342600698386823177;...
                  -0.082852283681797324065968279843 49, -0.093683290648218768462051538677453]
intJdn4dn4uvrs =[  0.22525137401606776447455758470085, -0.13943577781062161645243966285652;...
                   -0.13943577781062161645243966285652,  0.24822242723570359859415910167674]
intJdn4dn5uvrs =[   0.033881015844374607003400451583 06, -0.094765689307712237483661393080961;...
                   -0.094765689307712237483661393080961, -0.041736856619169902011341267601507]
intJdn4dn6uvrs =[ -0.0017609016019421764322513358165055, -0.10335794828229434487750587372274;...
                     -0.10335794828229434487750587372274, 0.050885917641966498462888866392175]
intJdn4dn7uvrs =[ -0.28073862677651621463393803503496, -0.40947892224231093561445031944987;...
                    0.25718774442435573105221634721679,  0.20486711030201898764584585313023]
intJdn4dn8uvrs =[ 0.085836729209541381287229242816599,  0.32020472214557132363883860307097;...
                   -0.34646194452109534302782806359 57, -0.46868216752790132335135919627624]
intJdn4dn9uvrs =[ -0.042962146348647921061586792195777,  0.40941168025551439401144362446668;...
                     0.40941168025551439401144362446668, 0.050027835935882456394917533267999]
```

Table-1e

(i=5,j=1(1)9)

```
intJdn5dn1uvrs =[ -0.42034020155248898256147959938904, -0.42831132160109804830548819630964;...
                    0.23835534506556861836117847035703,  0.10203781293970562330870803190571]
intJdn5dn2uvrs =[ -0.46868216752790132335135919627624,  0.32020472214557132363883860307097;...
                    -0.34646194452109534302782806359 57, 0.085836729209541381287229242816599]
intJdn5dn3uvrs =[ 0.15710844821544667058002407563081,  0.12945549547911074908308207666547;...
                    0.12945549547911074908308207666547, 0.043059644229262196507731226976332]
intJdn5dn4uvrs =[   0.033881015844374607003400451583 06, -0.094765689307712237483661393080961;...
                    -0.094765689307712237483661393080961, -0.041736856619169902011341267601507]
intJdn5dn5uvrs =[  1.0663384833344564856750376651589, 0.56556859863119073241392390206822;...
                   0.56556859863119073241392390206822,  1.1211333295329756651950424277428]
intJdn5dn6uvrs =[ -0.58306188548401601151324941680212, -0.54252818687130357695788942931127;...
                   -0.54252818687130357695788942931127, -0.22579554943131879593427777747996]
intJdn5dn7uvrs =[   -0.16767681050348731002396257406 51, -0.0017096478113284630633060979019 75;...
                  -0.0017096478113284630633060979019 75,   0.17132457681637017904834081106242]
intJdn5dn8uvrs =[ 0.090271486053820367920675971147722,  0.41024643803333829631989611851833;...
                   0.41024643803333829631989611851833, 0.090271486053820367920675971147722]
intJdn5dn9uvrs =[  0.29216163161979549627091262301201, -0.35816040869776877564539558371915;...
                  -0.35816040869776877564539558371915,  -1.3461311727311867153221086665701]
```

Table-1f

(i=6,j=1(1)9)

```
intJdn6dn1uvrs =[  0.1262687000516458834074677285113, 0.16865169113202404137334892155088]
                    0.16865169113202404137334892155088, 0.15224655673422927054938212412807;...
intJdn6dn2uvrs =[ 0.20486711030201898764584585313023, -0.40947892224231093561445031944987]
                   0.25718774442435573105221634721679, -0.28073862677651621463393803503496;...
intJdn6dn3uvrs =[  0.14159217838663875388665351330935,   0.3130647503591896582919021538196]
                   -0.35360191630747700837476451284707, -0.34086853405320051524855521401992;...
intJdn6dn4uvrs =[ -0.0017609016019421764322513358165055, -0.10335794828229434487750587372274]
                   -0.10335794828229434487750587372274, 0.050885917641966498462888866392175;...
intJdn6dn5uvrs =[ -0.58306188548401601151324941680212, -0.54252818687130357695788942931127;...
                    -0.54252818687130357695788942931127, -0.22579554943131879593427777747996]
intJdn6dn6uvrs =[ 1.5998510858095308546375928228576,   0.244693272250901099049116631754;...
                    0.244693272250901099049116631754, 0.64675514174853057100283264582228]
intJdn6dn7uvrs =[ -0.074612686016266341101136360446836,   0.42496611860422613701397774222377;...
                    0.42496611860422613701397774222377, -0.074612686016266341101136360446836]
intJdn6dn8uvrs =[   0.17132457681637017904834081106242, -0.001709647811328463063306097901975;...
                  -0.001709647811328463063306097901975,   -0.1676768105034873100239625740651]
intJdn6dn9uvrs =[  -1.5844681782639801295792636158054, -0.09430112713910361521519372896239;...
                  [ -0.09430112713910361521519372896239,  0.23980459065606283692676632470425]
```

Table-1g

(i=7,j=1(1)9)

```
intJdn7dn1uvrs =[ 0.15224655673422927054938212412807, 0.16865169113202404137334892155088;...
                    0.16865169113202404137334892155088,  0.1262687000516458834074677285113]
intJdn7dn2uvrs =[ 0.050885917641966498462888866392175,   -0.10335794828229434487750587372274;...
                   -0.10335794828229434487750587372274, -0.0017609016019421764322513358165054]
intJdn7dn3uvrs =[ -0.34086853405320051524855521401992, -0.35360191630747700837476451284707;...
                    0.3130647503591896582919021538196,  0.14159217838663875388665351330935]
intJdn7dn4uvrs =[ -0.28073862677651621463393803503496, 0.25718774442435573105221634721679;...
                   -0.40947892224231093561445031944987, 0.20486711030201898764584585313023]
intJdn7dn5uvrs =[   -0.1676768105034873100239625740651, -0.001709647811328463063306097901975;...
                   -0.001709647811328463063306097901975,   0.17132457681637017904834081106242]
intJdn7dn6uvrs =[ -0.074612686016266341101136360446836,   0.42496611860422613701397774222377;...
                    0.42496611860422613701397774222377, -0.074612686016266341101136360446836]
intJdn7dn7uvrs =[ 0.64675514174853057100283264582228,  0.244693272250901099049116631754;...
                    0.244693272250901099049116631754, 1.5998510858095308546375928228576]
intJdn7dn8uvrs =[ -0.22579554943131879593427777747996, -0.54252818687130357695788942931127;...
                    -0.54252818687130357695788942931127, -0.58306188548401601151324941680212]
intJdn7dn9uvrs =[   0.23980459065606283692676632470425, -0.094301127139103615215193728962391;...
                   -0.094301127139103615215193728962391,   -1.5844681782639801295792636158054]
```

Table-1h

(i=8,j=1(1)9)

```
intJdn8dn1uvrs =[  0.10203781293970562330870803190571,  0.23835534506556861836117847035703;...
                   -0.42831132160109804830548819630964, -0.42034020155248898256147959938904]
intJdn8dn2uvrs =[ -0.041736856619169902011341267601507, -0.094765689307712237483661393080961;...
                    -0.094765689307712237483661393080961,   0.03388101584437460700340045158306]
intJdn8dn3uvrs =[ 0.043059644229262196507731226976332, 0.12945549547911074908308207666547;...
                    0.12945549547911074908308207666547, 0.15710844821544667058002407563081]
intJdn8dn4uvrs =[ 0.085836729209541381287229242816599,  -0.3464619445210953430278280635957;...
                    0.32020472214557132363883860307097, -0.46868216752790132335135919627624]
intJdn8dn5uvrs =[ 0.090271486053820367920675971147722,  0.41024643803333829631989611851833;...
                    0.41024643803333829631989611851833, 0.090271486053820367920675971147722]
intJdn8dn6uvrs =[   0.17132457681637017904834081106242, -0.001709647811328463063306097901975;...
                   -0.001709647811328463063306097901975,   -0.1676768105034873100239625740651]
intJdn8dn7uvrs =[ -0.22579554943131879593427777747996, -0.54252818687130357695788942931127;...
                   -0.54252818687130357695788942931127, -0.58306188548401601151324941680212]
intJdn8dn8uvrs =[  1.1211333295329756651950424277428, 0.56556859863119073241392390206822;...
                     0.56556859863119073241392390206822,  1.0663384833344564856750376651589]
intJdn8dn9uvrs =[  -1.3461311727311867153221086665701, -0.35816040869776877564539558371915;...
                    -0.35816040869776877564539558371915,  0.29216163161979549627091262301201]
```

Table-1i

(i=9,j=1(1)9)

```
intJdn9dn1uvrs =[ -0.61200564253936530143881487152335, -0.72289148243880098858257938459775;...
                  -0.72289148243880098858257938459775, -0.61200564253936530143881487152335]
intJdn9dn2uvrs =[ 0.050027835935882456394917533267998,   0.40941168025551439401144362446668;...
                   0.40941168025551439401144362446668, -0.042962146348647921061586792195777]
intJdn9dn3uvrs =[ -0.24951262900756184435610227568234, -0.48105764260800696388913166477738;...
                  -0.48105764260800696388913166477738, -0.24951262900756184435610227568234]
intJdn9dn4uvrs =[ -0.042962146348647921061586792195777,  0.40941168025551439401144362446668;...
                   0.40941168025551439401144362446668, 0.050027835935882456394917533267999]
intJdn9dn5uvrs =[  0.29216163161979549627091262301201, -0.35816040869776877564539558371915;...
                  -0.35816040869776877564539558371915,  -1.3461311727311867153221086665701]
intJdn9dn6uvrs =[  -1.5844681782639801295792636158054, -0.09430112713910361521519372896239;...
                  -0.09430112713910361521519372896239,  0.23980459065606283692676632470425]
intJdn9dn7uvrs =[   0.23980459065606283692676632470425, -0.094301127139103615215193728962391;...
                  -0.094301127139103615215193728962391,   -1.5844681782639801295792636158054]
intJdn9dn8uvrs =[  -1.3461311727311867153221086665701, -0.35816040869776877564539558371915;...
                  -0.35816040869776877564539558371915,  0.29216163161979549627091262301201]
intJdn9dn9uvrs =[ 3.2530857106790011221652797407927, 1.2900488362095239461700024258049;...
                  1.2900488362095239461700024258049, 3.2530857106790011221652797407927]
```

SOME SAMPLE RESULTS (TABLES& FIGURES)

Table 2a(PRESENT PAPER)

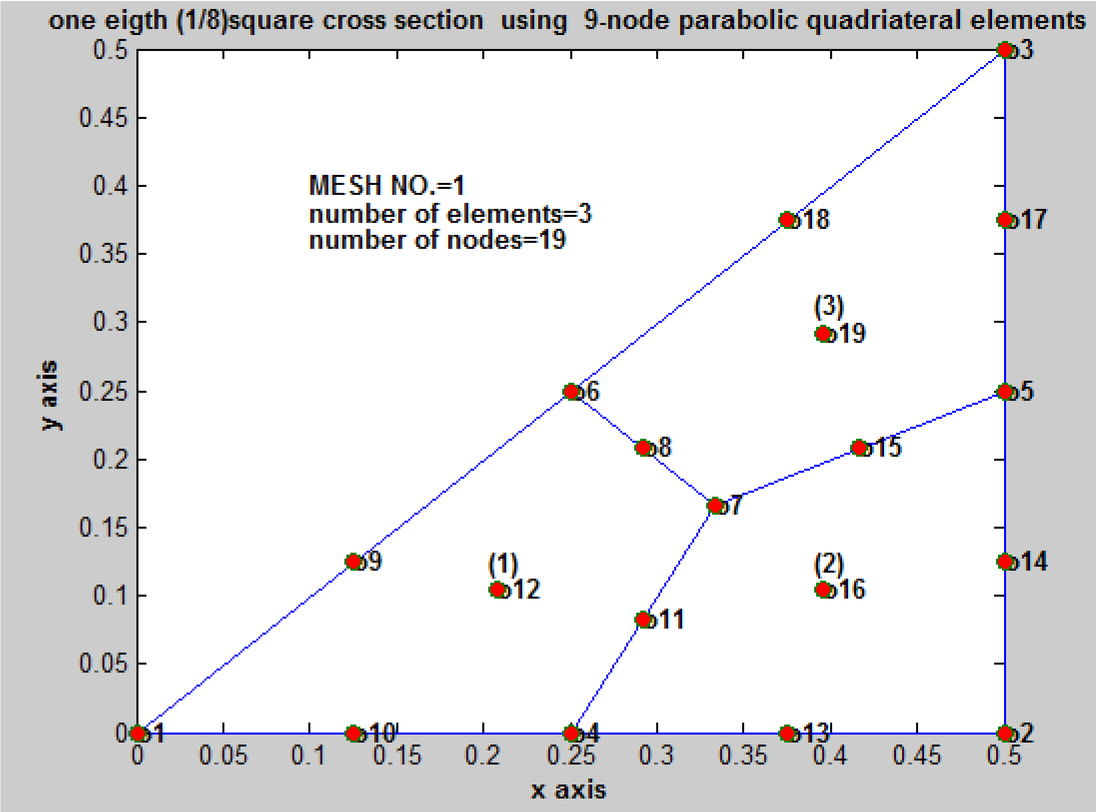
torsion of a square cross section modeled as a right isoscles triangle

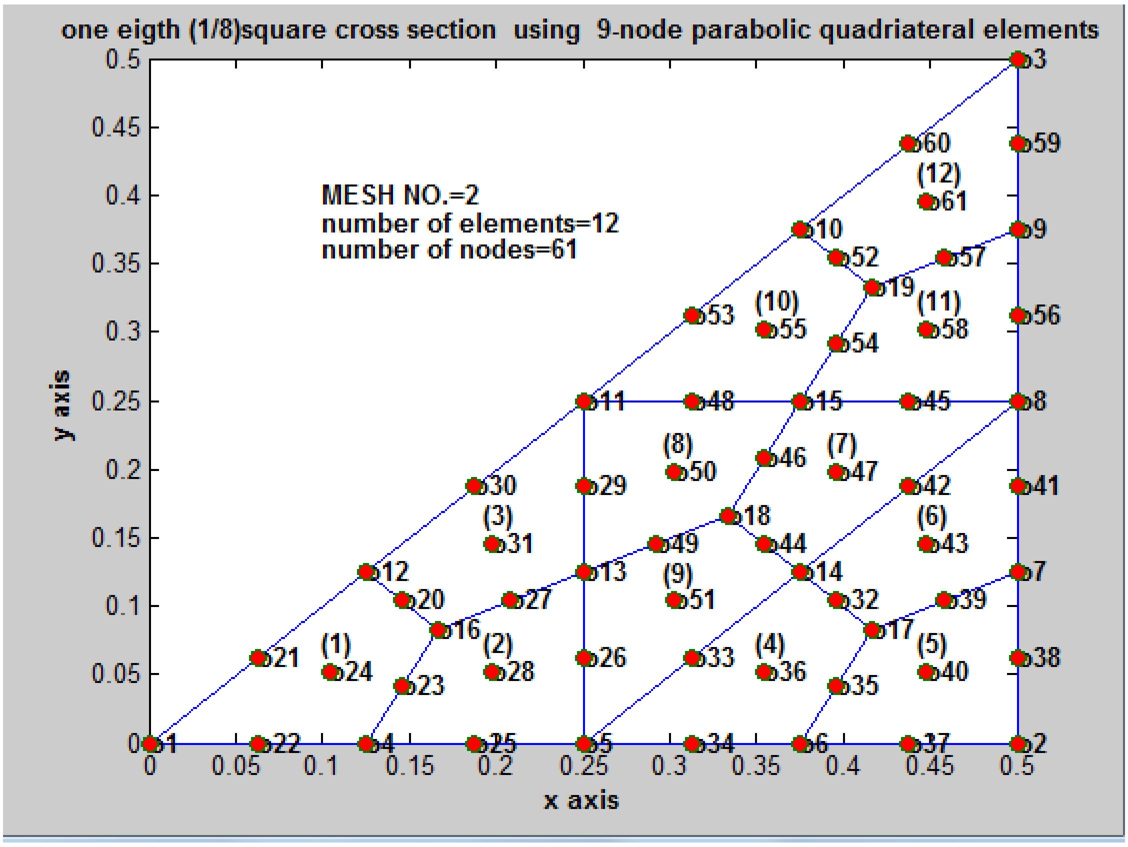
nine node -linear convex quadrilaterals(lagrange family elememts)

(exact solution of torisonal constant= 0.140577014955156)

nnode=number of nodes in theregion R ; nel=number of elements in theregion R;

Mesh No	nnode	nel	FEM solution for Torisonal constant	maximum absolute error of Prandtl Stress function values at element nodes
(1)	19	3	0.140226269123952	0.0011496140422405
(2)	61	12	0.140549995991093	0.000283838209277359
(3)	127	27	0.140571151900467	0.000125431921097136
(4)	217	48	0.140575044191592	7.22896125021954e-005
(5)	331	75	0.140576171269977	4.498537060151e-005
(6)	469	108	0.140576593774739	2.98546615531262e-005
(7)	631	147	0.140576781096277	2.23280855678916e-005
(8)	817	192	0.140576874567466	1.8370913205128e-005
(9)	1027	243	0 .140576925494304	1.58338405955379e-005
(10)	1261	300	0.140576955193951	1.38273942157716e-005


one eigth (1/8)square cross section using 9-node parabolic quadriateral elements
MESH NO.=1
number of elements=3
number of nodes=19
y axis
x axis
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
(1)
(2)
(3)
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19


one eigth (1/8)square cross section using 9-node parabolic quadriateral elements
MESH NO.=2
number of elements=12
number of nodes=61
y axis
x axis
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(9)
(10)
(11)
(12)
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
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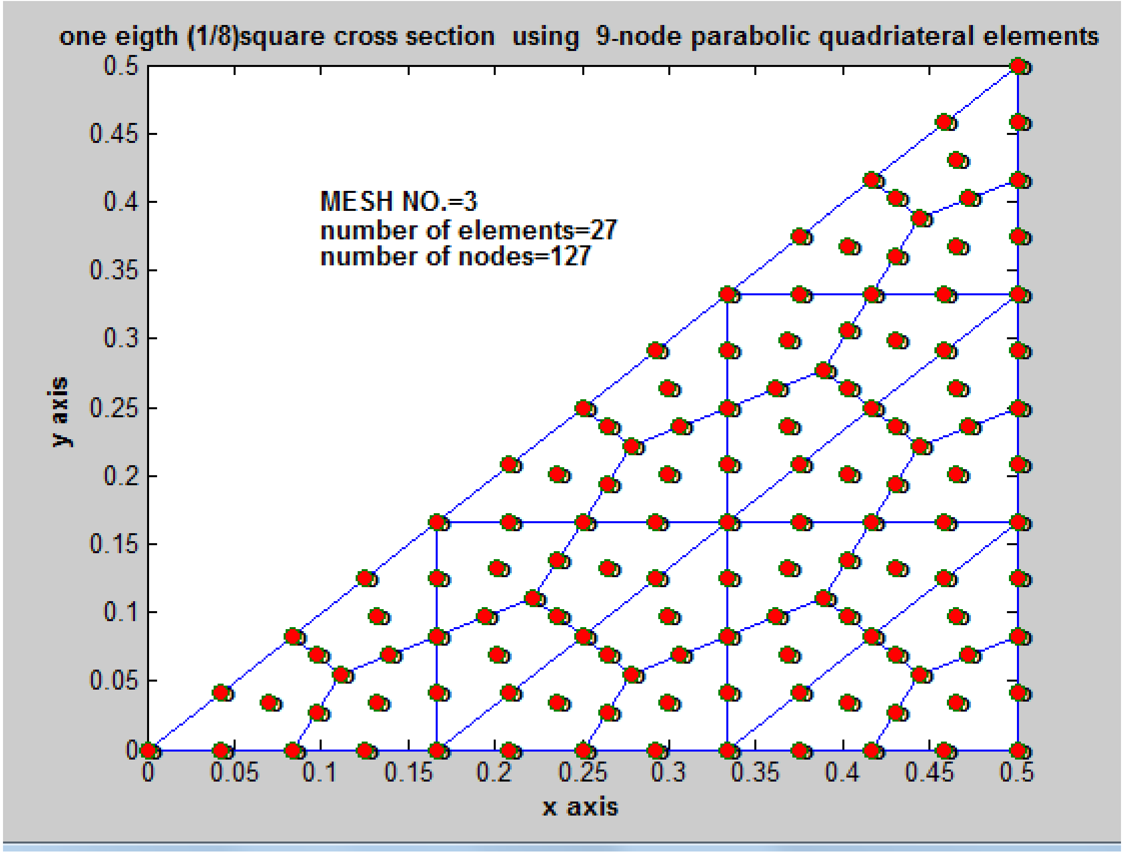
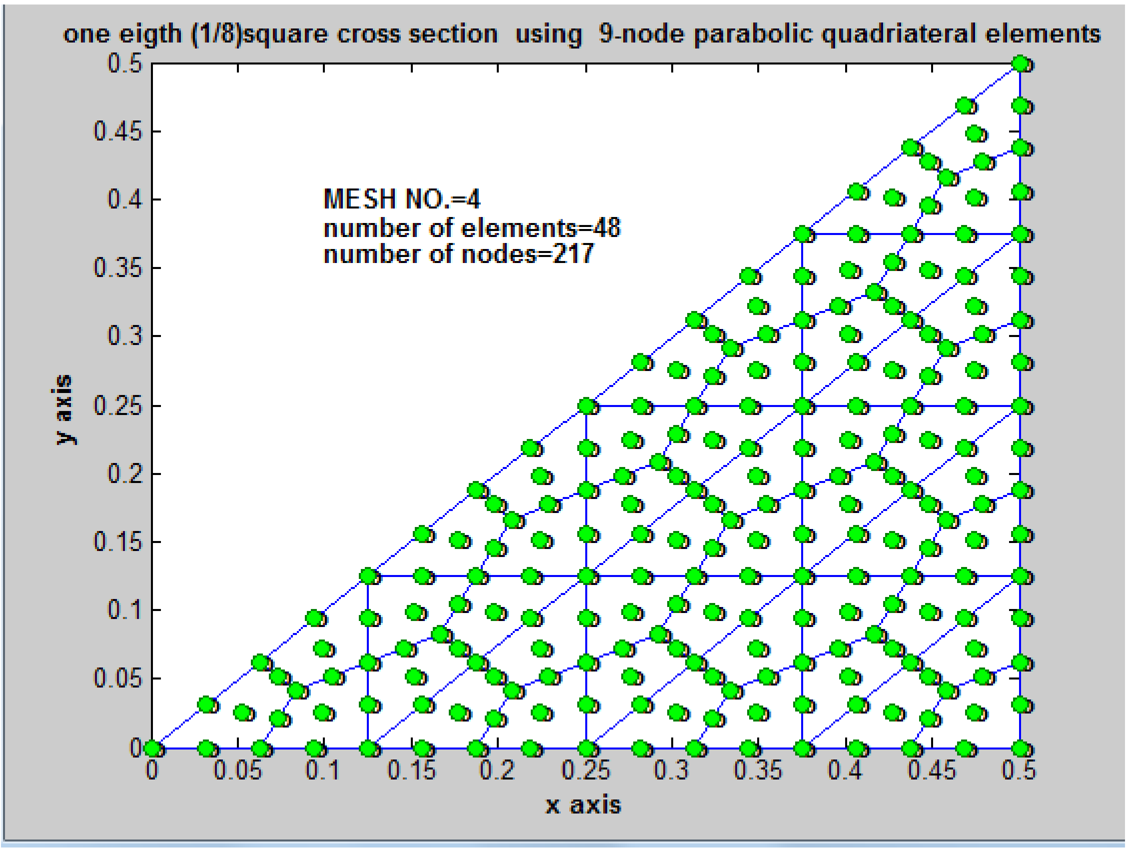
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60
61


one eigth (1/8)square cross section using 9-node parabolic quadriateral elements
MESH NO.=3
number of elements=27
number of nodes=127
y axis
x axis
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5


one eigth (1/8)square cross section using 9-node parabolic quadriateral elements
MESH NO.=4
number of elements=48
number of nodes=217
y axis
x axis
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

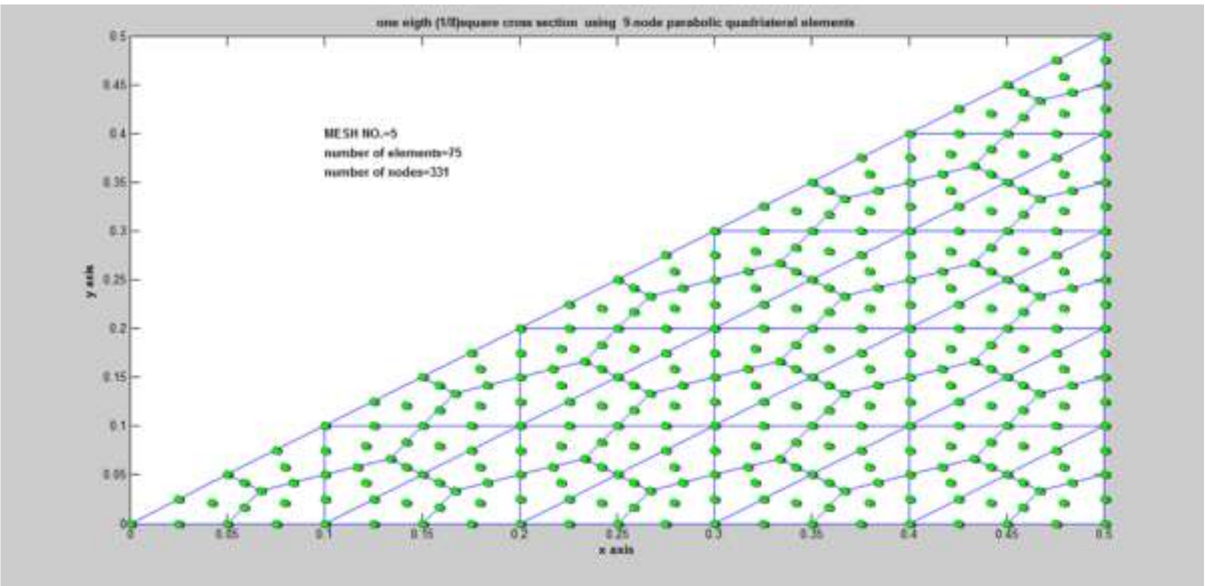


Table-2b

TORSION OF A SQUARE CROSS SECTION MODELED AS A RIGHT ISOSCLES TRIANGLE

EIGHT NODE -LINEAR CONVEX QUADRILATERALS(SERENDIPITY ELEMEMTS)

(exact solution of torisonal constant= 0.140577014955156)

nnode=number of nodes in theregion R ; nel=number of elements in theregion R;

(mesh no.)	nnode	nel	FEM Solution for torisonal constant REFERENCE[32]	maximum absolute error of Prandtl Stress function values at element nodes
(1)	16	3	0.139881192455598	0.00172214821773672
(2)	49	12	0.140511125235056	0.000228103295411064
(3)	100	27	0.140560146231912	0.000111318093756023
(4)	169	48	0.14057033232108	7.57868365891083e-005
(5)	256	75	0.14057364619955	5.71558452171442e-005
(6)	361	108	0.14057504105108	4.25097858819104e-005
(7)	484	147	0.14057573511073	3.45767462262801e-005
(8)	625	192	0.140576123341783	2.94391901602548e-005
(9)	784	243	0.140576360001611	2.42499917089884e-005
(10)	961	300	0.140576514028352	1.94431366115373e-005

(11)	1156	363	0.140576619547978	1.57123956216918e-005
(12)	1369	432	0.14057669486575	1.31847331217476e-005
(13)	1600	507	0.140576750456548	1.16333370254527e-005
(14)	1849	588	0.14057679264143	1.07440118374911e-005
(15)	2116	675	0.140576825408248	1.02449849120308e-005
(16)	2401	768	0.140576851369049	9.94170594790372e-006
(17)	2704	867	0.140576872290657	9.71043506567621e-006
(18)	3025	972	0.140576889401995	9.48027086123348e-006
(19)	3364	1083	0.140576903578949	9.21569212390368e-006
(20)	3721	1200	0.140576915459166	8.90311431203444e-006
(21)	4096	1323	0.140576925515857	8.54159601418455e-006
(22)	4489	1452	0.140576934105913	8.13682739378169e-006

Table-2c

TORSION OF A SQUARE CROSS SECTION MODELED AS A RIGHT ISOSCLES TRIANGLE

FOUR NODE-LINEAR CONVEX QUADRILATERALS {COMPUTED FROM REFERENCE[28])

(exact solution of torisonal constant= 0.140577014955156)

nnode=number of nodes in the region R ;

nel=number of elements in the region R;

			FOR FOUR NODE QUADRILATERAL MESHES	
nnode	nel	nnel	fem solution for torisonal constant] REFERENCE[28]	maximum absolute error of Prandtl Stress function values at element nodes
91	75	4	0.14016582079079	0.00118003991065224
331	300	4	0.140475648374825	0.000398482465845923
721	675	4	0.140532182472916	0.000204216075074687
1261	1200	4	0.140551856423775	0.000128187662553836
1951	1875	4	0.140560935627972	8.81447150146229e-005
2791	2700	4	0.140565858672553	6.3733272785143e-005

3781	3675	4	0.140568823558516	4.76848031509716e-005
4921	4800	4	0.14057074624724	3.6616338489467e-005
6211	6075	4	0.140572063602523	2.87090030208158e-005
7651	7500	4	0.140573005439404	2.28998710912698e-005

Table-3a(Present Work)

TORSION OF AN EQUILATERAL TRIANGULAR CROSS SECTION(NINE NODED ELEMENTS)

Mesh No	nodes	elements	TORISONAL CONSTANT VALUES FEM SOL	EXACT SOL	maximum absolute error of Prandtl Stress function values at element nodes
1	**19**	**3**	**3.10739407578924**	**3.11769145362398**	**0.00317246382279579**
2	**61**	**12**	**3.11704786750931**	**3.11769145362398**	**0.000396557977849543**
3	**127**	**27**	**3.11756432550257**	**3.11769145362398**	**0.000117498660104104**
4	**217**	**48**	**3.11765122949181**	**3.11769145362398**	**4.95697472316092e-005**
5	**331**	**75**	**3.11767497781946**	**3.11769145362398**	**2.53797105838616e-005**

Table-3b(refer [32])

TORSION OF AN EQUILATERAL TRIANGULAR CROSS SECTION(EIGHT NODED ELEMENTS)

Mesh No	nodes	elements	TORISONAL CONSTANT VALUES FEM SOL	EXACT SOL	maximum absolute error of Prandtl Stress function values at element nodes
1	**16**	**3**	**3.01771523127018**	**3.11769145362398**	**0.0326698575267504**

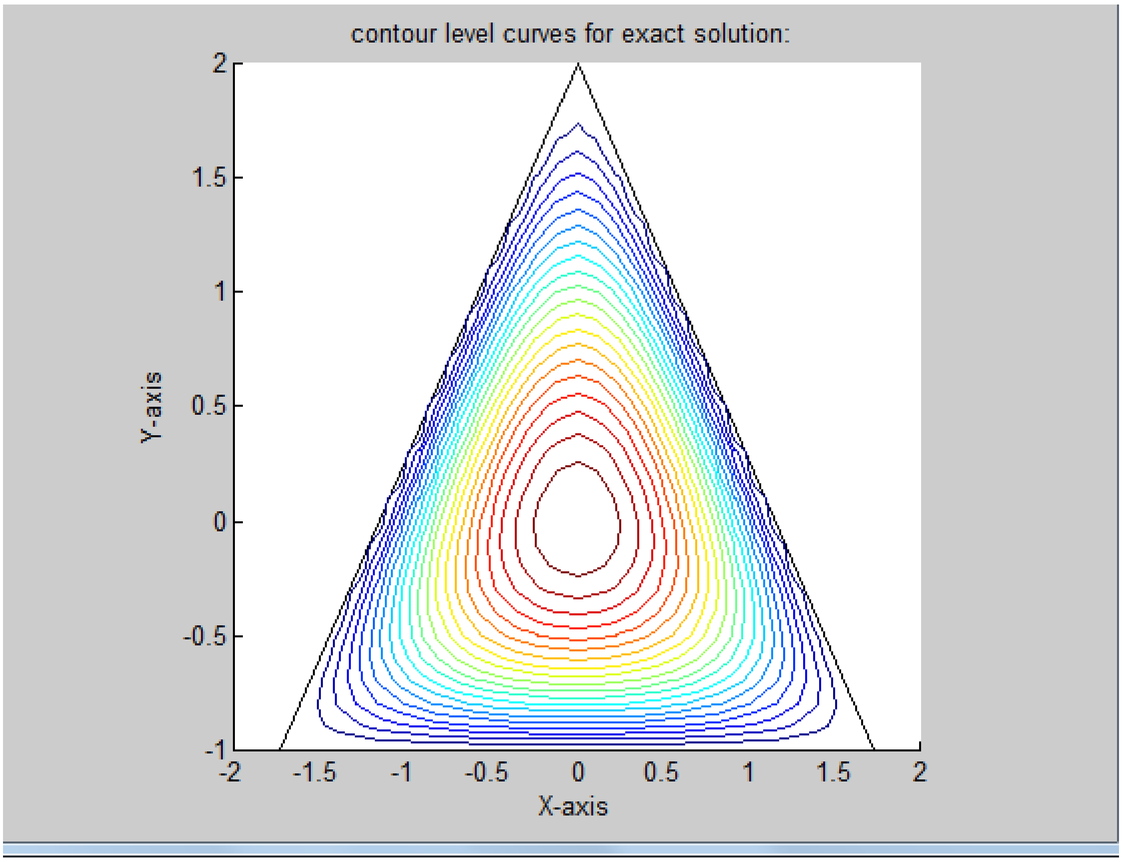
2	**49**	**12**	**3.10973432469345**	**3.11769145362398**	**0.00644969161285464**
3	**100**	**27**	**3.11557019336415**	**3.11769145362398**	**0.00209212772860204**
4	**169**	**48**	**3.1167795362198**	**3.11769145362398**	**0.00131858829294726**
5	**256**	**75**	**3.11719193923821**	**3.11769145362398**	**0.000908905458018264**

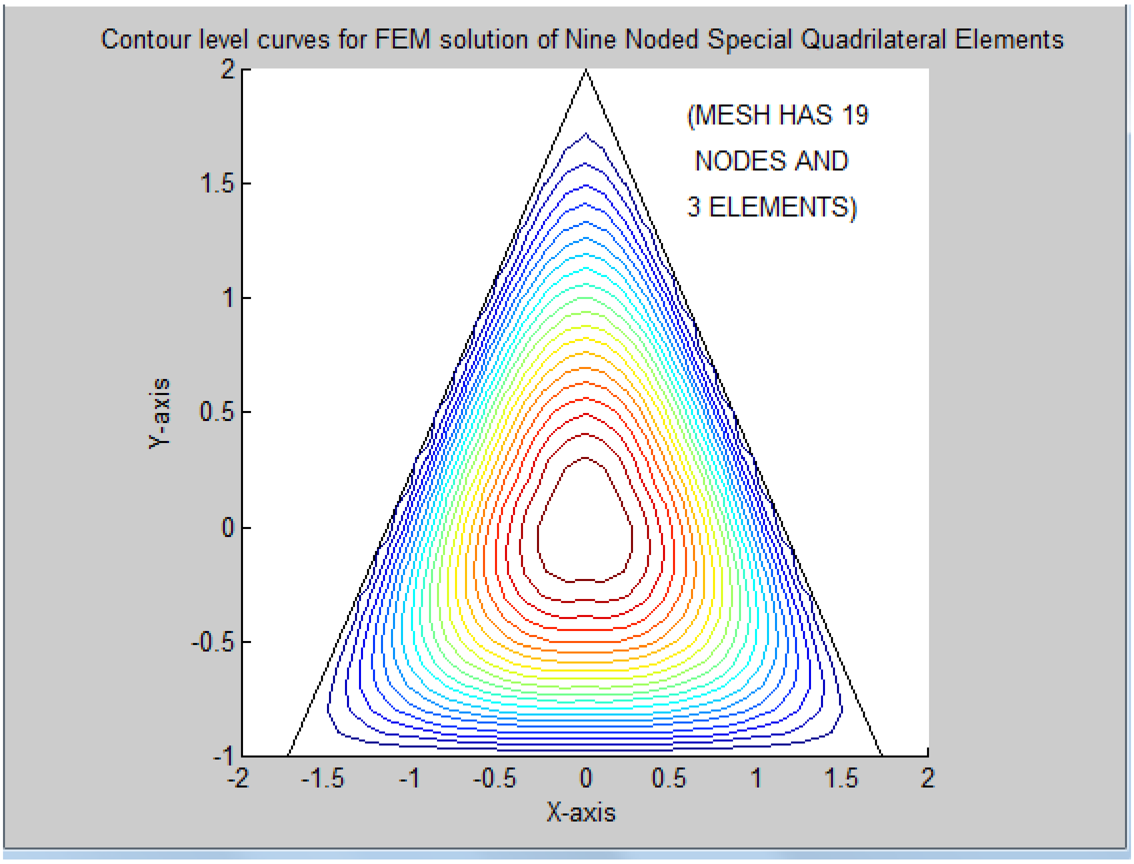
Table-3c(refer [28])

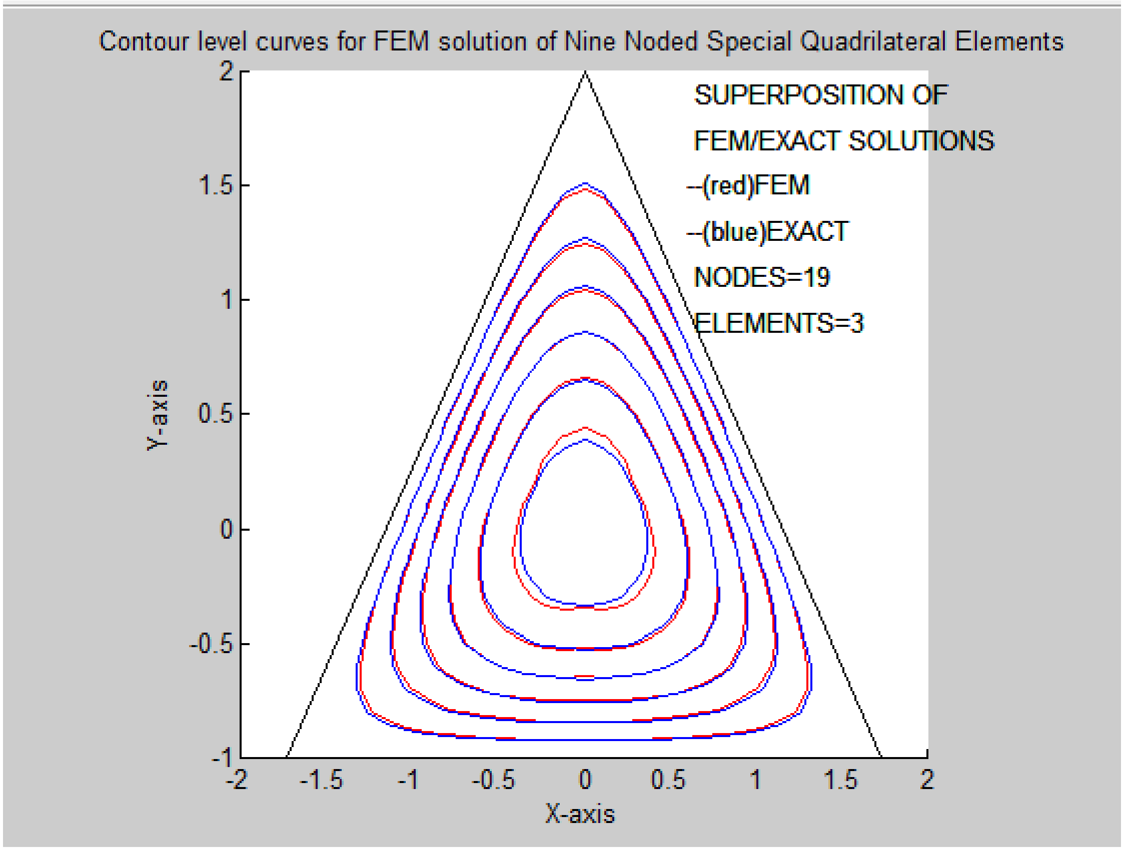
TORSION OF AN EQUILATERAL TRIANGULAR CROSS SECTION(FOUR NODED ELEMENTS)

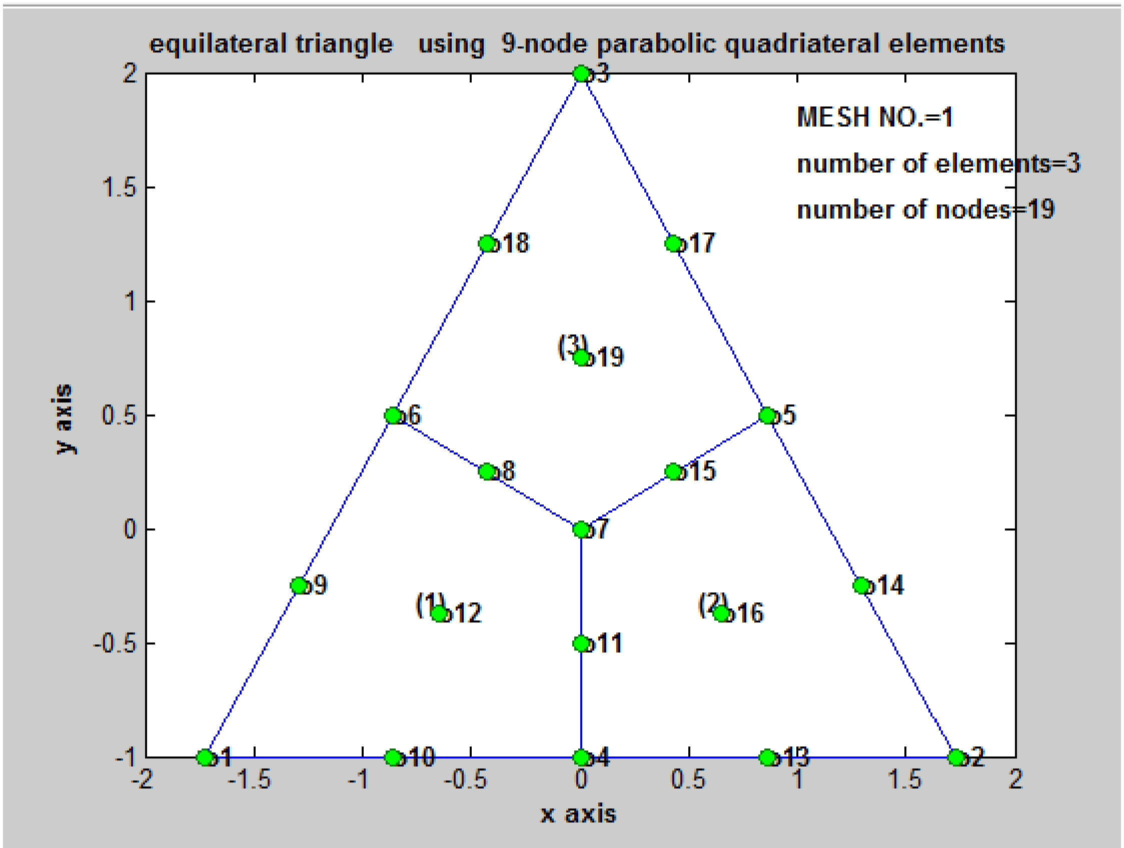
			TORISONAL CONSTANT VALUES		maximum absolute error
Mesh No	nodes	elements	FEM SOL	EXACT SOL	of Prandtl Stress function values at element nodes
1	7	3	1.8722884497	3.1176914536	1.2454030039
2	19	12	2.5125294041	3.1176914536	0.6051620495
3	37	27	2.8307089573	3.1176914536	0.2869824963
4	61	48	2.9539197663	3.1176914536	0.1637716873
5	91	75	3.0125356171	3.1176914536	0.1051558365
6	127	108	3.0446697570	3.1176914536	0.0730216966
7	169	147	3.0641026510	3.1176914536	0.0535888026
8	217	192	3.0767211933	3.1176914536	0.0409702604
9	271	243	3.0853673364	3.1176914536	0.0323241172
10	331	300	3.0915454781	3.1176914536	0.0261459755

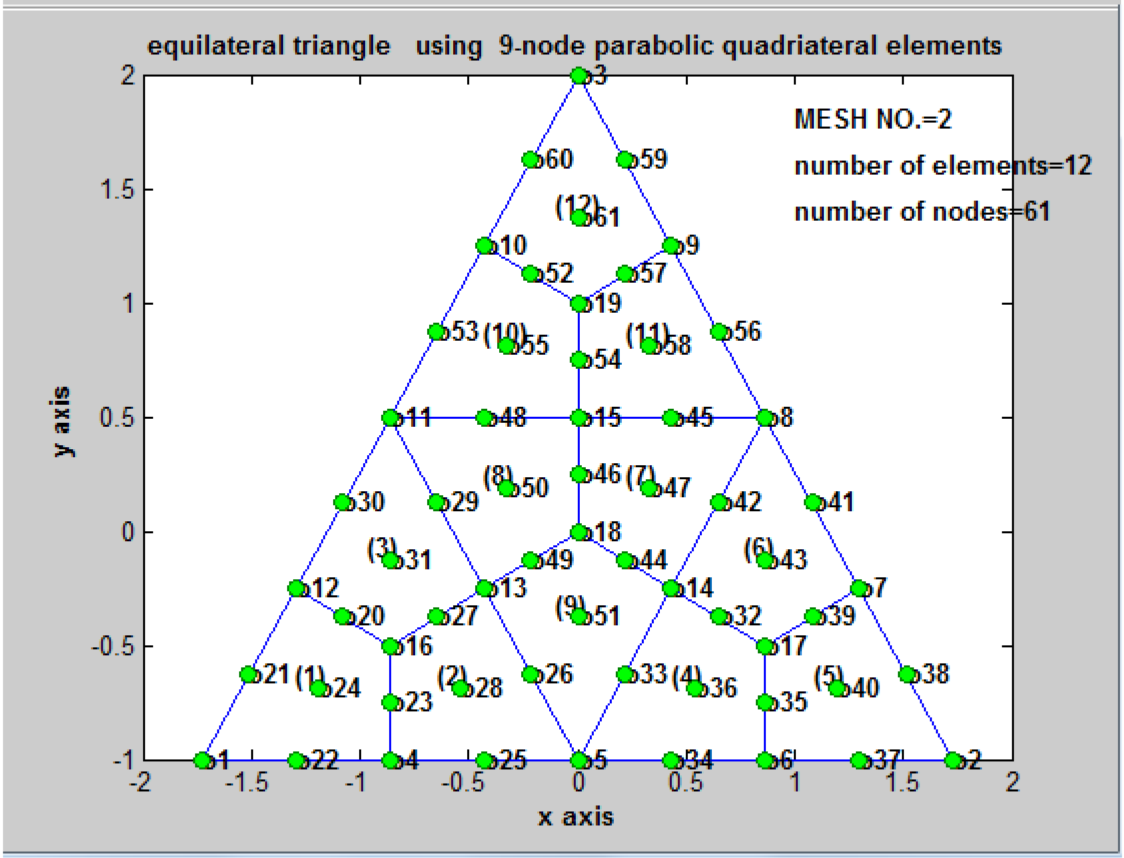
Example:TORSION OF A TRIANGULAR CROSS SECTION


contour level curves for exact solution:
Y-axis
X-axis
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
(MESH HAS 19
NODES AND
3 ELEMENTS)
Y-axis
X-axis
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
SUPERPOSITION OF
FEM/EXACT SOLUTIONS
--(red)FEM
--(blue)EXACT
NODES=19
ELEMENTS=3
Y-axis
X-axis
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


equilateral triangle using 9-node parabolic quadriateral elements
MESH NO.=1
number of elements=3
number of nodes=19
y axis
x axis
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
(1)
(2)
(3)


equilateral triangle using 9-node parabolic quadriateral elements
MESH NO.=2
number of elements=12
number of nodes=61
y axis
x axis
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(9)
(10)
(11)
(12)
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
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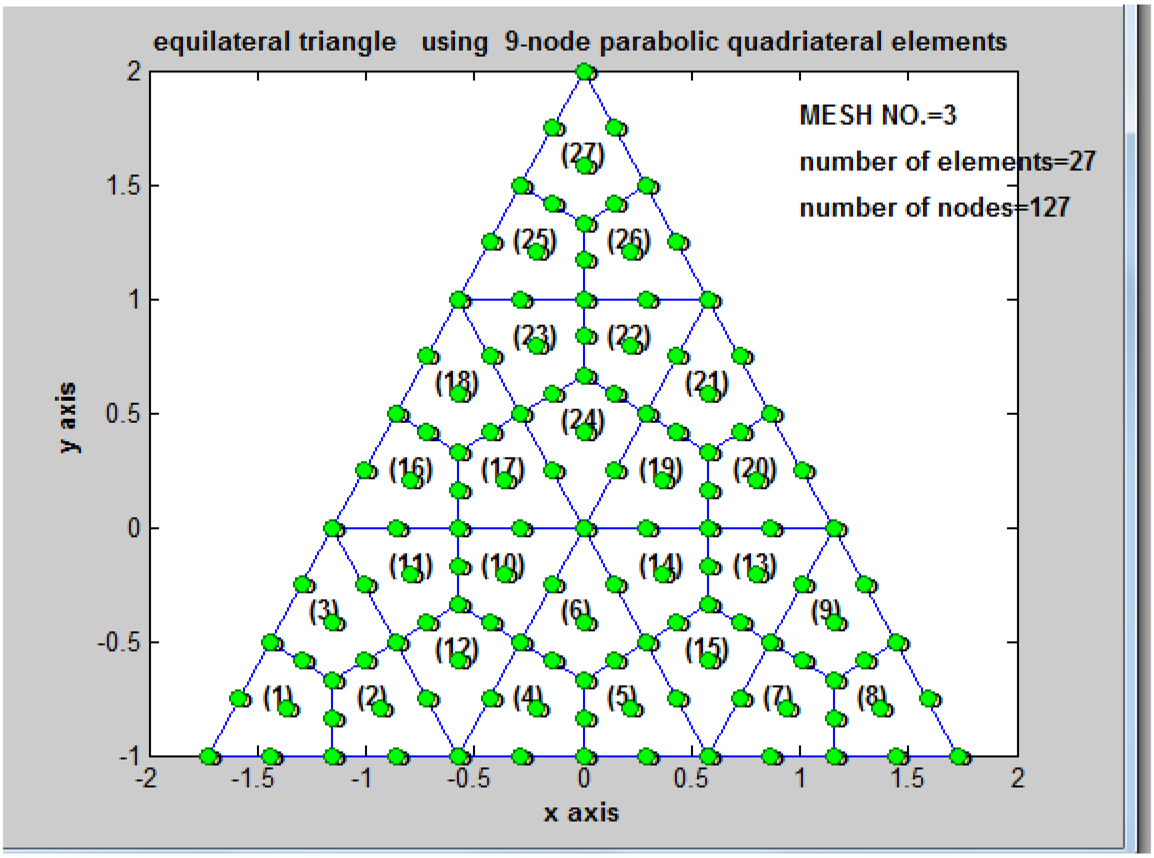
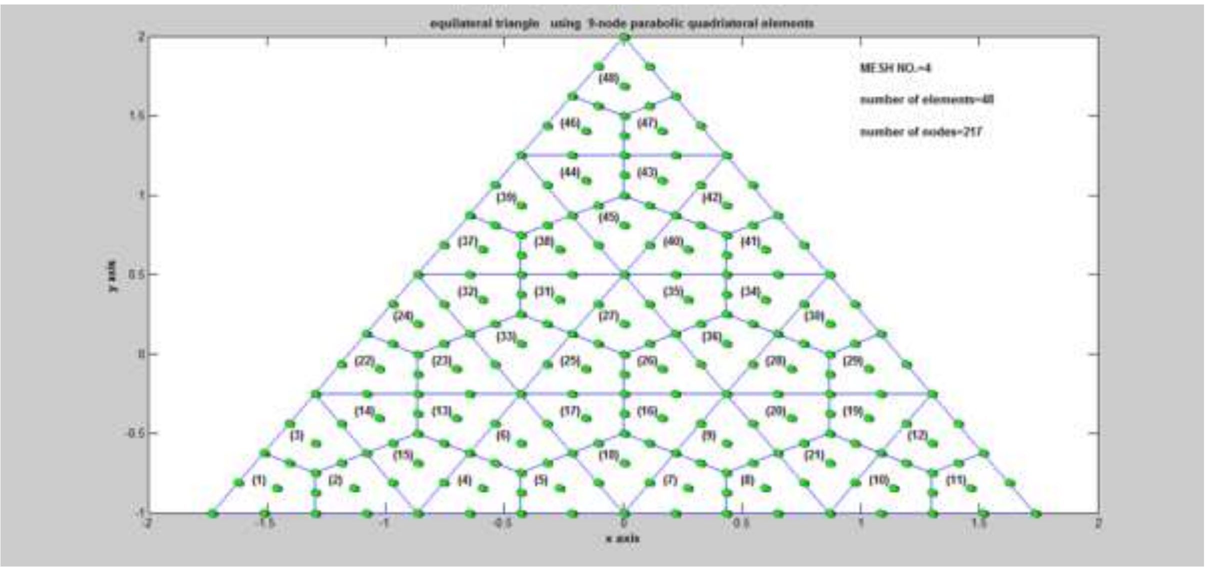
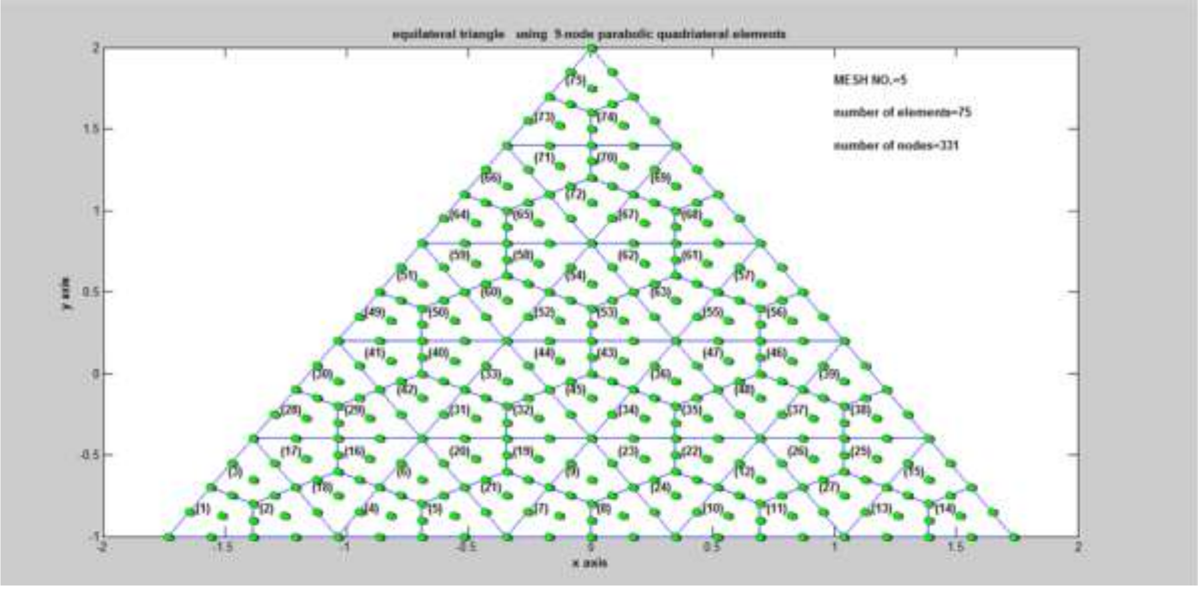
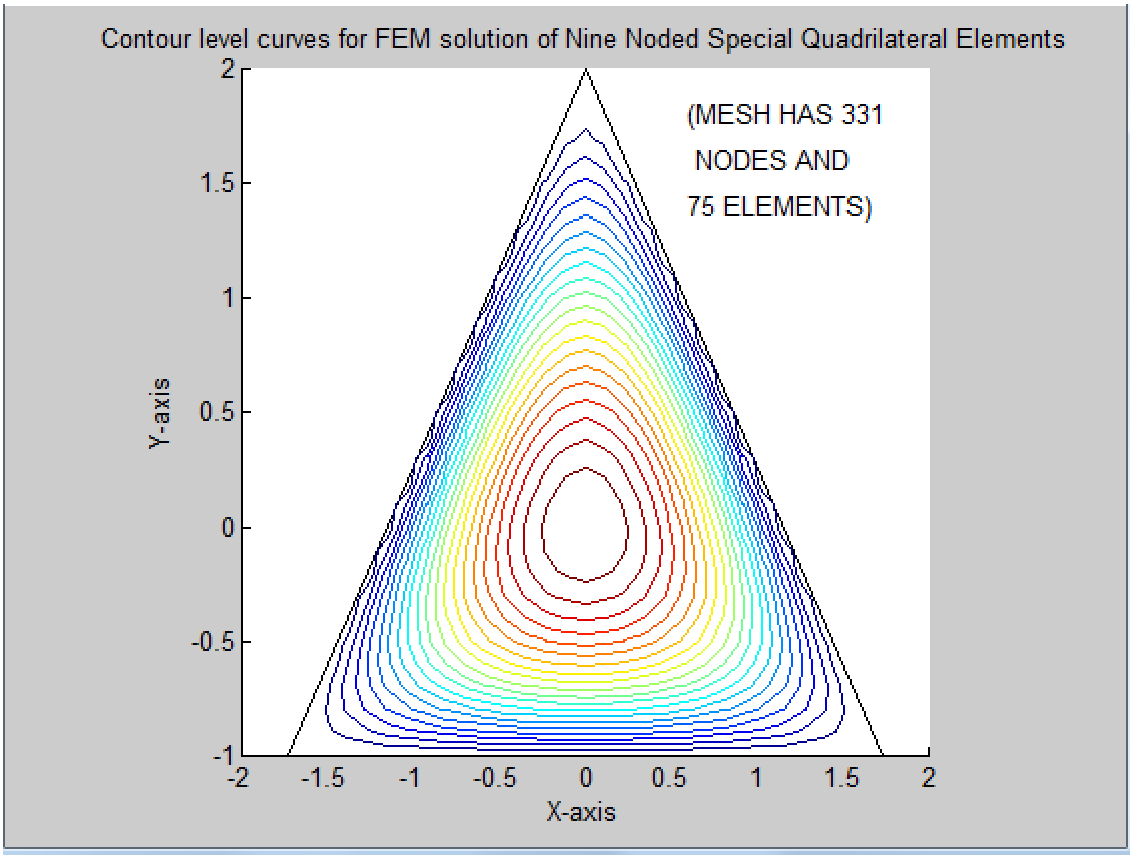
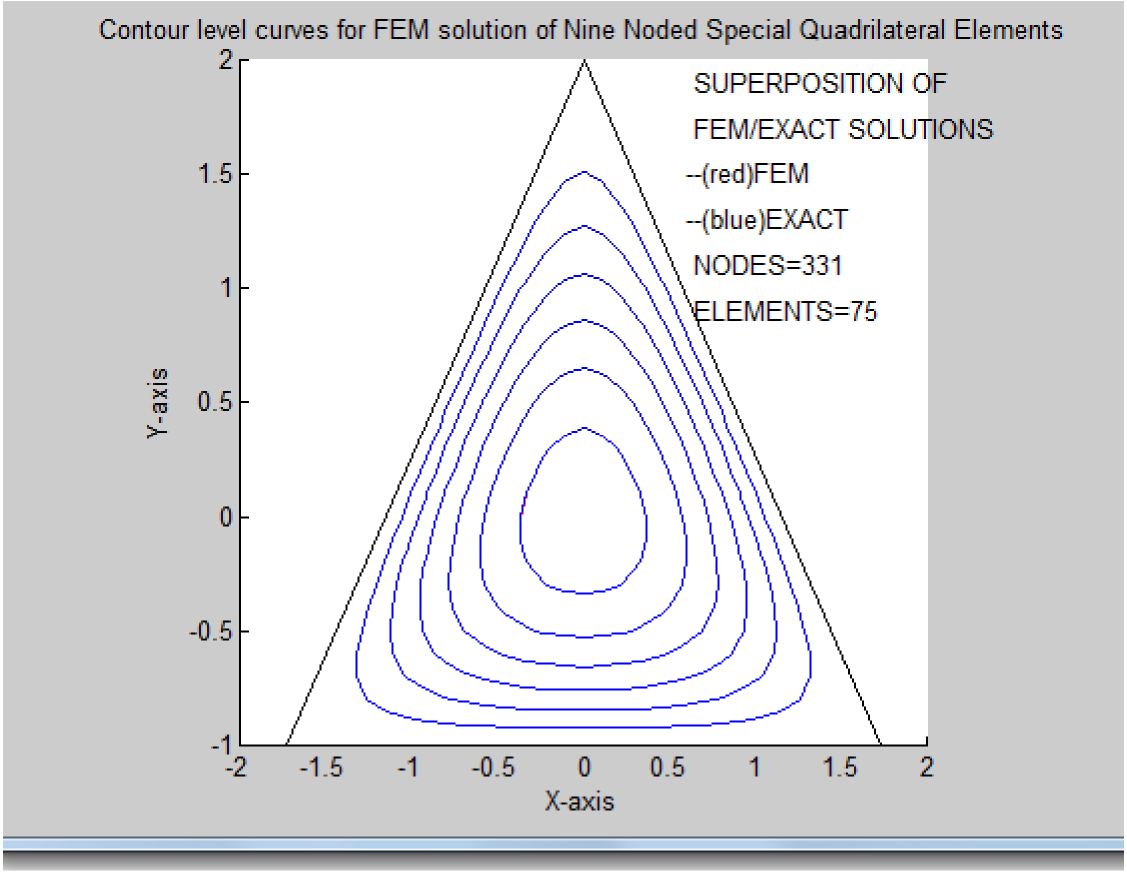
56
57
58
59
60
61


equilateral triangle using 9-node parabolic quadriateral elements
MESH NO.=3
number of elements=27
number of nodes=127
(27)
(25)
(26)
(23)
(22)
(18)
(21)
(24)
(16)
(17)
(19)
(20)
(11)
(10)
(14)
(13)
(3)
(6)
(9)
(12)
(15)
(1)
(2)
(4)
(5)
(7)
(8)
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
y axis
x axis


equilateral triangle using 9-node parabolic quadriateral elements
MESH NO.=4
number of elements=48
number of nodes=217
y axis
x axis


equilateral triangle using 9-node parabolic quadrilateral elements
MESH NO.=5
number of elements=75
number of nodes=331
(75)
(73) (74)
(71) (70)
(66) (69)
(72)
(64) (65) (67) (68)
(59) (58) (62) (61)
(51) (57)
(60) (54) (63)
(49) (50) (52) (53) (55) (56)
(41) (40) (44) (43) (47) (46)
(30) (33) (36) (39)
(42) (45) (48)
(28) (29) (31) (32) (34) (35) (37) (38)
(17) (16) (20) (19) (23) (22) (26) (25)
(3) (6) (9) (12) (15)
(18) (21) (24) (27)
(1) (2) (4) (5) (7) (8) (10) (11) (13) (14)
y axis
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x axis


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
(MESH HAS 331
NODES AND
75 ELEMENTS)
Y-axis
2
1.5
1
0.5
0
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
X-axis



Example5:TORSION OF A SQUARE CROSS SECTION

Table 4a
Torsion of a square cross section::Prandtl Stress Function Values
Mesh:Number of nodes=113,Number of nine noded elements=24

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
38	0.12058111589899	0.12051563971659	42	0.05815465754034	0.05825586477343	45	0.04329079617995	0.04418561671400
49	0.12058111589899	0.12051581611974	52	0.04329079617995	0.04418530139887	55	0.05815465754034	0.05825576227700
59	0.12058111589899	0.12051581611974	62	0.05815465754034	0.05825576227700	65	0.04329079617995	0.04418530139887
69	0.12058111589899	0.12051563971659	72	0.04329079617995	0.04418561671400	75	0.05815465754034	0.05825586477343
79	0.12058111589899	0.12051563971659	82	0.05815465754034	0.05825586477343	85	0.04329079617995	0.04418561671400
89	0.12058111589899	0.12051581611974	92	0.04329079617995	0.04418530139887	95	0.05815465754034	0.05825576227700
99	0.12058111589899	0.12051581611974	102	0.05815465754034	0.05825576227700	105	0.04329079617995	0.04418530139887

108	0.12058111589899	0.12051563971659	111	0.04329079617995	0.04418561671400	113	0.05815465754034	0.05825586477343

Table 4b
Torsion of a square cross section::Prandtl Stress Function Values
Mesh:Number of nodes=417,Number of nine noded elements=96

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
118	0.14058402174602	0.14057991011659	122	0.12606659834304	0.12607100298720	125	0.11819964167791	0.11822264404702
130	0.07829162319465	0.07828581395383	134	0.03230269835713	0.03230342146107	137	0.03061129750487	0.03065536488900
141	0.05313961558180	0.05317475469774	144	0.08633821959216	0.08635250522867	145	0.09538194827035	0.09536508048004
149	0.05554916029394	0.05549859083375	152	0.02370683846389	0.02371656300946	155	0.01561138492251	0.01583498199359
159	0.14058402174602	0.14057977294577	162	0.11819964167791	0.11822247696936	165	0.12606659834304	0.12607077188216
169	0.05554916029394	0.05549881852549	172	0.01561138492251	0.01583580314745	175	0.02370683846389	0.02371741735483
179	0.05313961558180	0.05317474613552	182	0.09538194827035	0.09536494479211	183	0.08633821959216	0.08635246347495
187	0.07829162319465	0.07828576878682	190	0.03061129750487	0.03065601040301	193	0.03230269835713	0.03230400294771
197	0.14058402174602	0.14057977294577	200	0.12606659834304	0.12607077188216	203	0.11819964167791	0.11822247696936
207	0.07829162319465	0.07828576878682	210	0.03230269835713	0.03230400294771	213	0.03061129750487	0.03065601040301
217	0.05313961558180	0.05317474613552	220	0.08633821959216	0.08635246347495	221	0.09538194827035	0.09536494479211
225	0.05554916029394	0.05549881852549	228	0.02370683846389	0.02371741735483	231	0.01561138492251	0.01583580314745
235	0.14058402174602	0.14057991011659	238	0.11819964167791	0.11822264404702	241	0.12606659834304	0.12607100298720
245	0.05554916029394	0.05549859083375	248	0.01561138492251	0.01583498199359	251	0.02370683846389	0.02371656300946
255	0.05313961558180	0.05317475469774	258	0.09538194827035	0.09536508048004	259	0.08633821959216	0.08635250522867
263	0.07829162319465	0.07828581395383	266	0.03061129750487	0.03065536488900	269	0.03230269835713	0.03230342146107
273	0.14058402174602	0.14057991011659	276	0.12606659834304	0.12607100298720	279	0.11819964167791	0.11822264404702
283	0.07829162319465	0.07828581395383	286	0.03230269835713	0.03230342146107	289	0.03061129750487	0.03065536488900

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  293  0.05313961558180  0.05317475469774  296  0.08633821959216  0.08635250522867  297  0.09538194827035
0.09536508048004
  301  0.05554916029394  0.05549859083375  304  0.02370683846389  0.02371656300946  307  0.01561138492251
0.01583498199359
  311  0.14058402174602  0.14057977294577  314  0.11819964167791  0.11822247696936  317  0.12606659834304
0.12607077188216
  321  0.05554916029394  0.05549881852549  324  0.01561138492251  0.01583580314745  327  0.02370683846389
0.02371741735483
  331  0.05313961558180  0.05317474613552  334  0.09538194827035  0.09536494479211  335  0.08633821959216
0.08635246347495
  339  0.07829162319465  0.07828576878682  342  0.03061129750487  0.03065601040301  345  0.03230269835713
0.03230400294771
  349  0.14058402174602  0.14057977294577  352  0.12606659834304  0.12607077188216  355  0.11819964167791
0.11822247696936
  359  0.07829162319465  0.07828576878682  362  0.03230269835713  0.03230400294771  365  0.03061129750487
0.03065601040301
  369  0.05313961558180  0.05317474613552  372  0.08633821959216  0.08635246347495  373  0.09538194827035
0.09536494479211
  377  0.05554916029394  0.05549881852549  380  0.02370683846389  0.02371741735483  383  0.01561138492251
0.01583580314745
  386  0.14058402174602  0.14057991011659  389  0.11819964167791  0.11822264404702  391  0.12606659834304
0.12607100298720
  395  0.05554916029394  0.05549859083375  398  0.01561138492251  0.01583498199359  401  0.02370683846389
0.02371656300946
  405  0.05313961558180  0.05317475469774  408  0.09538194827035  0.09536508048004  409  0.08633821959216
0.08635250522867
  412  0.07829162319465  0.07828581395383  415  0.03061129750487  0.03065536488900  417  0.03230269835713
0.03230342146107
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Table 4c

Torsion of a square cross section::Prandtl Stress Function Values

Mesh:Number of nodes=913,Number of nine noded elements=96

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NODE   FEM SOLUTION EXACT SOLUTION          NODE   FEM SOLUTION EXACT SOLUTION          NODE   FEM
SOLUTION EXACT SOLUTION

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  246  0.14433333219977  0.14433234669041  250  0.13796948006707  0.13797008232081  253  0.13412472870848
0.13412887952449
  258  0.11788082756962  0.11787916908081  262  0.09949468254560  0.09949520367557  265  0.09692684724613
0.09693472253870
  270  0.05652158769702  0.05651970194102  274  0.02219663795132  0.02219628932800  277  0.02170285319259
0.02171367177965
  281  0.10458047793871  0.10458730848097  284  0.11919187393044  0.11919400576816  285  0.12447574614932
0.12447195716961
  289  0.04021618706311  0.04022427127012  292  0.06919860279300  0.06920166800843  293  0.07193694214824
0.07193138178694
  297  0.10241983457569  0.10241275116633  300  0.08689301113676  0.08689467865435  303  0.07691875779399
0.07693039408998
  307  0.04997507629917  0.04996682333415  310  0.01980942844874  0.01980680670951  313  0.01786726473065
0.01788595244207
  317  0.03085537563365  0.03087043871473  320  0.05207864094458  0.05208350462548  321  0.06041612925117
0.06040621499874
  325  0.03130926700361  0.03128641504819  328  0.01290305114353  0.01290752628843  331  0.00818320410613
0.00828323169950
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335	0.14433333219977	0.14433246600812	338	0.13412472870848	0.13412908168048	341	0.13796948006707	0.13797018573330
345	0.10241983457569	0.10241283022985	348	0.07691875779399	0.07693025249116	351	0.08689301113676	0.08689462104669
355	0.03130926700361	0.03128590027657	358	0.00818320410613	0.00828304838643	361	0.01290305114353	0.01290762963338
365	0.10458047793871	0.10458713486162	368	0.12447574614932	0.12447168365388	369	0.11919187393044	0.11919374815765
373	0.03085537563365	0.03087119931359	376	0.06041612925117	0.06040655226683	377	0.05207864094458	0.05208397856475
381	0.11788082756962	0.11787909394632	384	0.09692684724613	0.09693460947652	387	0.09949468254560	0.09949499186990
391	0.04997507629917	0.04996672489415	394	0.01786726473065	0.01788638812152	397	0.01980942844874	0.01980732637832
401	0.04021618706311	0.04022471665081	404	0.07193694214824	0.07193152460283	405	0.06919860279300	0.06920182672954
409	0.05652158769702	0.05651944930300	412	0.02170285319259	0.02171413599477	415	0.02219663795132	0.02219665480144
419	0.14433333219977	0.14433246600812	422	0.13796948006707	0.13797018573330	425	0.13412472870848	0.13412908168048
429	0.11788082756962	0.11787909394632	432	0.09949468254560	0.09949499186990	435	0.09692684724613	0.09693460947652
439	0.05652158769702	0.05651944930300	442	0.02219663795132	0.02219665480144	445	0.02170285319259	0.02171413599477
449	0.10458047793871	0.10458713486162	452	0.11919187393044	0.11919374815765	453	0.12447574614932	0.12447168365388
457	0.04021618706311	0.04022471665081	460	0.06919860279300	0.06920182672954	461	0.07193694214824	0.07193152460283
465	0.10241983457569	0.10241283022985	468	0.08689301113676	0.08689462104669	471	0.07691875779399	0.07693025249116
475	0.04997507629917	0.04996672489415	478	0.01980942844874	0.01980732637832	481	0.01786726473065	0.01788638812152
485	0.03085537563365	0.03087119931359	488	0.05207864094458	0.05208397856475	489	0.06041612925117	0.06040655226683
493	0.03130926700361	0.03128590027657	496	0.01290305114353	0.01290762963338	499	0.00818320410613	0.00828304838643
503	0.14433333219977	0.14433234669041	506	0.13412472870848	0.13412887952449	509	0.13796948006707	0.13797008232081
513	0.10241983457569	0.10241275116633	516	0.07691875779399	0.07693039408998	519	0.08689301113676	0.08689467865435
523	0.03130926700361	0.03128641504819	526	0.00818320410613	0.00828323169950	529	0.01290305114353	0.01290752628843
533	0.10458047793871	0.10458730848097	536	0.12447574614932	0.12447195716961	537	0.11919187393044	0.11919400576816
541	0.03085537563365	0.03087043871473	544	0.06041612925117	0.06040621499874	545	0.05207864094458	0.05208350462548
549	0.11788082756962	0.11787916908081	552	0.09692684724613	0.09693472253870	555	0.09949468254560	0.09949520367557
559	0.04997507629917	0.04996682333415	562	0.01786726473065	0.01788595244207	565	0.01980942844874	0.01980680670951
569	0.04021618706311	0.04022427127012	572	0.07193694214824	0.07193138178694	573	0.06919860279300	0.06920166800843
577	0.05652158769702	0.05651970194102	580	0.02170285319259	0.02171367177965	583	0.02219663795132	0.02219628932800
587	0.14433333219977	0.14433234669041	590	0.13796948006707	0.13797008232081	593	0.13412472870848	0.13412887952449
597	0.11788082756962	0.11787916908081	600	0.09949468254560	0.09949520367557	603	0.09692684724613	0.09693472253870
607	0.05652158769702	0.05651970194102	610	0.02219663795132	0.02219628932800	613	0.02170285319259	0.02171367177965
617	0.10458047793871	0.10458730848097	620	0.11919187393044	0.11919400576816	621	0.12447574614932	0.12447195716961
625	0.04021618706311	0.04022427127012	628	0.06919860279300	0.06920166800843	629	0.07193694214824	0.07193138178694

633	0.10241983457569	0.10241275116633	636	0.08689301113676	0.08689467865435	639	0.07691875779399	0.07693039408998
643	0.04997507629917	0.04996682333415	646	0.01980942844874	0.01980680670951	649	0.01786726473065	0.01788595244207
653	0.03085537563365	0.03087043871473	656	0.05207864094458	0.05208350462548	657	0.06041612925117	0.06040621499874
661	0.03130926700361	0.03128641504819	664	0.01290305114353	0.01290752628843	667	0.00818320410613	0.00828323169950
671	0.14433333219977	0.14433246600812	674	0.13412472870848	0.13412908168048	677	0.13796948006707	0.13797018573330
681	0.10241983457569	0.10241283022985	684	0.07691875779399	0.07693025249116	687	0.08689301113676	0.08689462104669
691	0.03130926700361	0.03128590027657	694	0.00818320410613	0.00828304838643	697	0.01290305114353	0.01290762963338
701	0.10458047793871	0.10458713486162	704	0.12447574614932	0.12447168365388	705	0.11919187393044	0.11919374815765
709	0.03085537563365	0.03087119931359	712	0.06041612925117	0.06040655226683	713	0.05207864094458	0.05208397856475
717	0.11788082756962	0.11787909394632	720	0.09692684724613	0.09693460947652	723	0.09949468254560	0.09949499186990
727	0.04997507629917	0.04996672489415	730	0.01786726473065	0.01788638812152	733	0.01980942844874	0.01980732637832
737	0.04021618706311	0.04022471665081	740	0.07193694214824	0.07193152460283	741	0.06919860279300	0.06920182672954
745	0.05652158769702	0.05651944930300	748	0.02170285319259	0.02171413599477	751	0.02219663795132	0.02219665480144
755	0.14433333219977	0.14433246600812	758	0.13796948006707	0.13797018573330	761	0.13412472870848	0.13412908168048
765	0.11788082756962	0.11787909394632	768	0.09949468254560	0.09949499186990	771	0.09692684724613	0.09693460947652
775	0.05652158769702	0.05651944930300	778	0.02219663795132	0.02219665480144	781	0.02170285319259	0.02171413599477
785	0.10458047793871	0.10458713486162	788	0.11919187393044	0.11919374815765	789	0.12447574614932	0.12447168365388
793	0.04021618706311	0.04022471665081	796	0.06919860279300	0.06920182672954	797	0.07193694214824	0.07193152460283
801	0.10241983457569	0.10241283022985	804	0.08689301113676	0.08689462104669	807	0.07691875779399	0.07693025249116
811	0.04997507629917	0.04996672489415	814	0.01980942844874	0.01980732637832	817	0.01786726473065	0.01788638812152
821	0.03085537563365	0.03087119931359	824	0.05207864094458	0.05208397856475	825	0.06041612925117	0.06040655226683
829	0.03130926700361	0.03128590027657	832	0.01290305114353	0.01290762963338	835	0.00818320410613	0.00828304838643
838	0.14433333219977	0.14433234669041	841	0.13412472870848	0.13412887952449	843	0.13796948006707	0.13797008232081
847	0.10241983457569	0.10241275116633	850	0.07691875779399	0.07693039408998	853	0.08689301113676	0.08689467865435
857	0.03130926700361	0.03128641504819	860	0.00818320410613	0.00828323169950	863	0.01290305114353	0.01290752628843
867	0.10458047793871	0.10458730848097	870	0.12447574614932	0.12447195716961	871	0.11919187393044	0.11919400576816
875	0.03085537563365	0.03087043871473	878	0.06041612925117	0.06040621499874	879	0.05207864094458	0.05208350462548
882	0.11788082756962	0.11787916908081	885	0.09692684724613	0.09693472253870	887	0.09949468254560	0.09949520367557
891	0.04997507629917	0.04996682333415	894	0.01786726473065	0.01788595244207	897	0.01980942844874	0.01980680670951
901	0.04021618706311	0.04022427127012	904	0.07193694214824	0.07193138178694	905	0.06919860279300	0.06920166800843
908	0.05652158769702	0.05651970194102	911	0.02170285319259	0.02171367177965	913	0.02219663795132	0.02219628932800

Table 4d
Torsion of a square cross section::Prandtl Stress Function Values
Mesh:Number of nodes=1601,Number of nine noded elements=384

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
422	0.14564882193950	0.14564850166381	426	0.14208649815471	0.14208667583111	429	0.13985527731080	0.13985664851894
434	0.13101272754489	0.13101216481656	438	0.12108203921134	0.12108216535664	441	0.11925791310345	0.11926045648225
446	0.09848195119661	0.09848119842851	450	0.08091235217421	0.08091240989557	453	0.07979086601672	0.07979439352486
458	0.04405900279918	0.04405820257296	462	0.01688874318962	0.01688855310157	465	0.01668146633670	0.01668587125885
469	0.12310296984080	0.12310494525266	472	0.13130305077015	0.13130350351770	473	0.13452375325154	0.13452255122710
477	0.08852501053921	0.08852758581299	480	0.10382002358479	0.10382074166801	481	0.10622121710458	0.10621925474092
485	0.03167976690994	0.03168267057947	488	0.05597058300803	0.05597148564585	489	0.05713384426848	0.05713141455742
493	0.12127673851649	0.12127476157203	496	0.11225360105090	0.11225422334320	499	0.10550458118647	0.10550777492662
503	0.09162247079606	0.09161992564901	506	0.07547731092996	0.07547778781461	509	0.07128937674598	0.07129386969292
513	0.04132710409536	0.04132442898115	516	0.01590236529074	0.01590128555616	519	0.01513165919958	0.01513719627633
523	0.07669344895523	0.07669747832015	526	0.08951378454228	0.08951452455303	527	0.09631729400316	0.09631403414148
531	0.02793612832763	0.02794030007448	534	0.04899094354889	0.04899214129568	535	0.05232725976334	0.05232342964798
539	0.07377571467254	0.07377119917764	542	0.06125082843839	0.06125156071092	545	0.05304852373486	0.05305503958376
549	0.03411036569805	0.03410551405794	552	0.01328822289106	0.01328675186047	555	0.01174226234385	0.01175263634241
559	0.02023930647921	0.02024814929477	562	0.03476795312534	0.03477095544141	563	0.04116262484960	0.04115676184611
567	0.02025431875552	0.02024087103077	570	0.00820203998379	0.00820401150452	573	0.00509985809361	0.00515649923925
577	0.14564882193950	0.14564856305511	580	0.13985527731080	0.13985644002880	583	0.14208649815471	0.14208650531155
587	0.12127673851649	0.12127453045709	590	0.10550458118647	0.10550785252030	593	0.11225360105090	0.11225403661523
597	0.07377571467254	0.07377110019203	600	0.05304852373486	0.05305544972870	603	0.06125082843839	0.06125190256852
607	0.02025431875552	0.02024164756596	610	0.00509985809361	0.00515581715289	613	0.00820203998379	0.00820439611147
617	0.12310296984080	0.12310494605850	620	0.13452375325154	0.13452258444372	621	0.13130305077015	0.13130376864518
625	0.07669344895523	0.07669738967103	628	0.09631729400316	0.09631405816666	629	0.08951378454228	0.08951450419124
633	0.02023930647921	0.02024856279696	636	0.04116262484960	0.04115654922890	637	0.03476795312534	0.03477030198099
641	0.13101272754489	0.13101202830954	644	0.11925791310345	0.11926032639161	647	0.12108203921134	0.12108207323659

651 0.09162247079606 0.09161961957364 654 0.07128937674598 0.07129426878241 657 0.07547731092996 0.07547792258245

661 0.03411036569805 0.03410597899031 664 0.01174226234385 0.01175277769061 667 0.01328822289106 0.01328682492777

671 0.08852501053921 0.08852757556337 674 0.10622121710458 0.10621908086777 675 0.10382002358479 0.10382079970573

679 0.02793612832763 0.02794103507246 682 0.05232725976334 0.05232314207035 683 0.04899094354889 0.04899180933108

687 0.09848195119661 0.09848098696107 690 0.07979086601672 0.07979458493004 693 0.08091235217421 0.08091263927133

697 0.04132710409536 0.04132468682377 700 0.01513165919958 0.01513732656289 703 0.01590236529074 0.01590115159531

707 0.03167976690994 0.03168348397695 710 0.05713384426847 0.05713092908146 711 0.05597058300803 0.05597123208075

715 0.04405900279918 0.04405855502350 718 0.01668146633670 0.01668579391966 721 0.01688874318962 0.01688851372320

725 0.14564882193950 0.14564856305511 728 0.14208649815471 0.14208650531155 731 0.13985527731080 0.13985644002880

735 0.13101272754489 0.13101202830954 738 0.12108203921134 0.12108207323659 741 0.11925791310345 0.11926032639161

745 0.09848195119661 0.09848098696107 748 0.08091235217421 0.08091263927133 751 0.07979086601672 0.07979458493004

755 0.04405900279918 0.04405855502350 758 0.01688874318962 0.01688851372320 761 0.01668146633670 0.01668579391966

765 0.12310296984080 0.12310494605850 768 0.13130305077015 0.13130376864518 769 0.13452375325154 0.13452258444372

773 0.08852501053921 0.08852757556337 776 0.10382002358479 0.10382079970573 777 0.10622121710458 0.10621908086777

781 0.03167976690994 0.03168348397695 784 0.05597058300803 0.05597123208075 785 0.05713384426848 0.05713092908146

789 0.12127673851649 0.12127453045709 792 0.11225360105090 0.11225403661523 795 0.10550458118647 0.10550785252030

799 0.09162247079606 0.09161961957364 802 0.07547731092996 0.07547792258245 805 0.07128937674598 0.07129426878241

809 0.04132710409536 0.04132468682377 812 0.01590236529074 0.01590115159531 815 0.01513165919958 0.01513732656289

819 0.07669344895523 0.07669738967103 822 0.08951378454228 0.08951450419124 823 0.09631729400316 0.09631405816666

827 0.02793612832763 0.02794103507246 830 0.04899094354889 0.04899180933108 831 0.05232725976334 0.05232314207035

835 0.07377571467255 0.07377110019203 838 0.06125082843839 0.06125190256852 841 0.05304852373487 0.05305544972870

845 0.03411036569805 0.03410597899031 848 0.01328822289106 0.01328682492777 851 0.01174226234385 0.01175277769061

855 0.02023930647921 0.02024856279696 858 0.03476795312534 0.03477030198099 859 0.04116262484960 0.04115654922890

863 0.02025431875552 0.02024164756596 866 0.00820203998379 0.00820439611147 869 0.00509985809361 0.00515581715289

873 0.14564882193950 0.14564850166381 876 0.13985527731080 0.13985664851894 879 0.14208649815471 0.14208667583111

883 0.12127673851649 0.12127476157203 886 0.10550458118647 0.10550777492662 889 0.11225360105090 0.11225422334320

893 0.07377571467255 0.07377119917764 896 0.05304852373487 0.05305503958376 899 0.06125082843839 0.06125156071092

903 0.02025431875552 0.02024087103077 906 0.00509985809361 0.00515649923925 909 0.00820203998379 0.00820401150452

913 0.12310296984080 0.12310494525266 916 0.13452375325154 0.13452255122710 917 0.13130305077015 0.13130350351770

921 0.07669344895523 0.07669747832015 924 0.09631729400316 0.09631403414148 925 0.08951378454228 0.08951452455303

929 0.02023930647921 0.02024814929477 932 0.04116262484960 0.04115676184611 933 0.03476795312534 0.03477095544141

937 0.13101272754489 0.13101216481656 940 0.11925791310345 0.11926045648225 943 0.12108203921134 0.12108216535664

947 0.09162247079606 0.09161992564901 950 0.07128937674598 0.07129386969292 953 0.07547731092996 0.07547778781461

957 0.03411036569805 0.03410551405794 960 0.01174226234385 0.01175263634241 963 0.01328822289106 0.01328675186047

967 0.08852501053921 0.08852758581299 970 0.10622121710458 0.10621925474092 971 0.10382002358479 0.10382074166801

975 0.02793612832763 0.02794030007448 978 0.05232725976334 0.05232342964798 979 0.04899094354889 0.04899214129568

983 0.09848195119661 0.09848119842851 986 0.07979086601672 0.07979439352486 989 0.08091235217421 0.08091240989557

993 0.04132710409536 0.04132442898115 996 0.01513165919958 0.01513719627633 999 0.01590236529074 0.01590128555616

1003 0.03167976690994 0.03168267057947 1006 0.05713384426848 0.05713141455742 1007 0.05597058300803 0.05597148564585

1011 0.04405900279918 0.04405820257296 1014 0.01668146633670 0.01668587125885 1017 0.01688874318962 0.01688855310157

1021 0.14564882193950 0.14564850166381 1024 0.14208649815471 0.14208667583111 1027 0.13985527731080 0.13985664851894

1031 0.13101272754489 0.13101216481656 1034 0.12108203921134 0.12108216535664 1037 0.11925791310345 0.11926045648225

1041 0.09848195119661 0.09848119842851 1044 0.08091235217421 0.08091240989557 1047 0.07979086601672 0.07979439352486

1051 0.04405900279918 0.04405820257296 1054 0.01688874318962 0.01688855310157 1057 0.01668146633670 0.01668587125885

1061 0.12310296984080 0.12310494525266 1064 0.13130305077015 0.13130350351770 1065 0.13452375325154 0.13452255122710

1069 0.08852501053921 0.08852758581299 1072 0.10382002358479 0.10382074166801 1073 0.10622121710458 0.10621925474092

1077 0.03167976690994 0.03168267057947 1080 0.05597058300803 0.05597148564585 1081 0.05713384426848 0.05713141455742

1085 0.12127673851649 0.12127476157203 1088 0.11225360105090 0.11225422334320 1091 0.10550458118647 0.10550777492662

1095 0.09162247079606 0.09161992564901 1098 0.07547731092997 0.07547778781461 1101 0.07128937674598 0.07129386969292

1105 0.04132710409536 0.04132442898115 1108 0.01590236529074 0.01590128555616 1111 0.01513165919958 0.01513719627633

1115 0.07669344895523 0.07669747832015 1118 0.08951378454228 0.08951452455303 1119 0.09631729400316 0.09631403414148

1123 0.02793612832763 0.02794030007448 1126 0.04899094354889 0.04899214129568 1127 0.05232725976334 0.05232342964798

1131 0.07377571467255 0.07377119917764 1134 0.06125082843839 0.06125156071092 1137 0.05304852373487 0.05305503958376

1141 0.03411036569805 0.03410551405794 1144 0.01328822289106 0.01328675186047 1147 0.01174226234385 0.01175263634241

1151 0.02023930647921 0.02024814929477 1154 0.03476795312534 0.03477095544141 1155 0.04116262484960 0.04115676184611

1159 0.02025431875552 0.02024087103077 1162 0.00820203998379 0.00820401150452 1165 0.00509985809361 0.00515649923925

1169 0.14564882193950 0.14564856305511 1172 0.13985527731080 0.13985644002880 1175 0.14208649815471 0.14208650531155

1179 0.12127673851649 0.12127453045709 1182 0.10550458118647 0.10550785252030 1185 0.11225360105090 0.11225403661523

1189 0.07377571467255 0.07377110019203 1192 0.05304852373486 0.05305544972870 1195 0.06125082843839 0.06125190256852

1199 0.02025431875552 0.02024164756596 1202 0.00509985809361 0.00515581715289 1205 0.00820203998379 0.00820439611147

1209 0.12310296984080 0.12310494605850 1212 0.13452375325154 0.13452258444372 1213 0.13130305077015 0.13130376864518

1217 0.07669344895523 0.07669738967103 1220 0.09631729400316 0.09631405816666 1221 0.08951378454228 0.08951450419124

1225 0.02023930647921 0.02024856279696 1228 0.04116262484960 0.04115654922890 1229 0.03476795312534 0.03477030198099

1233 0.13101272754489 0.13101202830954 1236 0.11925791310345 0.11926032639161 1239 0.12108203921134 0.12108207323659

1243 0.09162247079606 0.09161961957364 1246 0.07128937674598 0.07129426878241 1249 0.07547731092997 0.07547792258245
1253 0.03411036569805 0.03410597899031 1256 0.01174226234385 0.01175277769061 1259 0.01328822289106 0.01328682492777
1263 0.08852501053921 0.08852757556337 1266 0.10622121710458 0.10621908086777 1267 0.10382002358479 0.10382079970573
1271 0.02793612832763 0.02794103507246 1274 0.05232725976334 0.05232314207035 1275 0.04899094354889 0.04899180933108
1279 0.09848195119661 0.09848098696107 1282 0.07979086601672 0.07979458493004 1285 0.08091235217421 0.08091263927133
1289 0.04132710409536 0.04132468682377 1292 0.01513165919958 0.01513732656289 1295 0.01590236529074 0.01590115159531
1299 0.03167976690994 0.03168348397695 1302 0.05713384426848 0.05713092908146 1303 0.05597058300803 0.05597123208075
1307 0.04405900279918 0.04405855502350 1310 0.01668146633670 0.01668579391966 1313 0.01688874318962 0.01688851372320
1317 0.14564882193950 0.14564856305511 1320 0.14208649815471 0.14208650531155 1323 0.13985527731080 0.13985644002880
1327 0.13101272754489 0.13101202830954 1330 0.12108203921134 0.12108207323659 1333 0.11925791310345 0.11926032639161
1337 0.09848195119661 0.09848098696107 1340 0.08091235217421 0.08091263927133 1343 0.07979086601672 0.07979458493004
1347 0.04405900279918 0.04405855502350 1350 0.01688874318962 0.01688851372320 1353 0.01668146633670 0.01668579391966
1357 0.12310296984080 0.12310494605850 1360 0.13130305077015 0.13130376864518 1361 0.13452375325154 0.13452258444372
1365 0.08852501053921 0.08852757556337 1368 0.10382002358479 0.10382079970573 1369 0.10622121710458 0.10621908086777
1373 0.03167976690994 0.03168348397695 1376 0.05597058300803 0.05597123208075 1377 0.05713384426848 0.05713092908146
1381 0.12127673851649 0.12127453045709 1384 0.11225360105090 0.11225403661523 1387 0.10550458118647 0.10550785252030
1391 0.09162247079606 0.09161961957364 1394 0.07547731092996 0.07547792258245 1397 0.07128937674598 0.07129426878241
1401 0.04132710409536 0.04132468682377 1404 0.01590236529074 0.01590115159531 1407 0.01513165919958 0.01513732656289
1411 0.07669344895523 0.07669738967103 1414 0.08951378454228 0.08951450419124 1415 0.09631729400316 0.09631405816666
1419 0.02793612832763 0.02794103507246 1422 0.04899094354889 0.04899180933108 1423 0.05232725976334 0.05232314207035
1427 0.07377571467255 0.07377110019203 1430 0.06125082843839 0.06125190256852 1433 0.05304852373486 0.05305544972870
1437 0.03411036569805 0.03410597899031 1440 0.01328822289106 0.01328682492777 1443 0.01174226234385 0.01175277769061
1447 0.02023930647921 0.02024856279696 1450 0.03476795312534 0.03477030198099 1451 0.04116262484960 0.04115654922890
1455 0.02025431875552 0.02024164756596 1458 0.00820203998379 0.00820439611147 1461 0.00509985809361 0.00515581715289
1464 0.14564882193950 0.14564850166381 1467 0.13985527731080 0.13985664851894 1469 0.14208649815471 0.14208667583111
1473 0.12127673851649 0.12127476157203 1476 0.10550458118647 0.10550777492662 1479 0.11225360105090 0.11225422334320
1483 0.07377571467255 0.07377119917764 1486 0.05304852373487 0.05305503958376 1489 0.06125082843839 0.06125156071092
1493 0.02025431875552 0.02024087103077 1496 0.00509985809361 0.00515649923925 1499 0.00820203998379 0.00820401150452
1503 0.12310296984080 0.12310494525266 1506 0.13452375325154 0.13452255122710 1507 0.13130305077015 0.13130350351770
1511 0.07669344895523 0.07669747832015 1514 0.09631729400316 0.09631403414148 1515 0.08951378454228 0.08951452455303
1519 0.02023930647921 0.02024814929477 1522 0.04116262484960 0.04115676184611 1523 0.03476795312534 0.03477095544141
1526 0.13101272754489 0.13101216481656 1529 0.11925791310345 0.11926045648225 1531 0.12108203921134 0.12108216535664

1535	0.09162247079606	0.09161992564901	1538	0.07128937674598	0.07129386969292	1541	0.07547731092997	0.07547778781461
1545	0.03411036569805	0.03410551405794	1548	0.01174226234385	0.01175263634241	1551	0.01328822289106	0.01328675186047
1555	0.08852501053921	0.08852758581299	1558	0.10622121710458	0.10621925474092	1559	0.10382002358479	0.10382074166801
1563	0.02793612832763	0.02794030007448	1566	0.05232725976334	0.05232342964798	1567	0.04899094354889	0.04899214129568
1570	0.09848195119661	0.09848119842851	1573	0.07979086601672	0.07979439352486	1575	0.08091235217421	0.08091240989557
1579	0.04132710409536	0.04132442898115	1582	0.01513165919958	0.01513719627633	1585	0.01590236529074	0.01590128555616
1589	0.03167976690994	0.03168267057947	1592	0.05713384426848	0.05713141455742	1593	0.05597058300803	0.05597148564585
1596	0.04405900279918	0.04405820257296	1599	0.01668146633670	0.01668587125885	1601	0.01688874318962	0.01688855310157

Table 4e

Torsion of a square cross section::Prandtl Stress Function Values

Mesh:Number of nodes=2481,Number of nine noded elements=600

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
646	0.14625828214478	0.14625799334796	650	0.14398352999959	0.14398344465129	653	0.14253524587494	0.14253564500567
658	0.13696332947999	0.13696293746972	662	0.13072998931971	0.13072987760096	665	0.12944879535718	0.12944967327349
670	0.11673713166364	0.11673663554891	674	0.10602115913807	0.10602101141591	677	0.10503329287795	0.10503461041355
682	0.08391873690481	0.08391817314443	686	0.06788706591390	0.06788688630284	689	0.06730349895817	0.06730515385109
694	0.03605924523629	0.03605867528529	698	0.01362510608026	0.01362484287590	701	0.01351922271267	0.01352125998214
705	0.13177364211724	0.13177436324512	708	0.13701663213118	0.13701672541751	709	0.13915146714621	0.13915080245345
713	0.11011388417765	0.11011489278544	716	0.11961108317550	0.11961128162549	717	0.12140866920393	0.12140766383844
721	0.07573336777530	0.07573456786920	724	0.09026847026665	0.09026875898306	725	0.09154179389298	0.09154050438680
729	0.02600341715387	0.02600476506314	732	0.04664722489382	0.04664757101135	733	0.04724465149876	0.04724320568487
737	0.13042525501443	0.13042425445322	740	0.12456395040140	0.12456399956828	743	0.11991353923327	0.11991473153453
747	0.11138341046992	0.11138216099422	750	0.10126335998705	0.10126336575411	753	0.09766250042398	0.09766417558510
757	0.08032438624530	0.08032298689430	760	0.06507691428851	0.06507683804024	763	0.06294155634344	0.06294360755461
767	0.03466765313827	0.03466621657555	770	0.01312479947849	0.01312402508314	773	0.01274071714261	0.01274309887747
777	0.10061218807971	0.10061366302537	780	0.10909460241907	0.10909473031363	781	0.11405271354075	0.11405130281864
785	0.06964965378226	0.06965129896048	788	0.08279056508540	0.08279080947843	789	0.08632424541746	0.08632249084902
793	0.02413166908930	0.02413330939773	796	0.04313148049314	0.04313181767747	797	0.04479717860004	0.04479525024587

801	0.09789612928671	0.09789431467968	804	0.08924555254664	0.08924562616294	807	0.08260013408198	0.08260222305115
811	0.07120017028542	0.07119808804983	814	0.05792581151041	0.05792580050629	817	0.05393158489629	0.05393438477477
821	0.03111299708174	0.03111096543047	824	0.01184501107289	0.01184408484193	827	0.01112203233083	0.01112544586079
831	0.05796566866476	0.05796812695941	834	0.06849919642491	0.06849942867776	835	0.07476009548352	0.07475775859274
839	0.02050178906139	0.02050427002337	842	0.03633194224359	0.03633249126146	843	0.03932752253618	0.03932495408980
847	0.05526377695512	0.05526060037598	850	0.04534846995487	0.04534871604109	853	0.03881186270325	0.03881586882921
857	0.02480313499499	0.02479981603325	860	0.00956655088363	0.00956543089302	863	0.00835406477241	0.00836052242587
867	0.01438388530817	0.01438903034313	870	0.02496912805505	0.02497051748330	871	0.02990618899601	0.02990229756286
875	0.01427479196745	0.01426593988594	878	0.00571792128896	0.00571894559182	881	0.00351053591177	0.00354544447856
885	0.14625828214478	0.14625800572427	888	0.14253524587494	0.14253563018103	891	0.14398352999959	0.14398344064660
895	0.13042525501443	0.13042426542688	898	0.11991353923327	0.11991471180672	901	0.12456395040140	0.12456398526623
905	0.09789612928671	0.09789432420388	908	0.08260013408198	0.08260219169659	911	0.08924555254664	0.08924559062991
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925	0.01427479196745	0.01426588254934	928	0.00351053591177	0.00354461935573	931	0.00571792128896	0.00571769637168
935	0.13177364211724	0.13177436378611	938	0.13915146714621	0.13915078118767	939	0.13701663213118	0.13701672053274
943	0.10061218807971	0.10061364601075	946	0.11405271354076	0.11405126470678	947	0.10909460241907	0.10909471747749
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1705	0.06964965378226	0.06965129896048	1708	0.08279056508540	0.08279080947843	1709	0.08632424541746	0.08632249084902
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1943	0.02050178906139	0.02050356024535	1946	0.03932752253618	0.03932462391872	1947	0.03633194224359	0.03633220614761
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2137 0.08032438624530 0.08032289486677 2140 0.06507691428851 0.06507663138857 2143 0.06294155634344 0.06294339547718
2147 0.03466765313827 0.03466581145387 2150 0.01312479947849 0.01312242845916 2153 0.01274071714261 0.01274149683065
2157 0.10061218807971 0.10061364601075 2160 0.10909460241907 0.10909471747749 2161 0.11405271354075 0.11405126470678
2165 0.06964965378226 0.06965117955198 2168 0.08279056508540 0.08279072767688 2169 0.08632424541746 0.08632238377176
2173 0.02413166908930 0.02413255125063 2176 0.04313148049314 0.04313148419444 2177 0.04479717860003 0.04479489148713
2181 0.09789612928671 0.09789432420388 2184 0.08924555254664 0.08924559062991 2187 0.08260013408198 0.08260219169659
2191 0.07120017028542 0.07119802316049 2194 0.05792581151041 0.05792562099281 2197 0.05393158489629 0.05393420943977
2201 0.03111299708174 0.03111058744699 2204 0.01184501107289 0.01184251532034 2207 0.01112203233083 0.01112388050008
2211 0.05796566866476 0.05796805592010 2214 0.06849919642491 0.06849939524539 2215 0.07476009548352 0.07475768010314
2219 0.02050178906139 0.02050356024535 2222 0.03633194224359 0.03633220614761 2223 0.03932752253618 0.03932462391872
2227 0.05526377695512 0.05526060445204 2230 0.04534846995487 0.04534860549301 2233 0.03881186270325 0.03881579588808
2237 0.02480313499499 0.02479950701517 2240 0.00956655088363 0.00956393024905 2243 0.00835406477241 0.00835905930447
2247 0.01438388530817 0.01438846454530 2250 0.02496912805505 0.02497037634989 2251 0.02990618899601 0.02990204180532
2255 0.01427479196745 0.01426588254934 2258 0.00571792128896 0.00571769637168 2261 0.00351053591177 0.00354461935573
2264 0.14625828214478 0.14625799334796 2267 0.14253524587494 0.14253564500567 2269 0.14398352999959 0.14398344465129

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 2273    0.13042525501443    0.13042425445322  2276    0.11991353923327    0.11991473153453  2279    0.12456395040140
0.12456399956828
 2283    0.09789612928671    0.09789431467968  2286    0.08260013408198    0.08260222305115  2289    0.08924555254664
0.08924562616294
 2293    0.05526377695512    0.05526060037598  2296    0.03881186270325    0.03881586882921  2299    0.04534846995487
0.04534871604109
 2303    0.01427479196745    0.01426593988594  2306    0.00351053591177    0.00354544447856  2309    0.00571792128896
0.00571894559182
 2313    0.13177364211724    0.13177436324512  2316    0.13915146714621    0.13915080245345  2317    0.13701663213118
0.13701672541751
 2321    0.10061218807971    0.10061366302537  2324    0.11405271354075    0.11405130281864  2325    0.10909460241907
0.10909473031363
 2329    0.05796566866476    0.05796812695941  2332    0.07476009548352    0.07475775859274  2333    0.06849919642491
0.06849942867776
 2337    0.01438388530817    0.01438903034313  2340    0.02990618899601    0.02990229756286  2341    0.02496912805505
0.02497051748330
 2344    0.13696332947999    0.13696293746972  2347    0.12944879535718    0.12944967327349  2349    0.13072998931971
0.13072987760096
 2353    0.11138341046992    0.11138216099422  2356    0.09766250042398    0.09766417558510  2359    0.10126335998705
0.10126336575411
 2363    0.07120017028542    0.07119808804983  2366    0.05393158489629    0.05393438477477  2369    0.05792581151041
0.05792580050629
 2373    0.02480313499499    0.02479981603325  2376    0.00835406477241    0.00836052242587  2379    0.00956655088363
0.00956543089302
 2383    0.11011388417766    0.11011489278544  2386    0.12140866920393    0.12140766383844  2387    0.11961108317550
0.11961128162549
 2391    0.06964965378226    0.06965129896048  2394    0.08632424541746    0.08632249084902  2395    0.08279056508540
0.08279080947843
 2399    0.02050178906139    0.02050427002337  2402    0.03932752253618    0.03932495408980  2403    0.03633194224359
0.03633249126146
 2406    0.11673713166364    0.11673663554891  2409    0.10503329287795    0.10503461041355  2411    0.10602115913807
0.10602101141591
 2415    0.08032438624530    0.08032298689430  2418    0.06294155634344    0.06294360755461  2421    0.06507691428851
0.06507683804024
 2425    0.03111299708174    0.03111096543047  2428    0.01112203233083    0.01112544586079  2431    0.01184501107289
0.01184408484193
 2435    0.07573336777530    0.07573456786920  2438    0.09154179389298    0.09154050438680  2439    0.09026847026665
0.09026875898306
 2443    0.02413166908930    0.02413330939773  2446    0.04479717860004    0.04479525024587  2447    0.04313148049314
0.04313181767747
 2450    0.08391873690481    0.08391817314443  2453    0.06730349895817    0.06730515385109  2455    0.06788706591391
0.06788688630284
 2459    0.03466765313827    0.03466621657555  2462    0.01274071714261    0.01274309887747  2465    0.01312479947849
0.01312402508314
 2469    0.02600341715387    0.02600476506314  2472    0.04724465149876    0.04724320568487  2473    0.04664722489382
0.04664757101135
 2476    0.03605924523629    0.03605867528529  2479    0.01351922271267    0.01352125998214  2481    0.01362510608026
0.01362484287590
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----------------------------------------------
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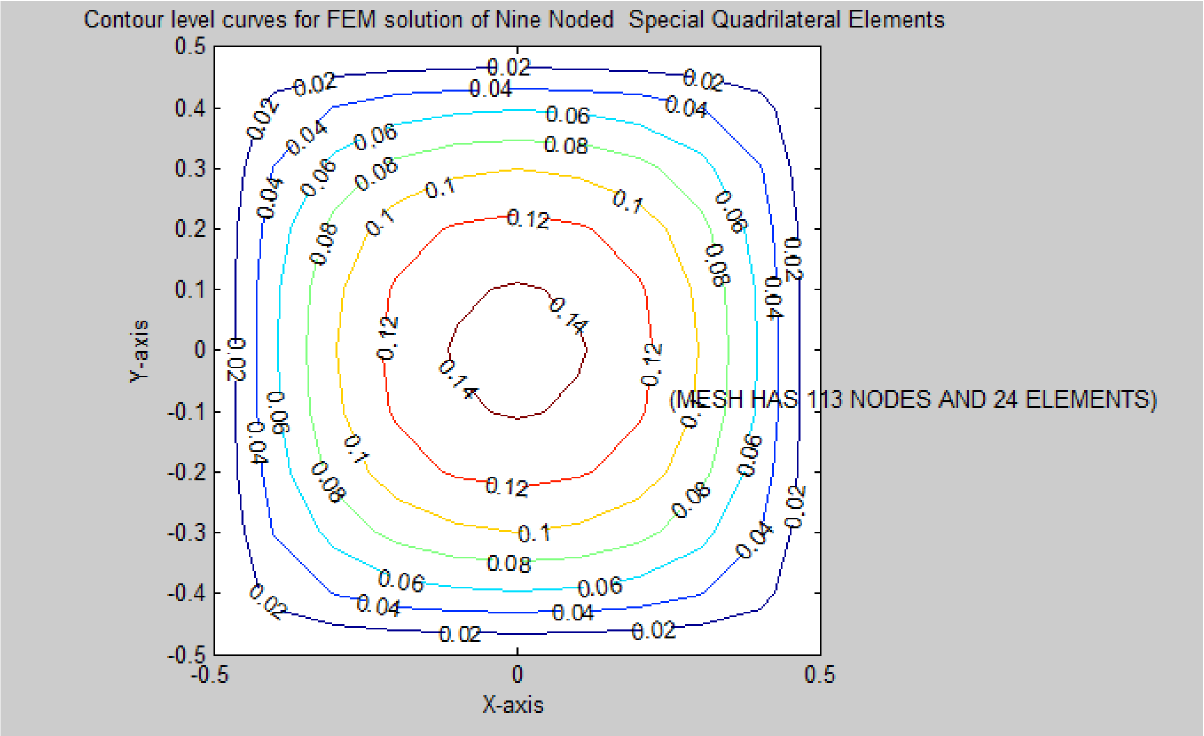
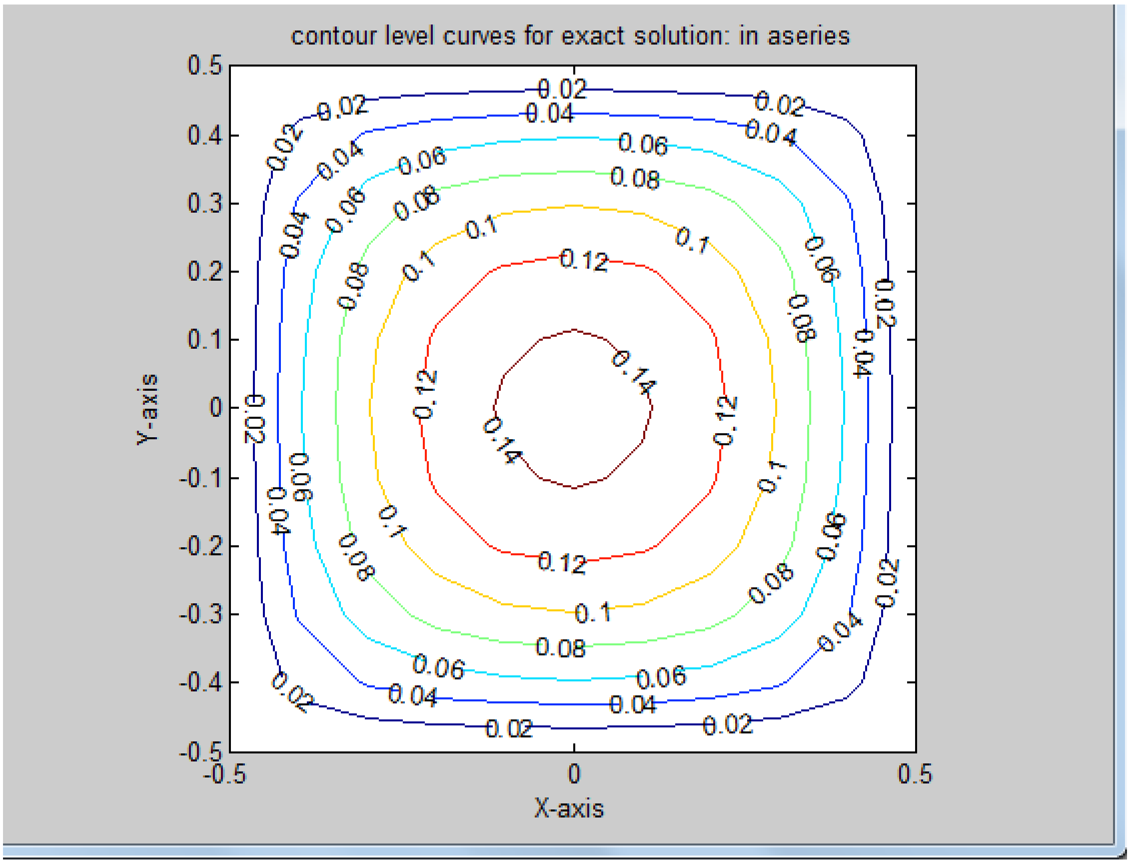
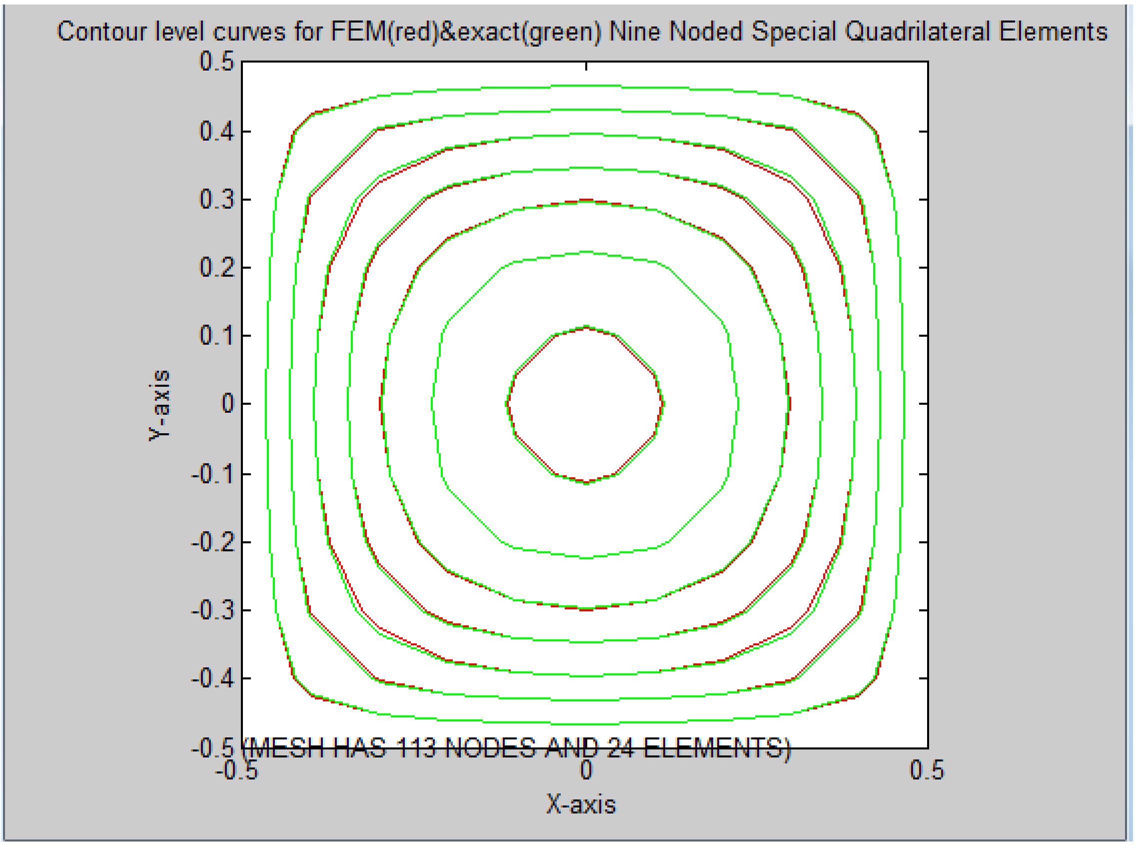
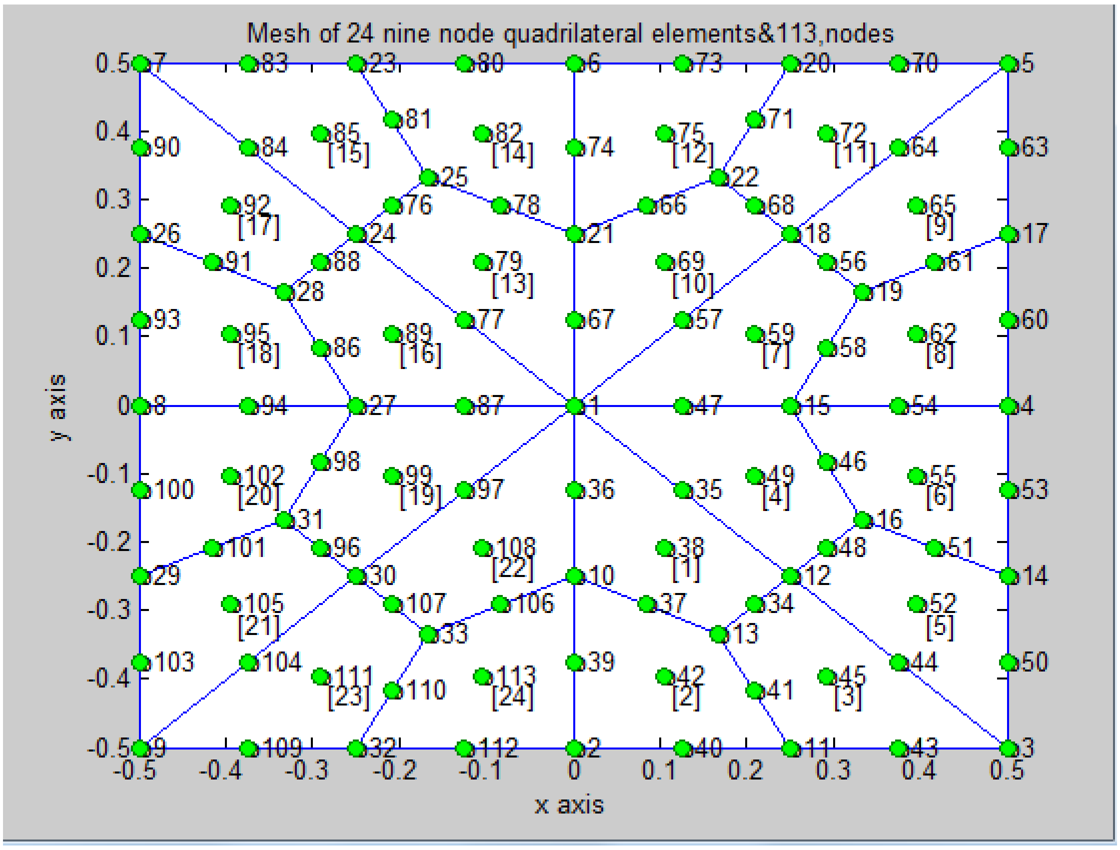
Mesh No.1

Mesh with 24 nine noded quadrilateral elements & no. of nodes = 113



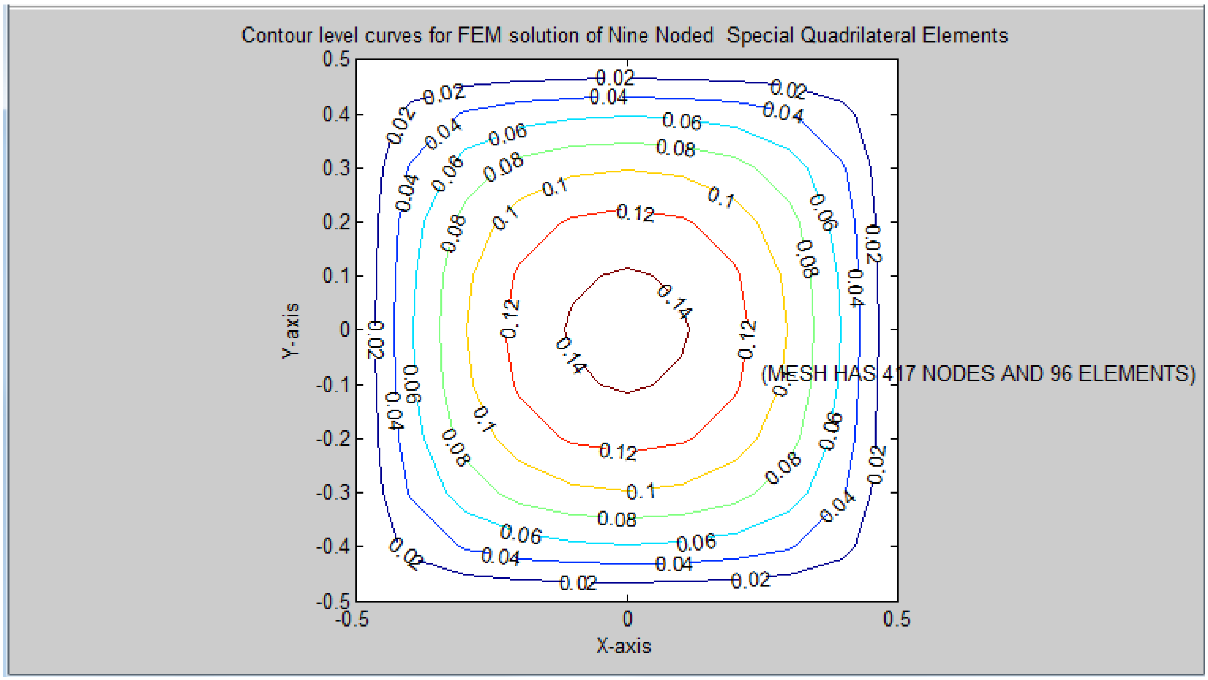


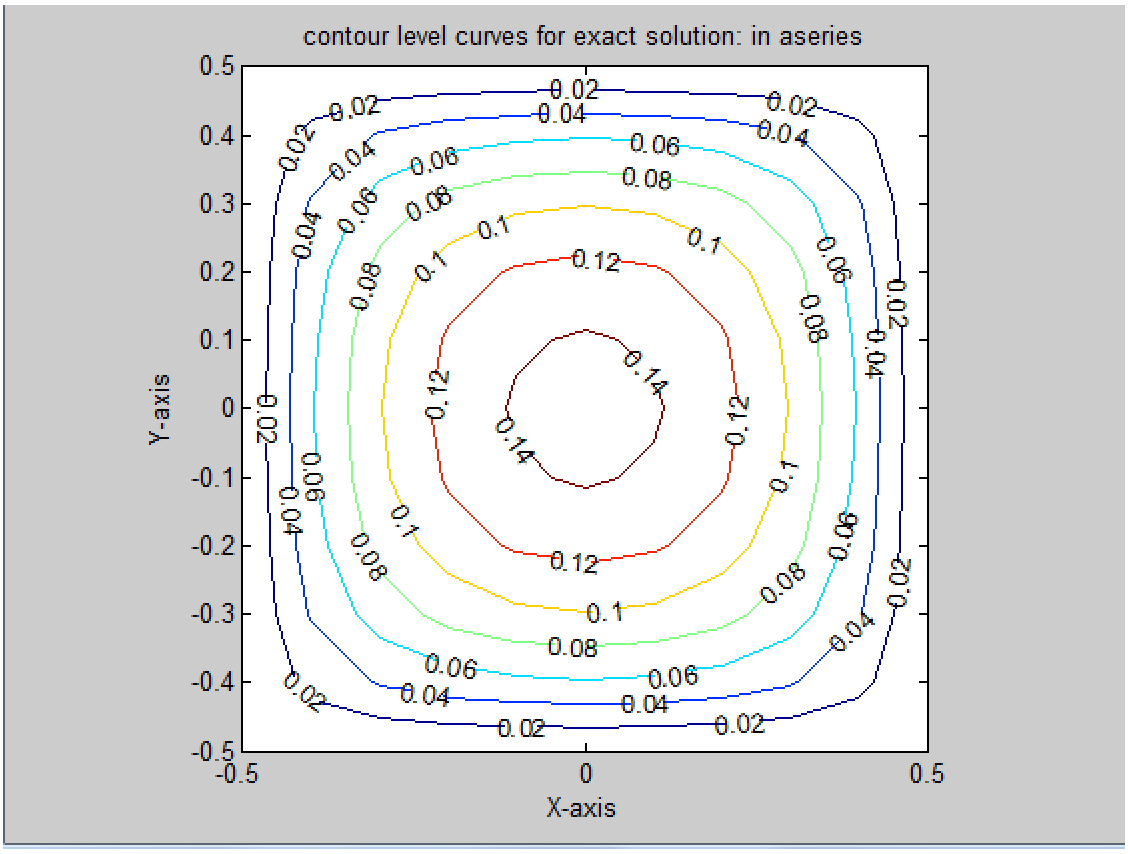


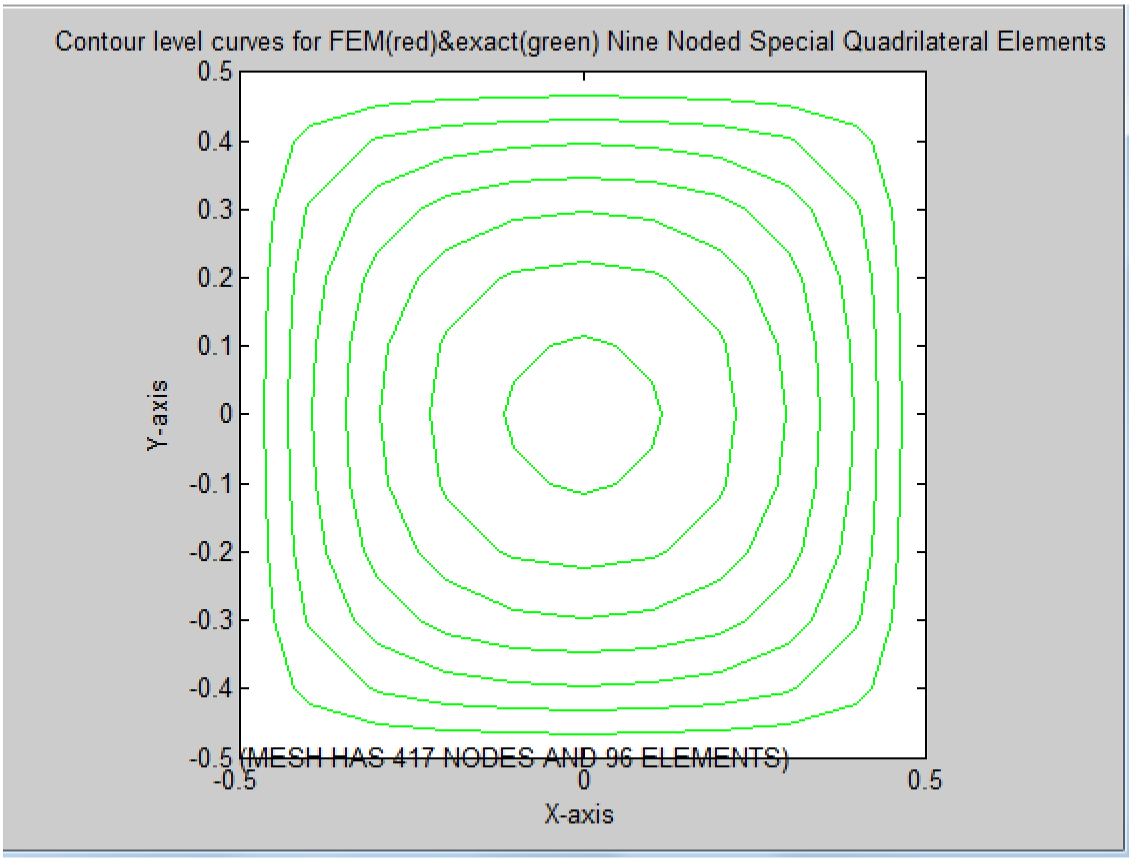


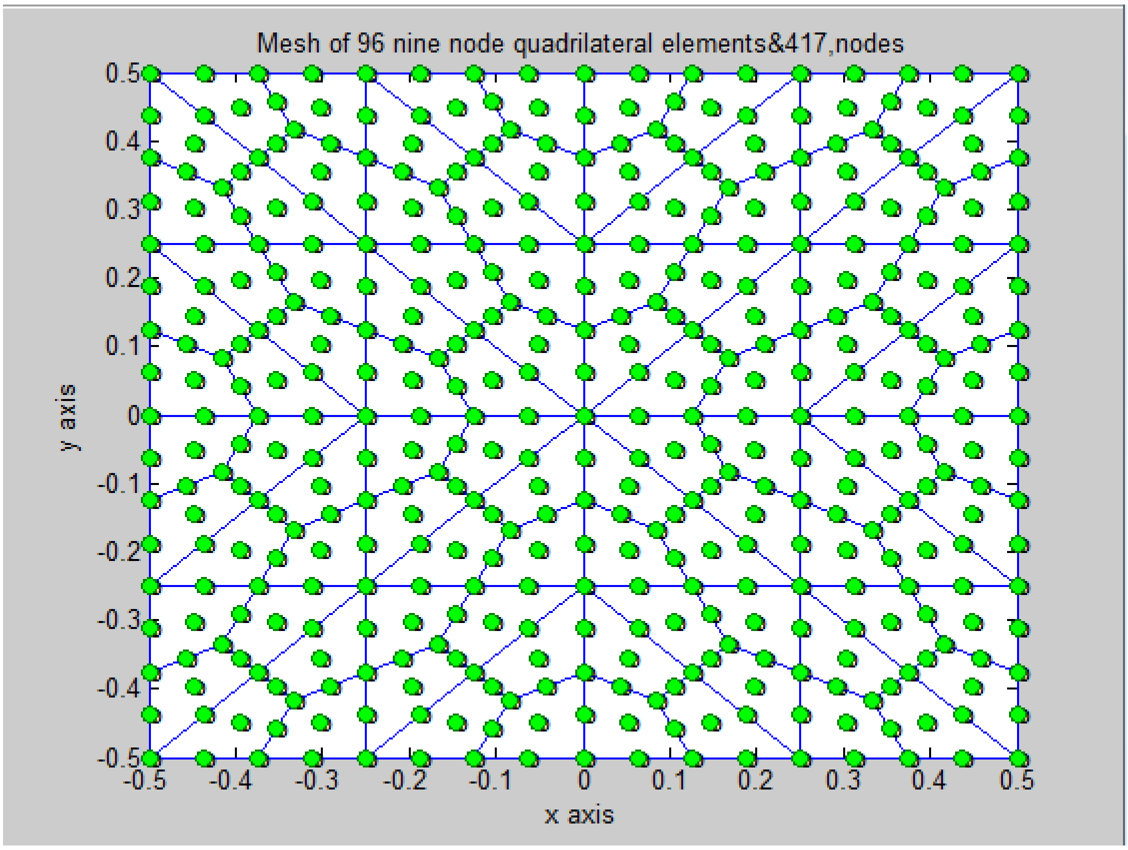
Mesh No.2

Mesh with 96 nine noded quadrilateral elements & no. of nodes = 417


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
(MESH HAS 417 NODES AND 96 ELEMENTS)
Y-axis
X-axis
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5
0.02
0.04
0.06
0.08
0.1
0.12
0.14

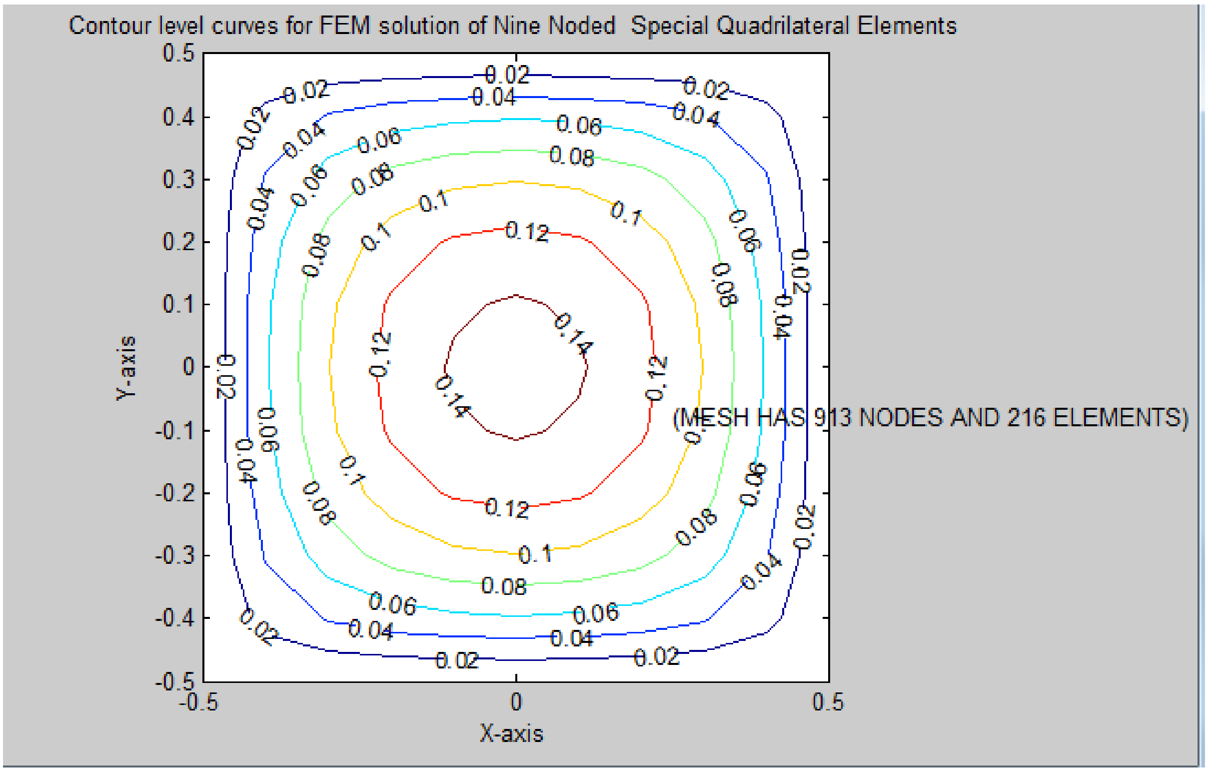

contour level curves for exact solution: in aseries
Y-axis
X-axis
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5
0.02
0.04
0.06
0.08
0.1
0.12
0.14

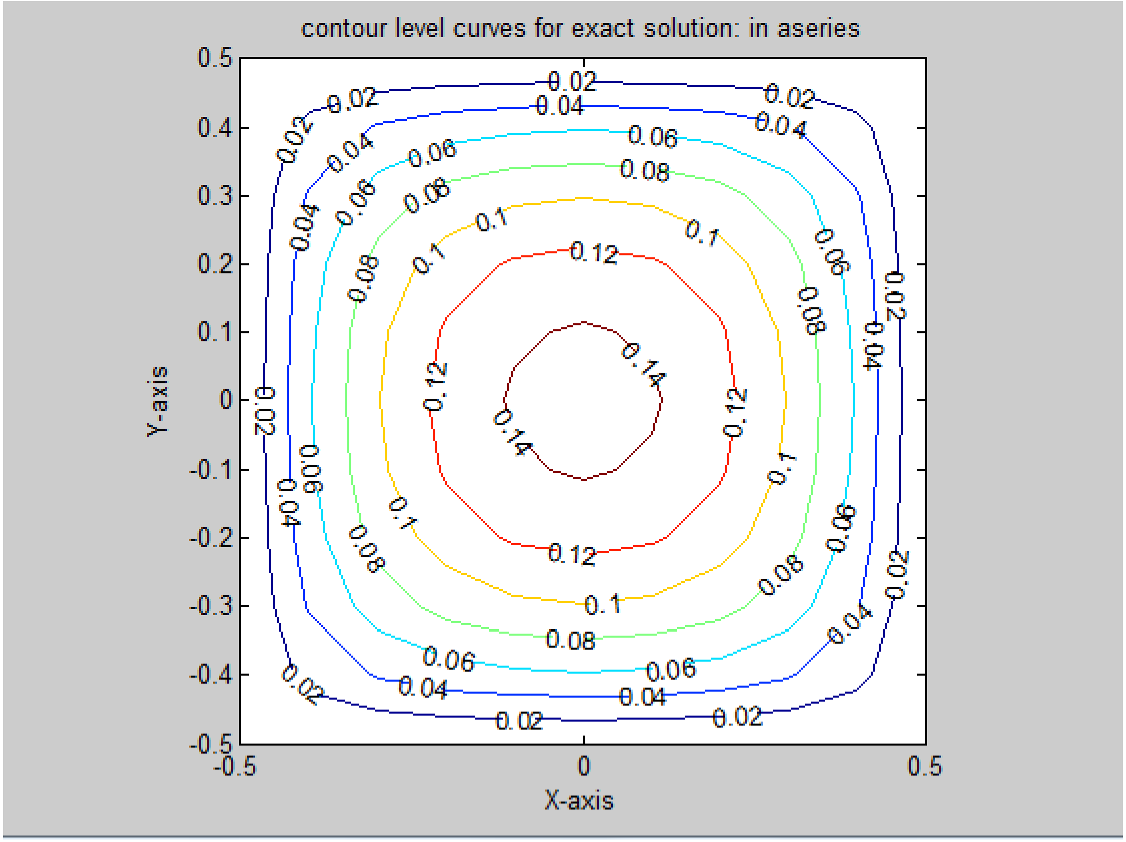


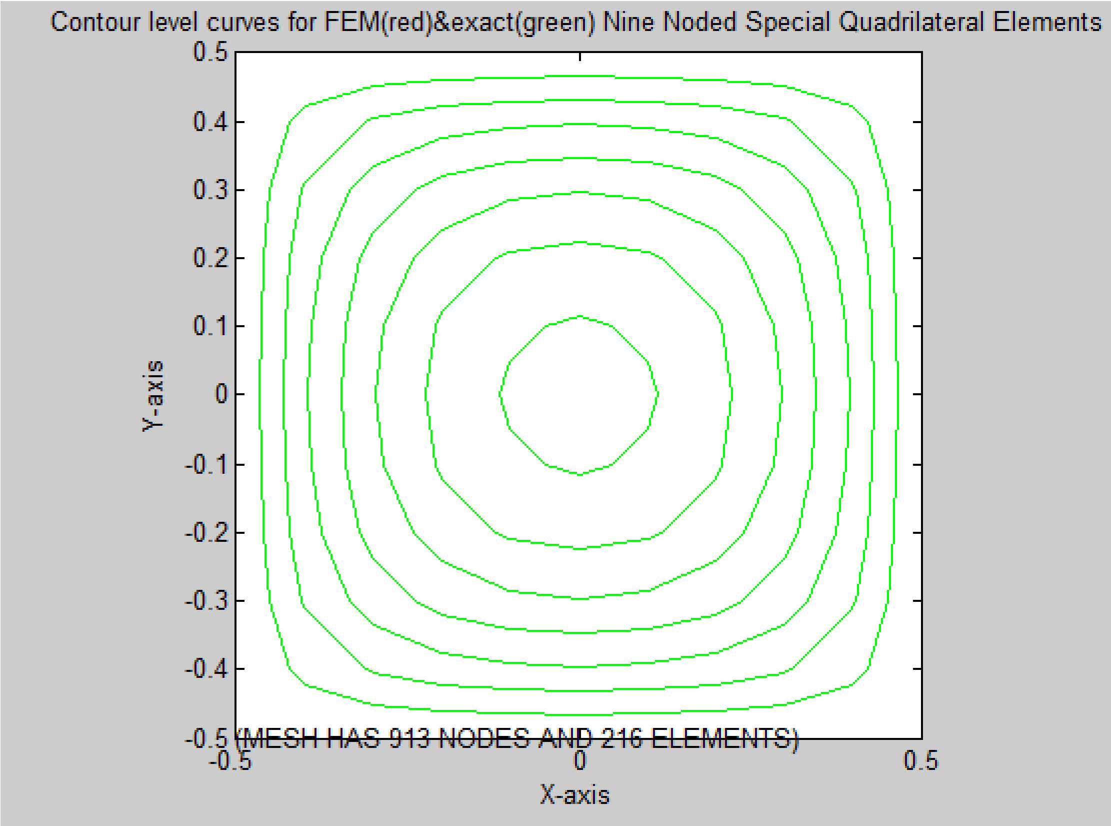


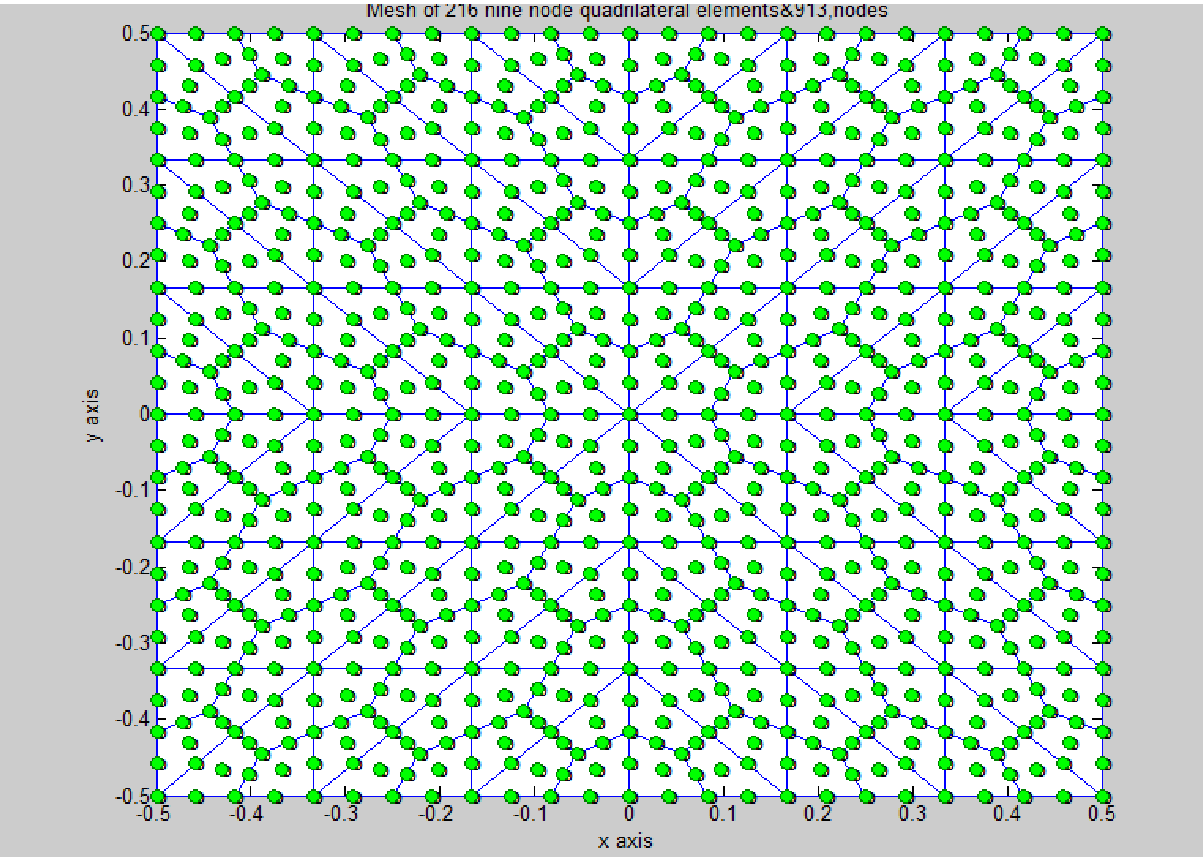
Mesh No.3

Mesh with 216 nine noded quadrilateral elements & no. of nodes = 913


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
(MESH HAS 913 NODES AND 216 ELEMENTS)
Y-axis
X-axis
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5
0.02
0.04
0.06
0.08
0.1
0.12
0.14

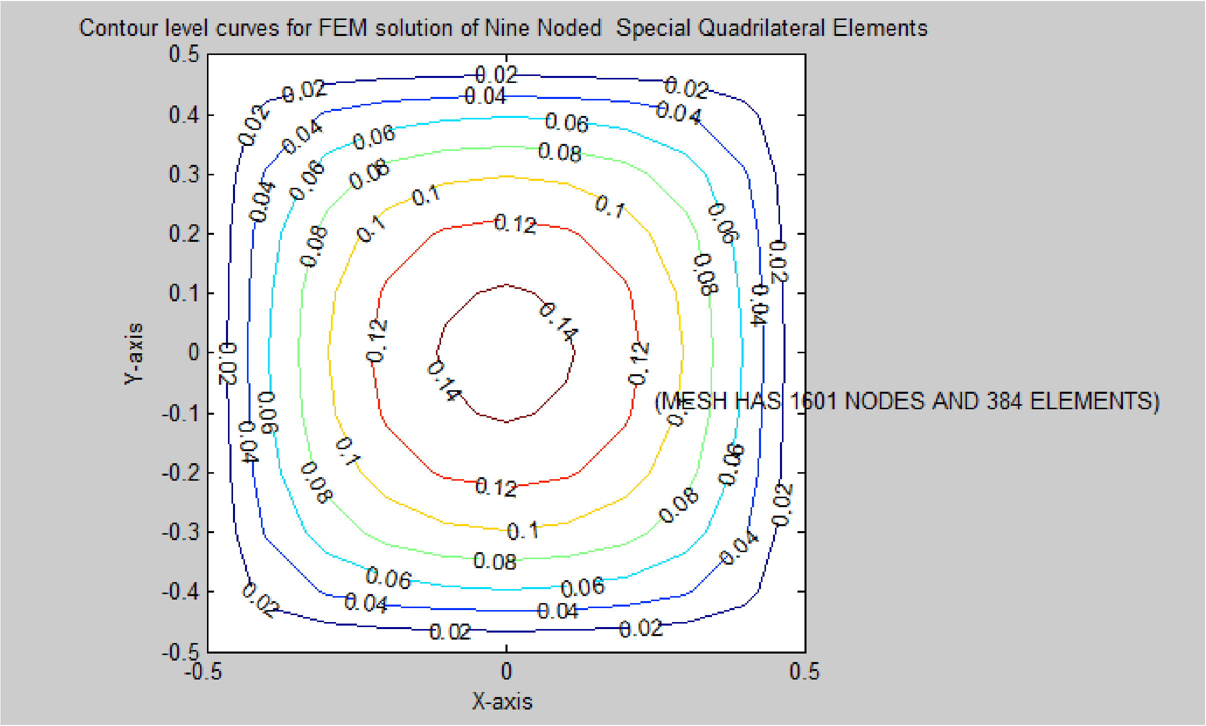

contour level curves for exact solution: in aseries
Y-axis
X-axis
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5
0.02
0.04
0.06
0.08
0.1
0.12
0.14

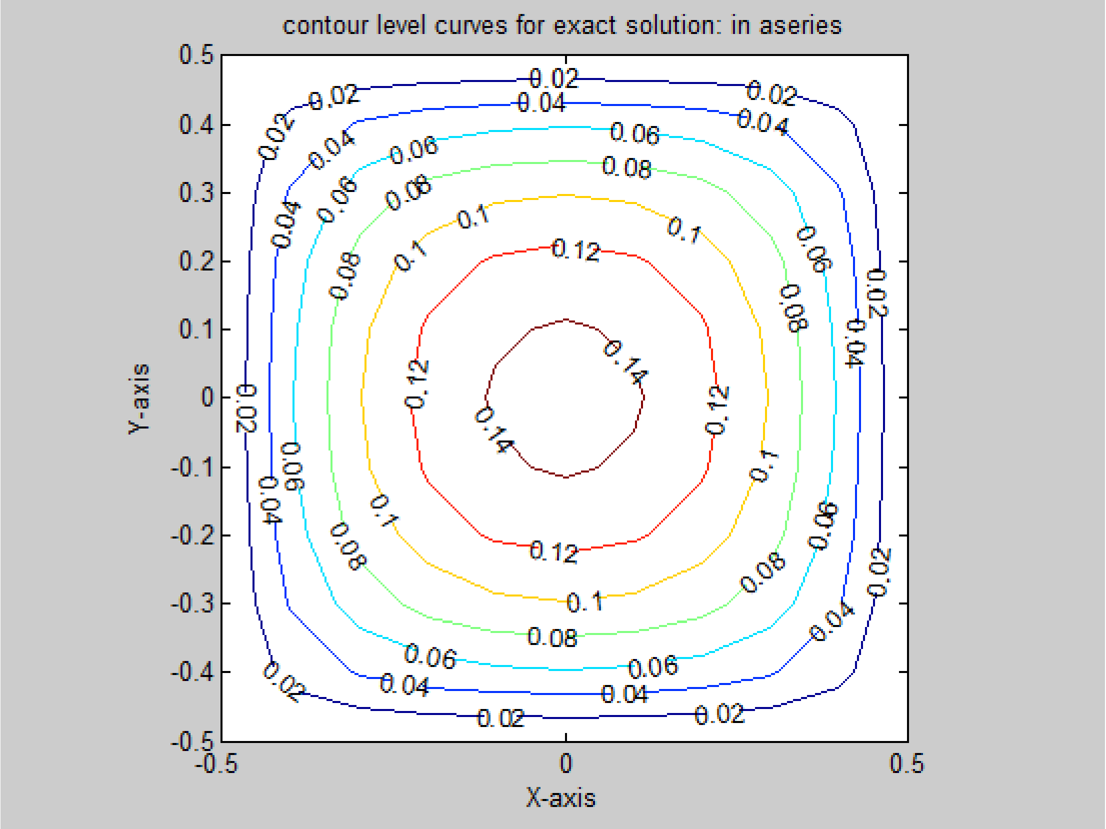


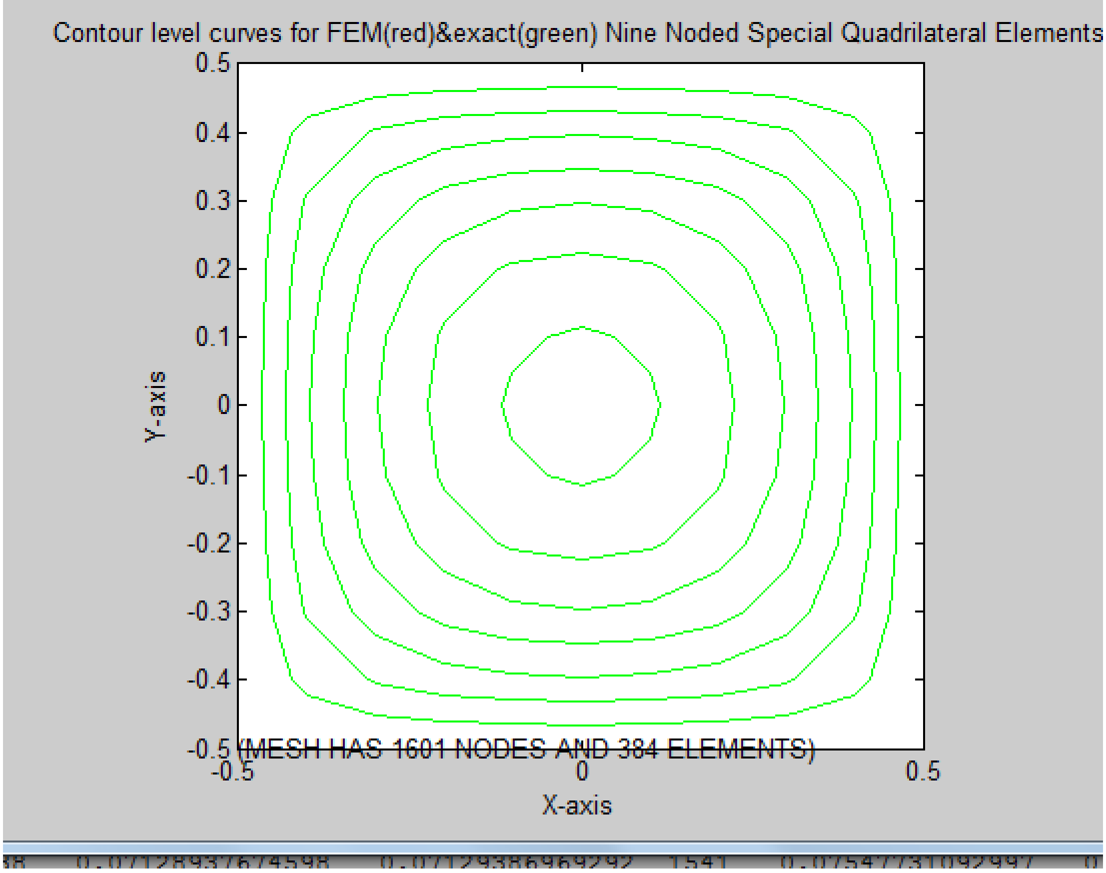


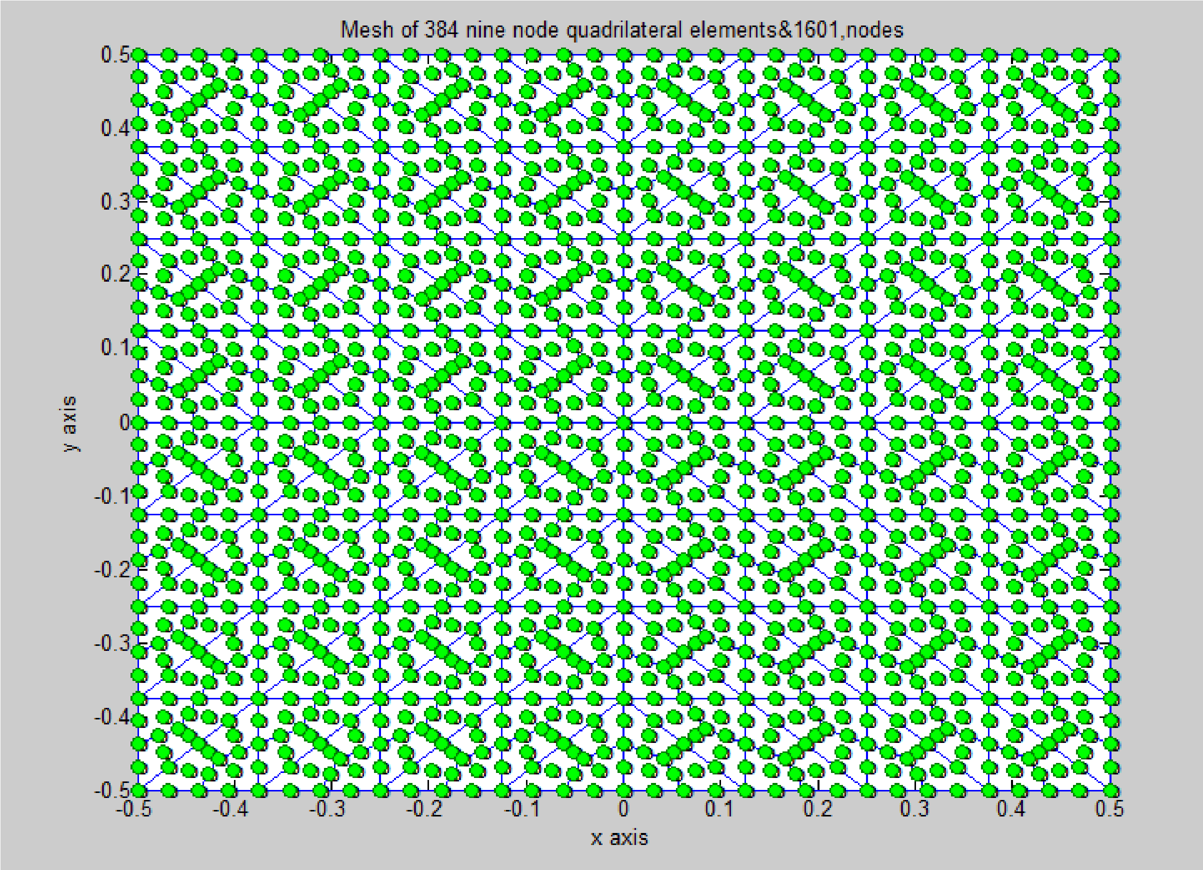
Mesh No.4

Mesh with 384 nine noded quadrilateral elements & no. of nodes = 1601


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
(MESH HAS 1601 NODES AND 384 ELEMENTS)
X-axis
Y-axis
-0.5
0
0.5
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.4
0.02
0.04
0.06
0.08
0.1
0.12
0.14

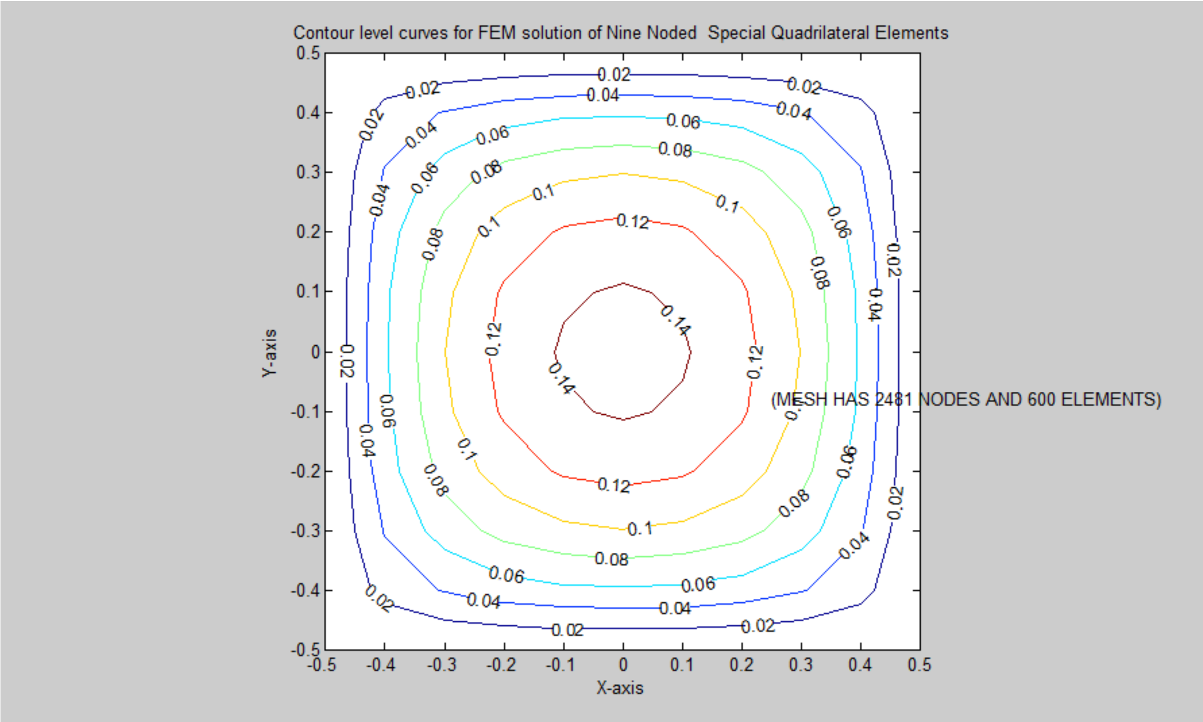

contour level curves for exact solution: in aseries
X-axis
Y-axis
-0.5
0
0.5
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.4
0.02
0.04
0.06
0.08
0.1
0.12
0.14

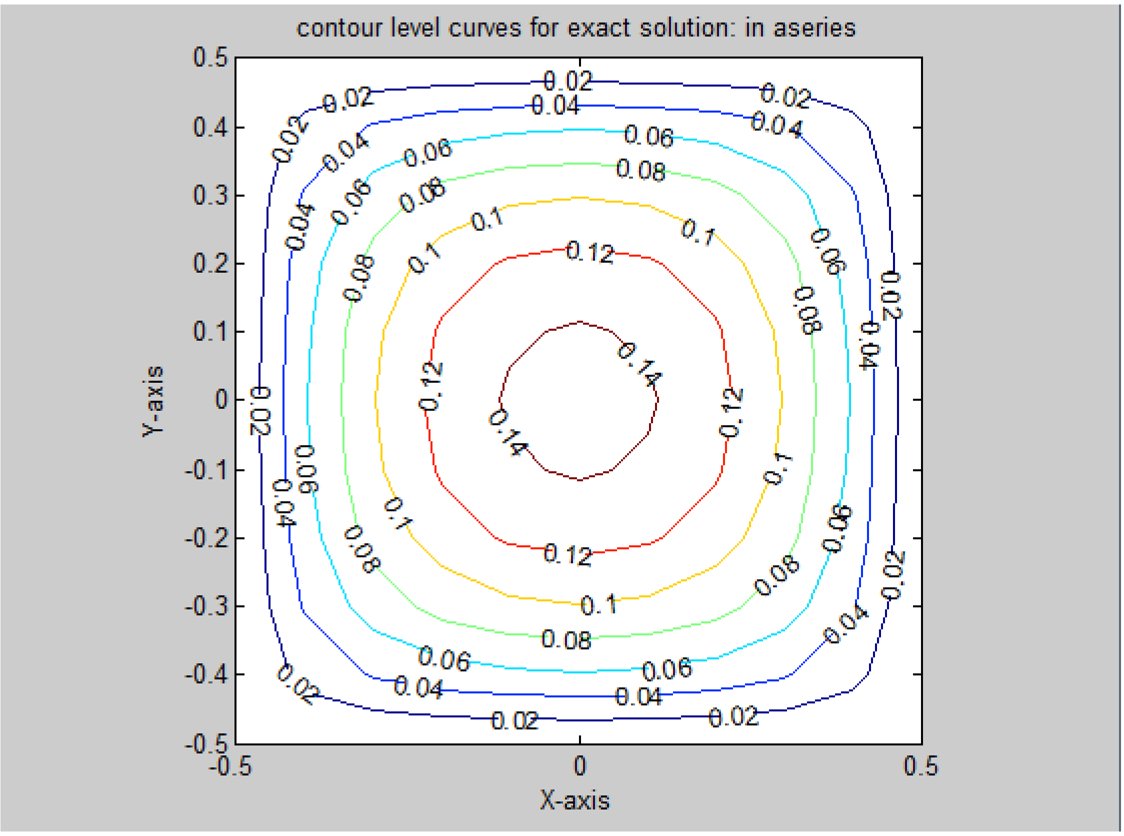


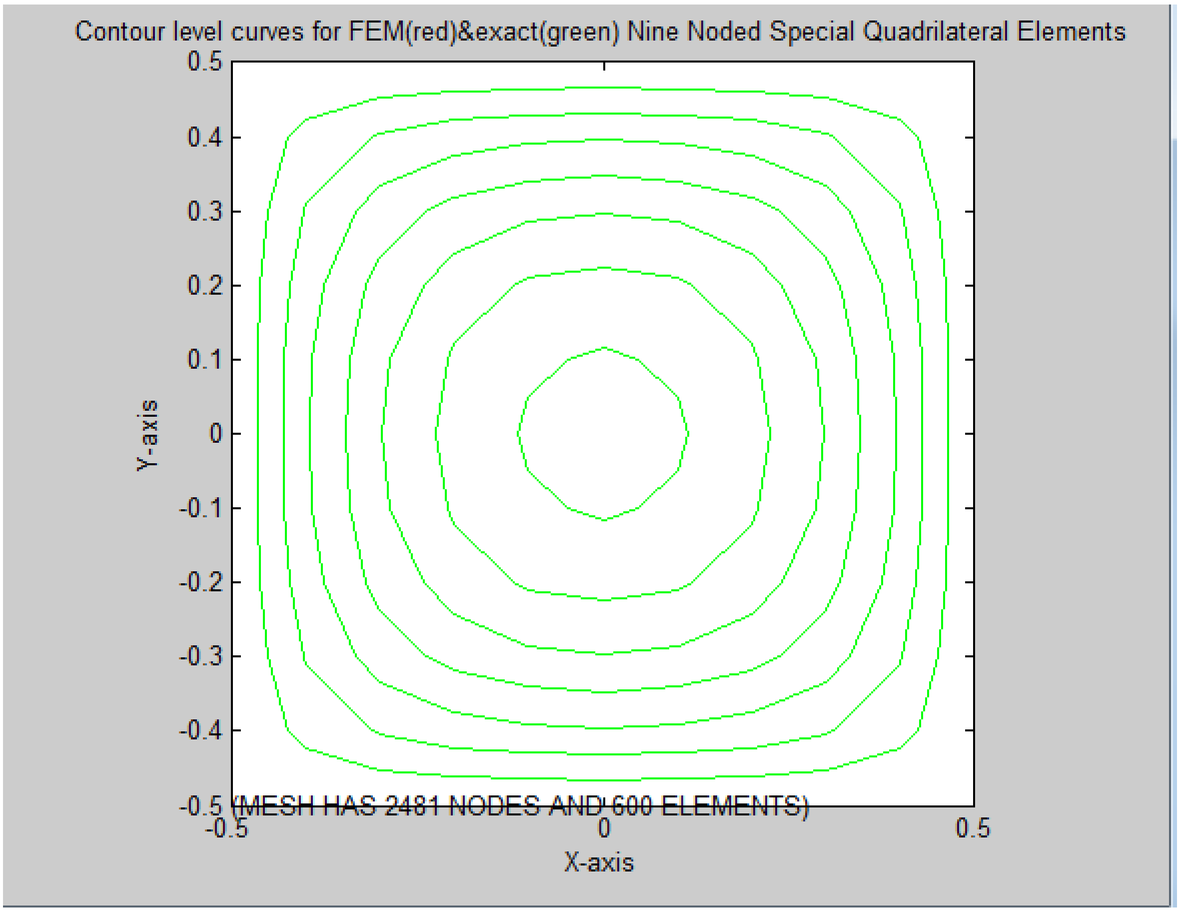


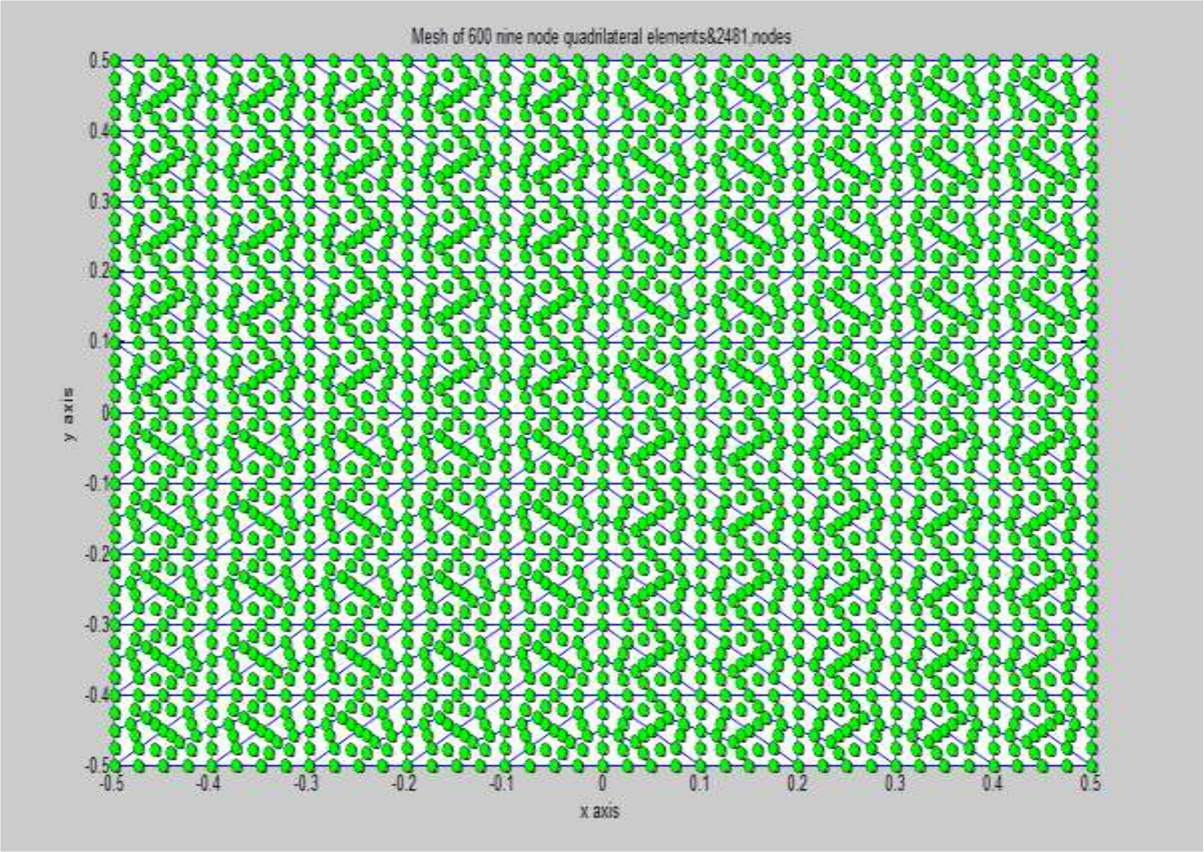
Mesh No.5

Mesh with 600 nine noded quadrilateral elements & no. of nodes = 2481


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
(MESH HAS 2481 NODES AND 600 ELEMENTS)
0.02
0.04
0.06
0.08
0.1
0.12
0.14
X-axis
Y-axis
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5


contour level curves for exact solution: in aseries
0.02
0.04
0.06
0.08
0.1
0.12
0.14
X-axis
Y-axis
-0.5
0
0.5
-0.4
-0.3
-0.2
-0.1
0.1
0.2
0.3
0.4





Solution of Poisson Boundary Value Problems Over Polygonal Domains

Example 3(pentagonal domain,refer section 7.5.2)

Table-5a

Mesh:number of nodes=2171,number of nine node elements= 525

fem computed values and exact values at centroid points

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
566	0.97339958515641	0.98932210182092	570	0.95128879706595	0.96715571192955	573	0.93817819513447	0.95300104853184
578	0.88500594743287	0.90065277997236	582	0.82919571310024	0.84391705351846	585	0.81814735147050	0.83156603115381
590	0.71011969844479	0.72382128880320	594	0.62610883478952	0.63806991399311	597	0.61825668593475	0.62873153678521
602	0.46598426618935	0.47613712672652	606	0.36208882054412	0.36976404559162	609	0.35818126088466	0.36435241896587
614	0.17690823155835	0.18184534524342	618	0.06378624279181	0.06526309611003	621	0.06363375240841	0.06430794778558
625	0.83842186999125	0.85177670957122	628	0.88604947680321	0.90023092787438	629	0.90539039234496	0.92086345838978
633	0.65646682745044	0.66717453160046	636	0.73289843597119	0.74518526832117	637	0.74846649660232	0.76226428361846
641	0.41053710106666	0.41726466199836	644	0.50826615132126	0.51719568269362	645	0.51852140520908	0.52904936975901
649	0.12481313491872	0.12651002002600	652	0.23424557185042	0.23857938012980	653	0.23831180355192	0.24404741748384
657	0.82511110496604	0.83832983438577	660	0.77308797097766	0.78552007992821	663	0.73267392432981	0.74324008236778
667	0.66256766386496	0.67373464520468	670	0.58427563230040	0.59391705351846	673	0.55409502597724	0.56194993744403
677	0.43541701487237	0.44318961476565	680	0.33850562840281	0.34417728784688	683	0.32144462583117	0.32565221730781
687	0.16586680549736	0.16926209694974	690	0.05987946679785	0.06074705121667	693	0.05721403528346	0.05747738918909
697	0.57417840859553	0.58216061586144	700	0.64094709214368	0.65023092787438	701	0.68273685835609	0.69394532362209
705	0.35946587895553	0.36409521212307	708	0.44490196186630	0.45129264217493	709	0.47350325860941	0.48163260958091
713	0.10945050000013	0.11038963222158	716	0.20533875887391	0.20817868831867	717	0.21807345125743	0.22217433998223
721	0.54952152304417	0.55769816034822	724	0.48464462950258	0.49162745378198	727	0.43507174056087	0.44016076289038
731	0.36143500154923	0.36685961542790	734	0.28106288593779	0.28490005914351	737	0.25261886583279	0.25507490766725
741	0.13784329931033	0.14011029528825	744	0.04977249279193	0.05028466170064	747	0.04500592766131	0.04502054327026
751	0.27260533083388	0.27528558608466	754	0.33735981941603	0.34121393349943	755	0.38144239886448	0.38707029384531
759	0.08308995827599	0.08346353808552	762	0.15586612644483	0.15740001602869	763	0.17585406406789	0.17855329010351
767	0.25149612002628	0.25461884087082	770	0.19559888458441	0.19773482763574	773	0.15856891026921	0.15952908885283
777	0.09602824615462	0.09724352171812	780	0.03467986620585	0.03490005914351	783	0.02828773746520	0.02815677289956
787	0.04838860559931	0.04836745131698	790	0.09076720244006	0.09121393349943	791	0.11605387179739	0.11745420013553
795	0.04449130719781	0.04485787470676	798	0.01607078784449	0.01609919563440	801	0.00864157206935	0.00853682141769
805	0.97386435069014	0.98932210182092	808	0.93855726981600	0.95300104853184	811	0.95272931193375	0.96715571192955
815	0.82539562673207	0.83832983438577	818	0.73286050846179	0.74324008236778	821	0.77449833403681	0.78552007992821
825	0.54959545755408	0.55769816034822	828	0.43507596060127	0.44016076289038	831	0.48567114575503	0.49162745378198
835	0.25144384079420	0.25461884087082	838	0.15850350099382	0.15952908885283	841	0.19617345173084	0.19773482763574
845	0.04443254204524	0.04485787470676	848	0.00858368179911	0.00853682141769	851	0.01618249531988	0.01609919563440
855	0.83984776287391	0.85177670957122	858	0.90720788005848	0.92086345838978	859	0.88651317081068	0.90023092787438
863	0.57496561675682	0.58216061586144	866	0.68399553729074	0.69394532362209	867	0.64085763216907	0.65023092787438
871	0.27282119980680	0.27528558608466	874	0.38206595977850	0.38707029384531	875	0.33695306955408	0.34121393349943
879	0.04834918369049	0.04836745131698	882	0.11629010887523	0.11745420013553	883	0.09042756536073	0.09121393349943
887	0.88763836897989	0.90065277997236	890	0.82017306966673	0.83156603115380	893	0.83268063276447	0.84391705351846
897	0.66418083232643	0.67373464520468	900	0.55504530409589	0.56194993744403	903	0.58666317242474	0.59391705351846
907	0.36202536997823	0.36685961542790	910	0.25271759530647	0.25507490766725	913	0.28218201733783	0.28490005914351
917	0.09610574114357	0.09724352171812	920	0.02812097307258	0.02815677289956	923	0.03492600749796	0.03490005914351
927	0.65894418163777	0.66717453160046	930	0.75196970225667	0.76226428361846	931	0.73468830577483	0.74518526832117
935	0.36033783242736	0.36409521212307	938	0.47536521559936	0.48163260958091	939	0.44530734289948	0.45129264217493
943	0.08309231886438	0.08346353808552	946	0.17648183010941	0.17855329010351	947	0.15556391558919	0.15740001602869
951	0.71429962000763	0.72382128880320	954	0.62113604447764	0.62873153678521	957	0.63083333624229	0.63806991399311
961	0.43736920633676	0.44318961476565	964	0.32223468971500	0.32565221730781	967	0.34074323802412	0.34417728784688
971	0.13825243617810	0.14011029528825	974	0.04481125801626	0.04502054327026	977	0.05020451441466	0.05028466170064
981	0.41294460457120	0.41726466199836	984	0.52268215554921	0.52904936975901	985	0.51045001444616	0.51719568269362
989	0.10968343957875	0.11038963222158	992	0.21946982814940	0.22217433998223	993	0.20544263961316	0.20817868831867
997	0.47057067245229	0.47613712672652	1000	0.36064963920186	0.36435241896587	1003	0.36661318359047	0.36976404559162
1007	0.16696611260443	0.16926209694974	1010	0.05712203332544	0.05747738918909	1013	0.06060608874387	0.06074705121667
1017	0.12568970617443	0.12651002002600	1020	0.24140360808720	0.24404741748384	1021	0.23547509532103	0.23857938012980
1025	0.17977794383315	0.18184534524342	1028	0.06394425799922	0.06430794778558	1031	0.06523140277810	0.06526309611003
1035	0.98037649712264	0.99572243068691	1038	0.96660809290317	0.98114968333084	1041	0.96669838344885	0.98114968333084
1045	0.91354883852343	0.92682090032251	1048	0.86529084474814	0.87693549776453	1051	0.90093032737360	0.91325655105360
1055	0.75723858795105	0.76719568269362	1058	0.67915247782640	0.68688075570814	1061	0.74708046864716	0.75596750452669
1065	0.52651258083929	0.53247200627604	1068	0.42609958293322	0.42958933950340	1071	0.51993637518262	0.52467909152139
1075	0.24342459883105	0.24562625993355	1078	0.12996806669906	0.13024672562314	1081	0.24138491744629	0.24203143338379
1085	0.92648238808501	0.93940855633098	1088	0.94726712576208	0.96093900491343	1089	0.94712609371525	0.96093900491343
1093	0.80999853004457	0.81970554599445	1096	0.86350708818213	0.87481355623834	1097	0.82777457798311	0.83849250294927
1101	0.61429373417767	0.61976404559162	1104	0.69538425364551	0.70305526149420	1105	0.62730184605860	0.63396851267565
1109	0.35815431506836	0.35915572225626	1112	0.45901751301874	0.46247701928097	1113	0.36497342022337	0.36738726726297

1117	0.91373453791467	0.92682090032251	1120	0.90069929493313	0.91325655105360	1123	0.86545175736366	0.87693549776453
1127	0.85162923933802	0.86268718550614	1130	0.80665024537306	0.81625372946776	1133	0.80692838006340	0.81625372946776
1137	0.70657490515227	0.71410763827738	1140	0.63399102807601	0.63935030566746	1143	0.67004746759671	0.67567135895653
1147	0.49246827769827	0.49562625993355	1150	0.39909040976747	0.39986281933291	1153	0.46799614794364	0.46894956815147

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
1157	0.80981406936323	0.81970554599445	1160	0.82805457840924	0.83849250294927	1161	0.86329989709724	0.87481355623834
1165	0.70877876861560	0.71525554840415	1168	0.75540323282716	0.76334148643527	1169	0.75507877531569	0.76334148643527
1173	0.53912345016228	0.54079135425760	1176	0.60956888929016	0.61346928671611	1177	0.57338426443085	0.57714823342704
1181	0.75759268855316	0.76719568269362	1184	0.74673214624403	0.75596750452669	1187	0.67936896453094	0.68688075570814
1191	0.70676086644772	0.71410763827738	1194	0.66966819207946	0.67567135895653	1197	0.63430026914298	0.63935030566746
1201	0.58774016151010	0.59111776274607	1204	0.52814120344415	0.52923579309253	1207	0.52825607560608	0.52923579309253
1211	0.61394321993248	0.61976404559162	1214	0.62769243704886	0.63396851267565	1215	0.69513335802393	0.70305526149420
1219	0.53893847721870	0.54079135425760	1222	0.57383511680181	0.57714823342704	1223	0.60919035193814	0.61346928671611
1227	0.52700158123462	0.53247200627604	1230	0.51949428454624	0.52467909152139	1233	0.42636348052741	0.42958933950340
1237	0.49282158462950	0.49562625993355	1240	0.46780969431196	0.46894956815147	1243	0.39914194398419	0.39986281933291
1247	0.35767155330089	0.35915572225626	1250	0.36547902610722	0.36738726726297	1251	0.45870642906768	0.46247701928097
1255	0.24400693934599	0.24562625993355	1258	0.24114873831592	0.24203143338379	1261	0.13000697466807	0.13024672562314
1265	0.97387704962309	0.98932210182092	1268	0.95254404447292	0.96715571192955	1271	0.93861638615850	0.95300104853184
1275	0.88757007468902	0.90065277997236	1278	0.83232956248008	0.84391705351846	1281	0.82037898979156	0.83156603115380
1285	0.71415987413367	0.72382128880320	1288	0.63035184166918	0.63806991399311	1291	0.62146880074990	0.62873153678521
1295	0.47037639998466	0.47613712672652	1298	0.36605095854763	0.36976404559162	1301	0.36107454785849	0.36435241896587
1305	0.17953250463562	0.18184534524342	1308	0.06479250704588	0.06526309611003	1311	0.06429958697921	0.06430794778558
1315	0.83987465140379	0.85177670957122	1318	0.88686565077639	0.90023092787438	1319	0.90704628816827	0.92086345838978
1323	0.65905733643477	0.66717453160046	1326	0.73517539900436	0.74518526832117	1327	0.75167505089209	0.76226428361846
1331	0.41313274150346	0.41726466199836	1334	0.51102510547620	0.51719568269362	1335	0.52228479981672	0.52904936975901
1339	0.12585719833827	0.12651002002600	1342	0.23606838331681	0.23857938012980	1343	0.24093456616925	0.24404741748384
1347	0.82547229924464	0.83832983438577	1350	0.77401434917184	0.78552007992821	1353	0.73297137774705	0.74324008236778
1357	0.66416699597564	0.67373464520468	1360	0.58609706714044	0.59391705351846	1363	0.55528532780608	0.56194993744403
1367	0.43726787048210	0.44318961476565	1370	0.34015186271974	0.34417728784688	1373	0.32258346770288	0.32565221730781
1377	0.16681073635697	0.16926209694974	1380	0.06023305056468	0.06074705121667	1383	0.05745760463761	0.05747738918909
1387	0.57496301365784	0.58216061586144	1390	0.64142893172425	0.65023092787438	1391	0.68378307471938	0.69394532362209
1395	0.36044724147169	0.36409521212307	1398	0.44591309585203	0.45129264217493	1399	0.47503999624731	0.48163260958091
1403	0.10983555686392	0.11038963222158	1406	0.20602122635880	0.20817868831867	1407	0.21907276212998	0.22217433998223
1411	0.54970715219894	0.55769816034822	1414	0.48507463933154	0.49162745378198	1417	0.43519732448210	0.44016076289038
1421	0.36203009763207	0.36685961542790	1424	0.28162484644632	0.28490005914351	1427	0.25295333133180	0.25507490766725
1431	0.13817518341735	0.14011029528825	1434	0.04989702217619	0.05028466170064	1437	0.04508521089877	0.04502054327026
1441	0.27283735622861	0.27528558608466	1444	0.33752684299921	0.34121393349943	1445	0.38184428031474	0.38707029384531
1449	0.08320609413157	0.08346353808552	1452	0.15606349007450	0.15740001602869	1453	0.17618570301572	0.17855329010351
1457	0.25154763974305	0.25461884087082	1460	0.19569786515867	0.19773482763574	1463	0.15859344549972	0.15952908885283
1467	0.09610774271335	0.09724352171812	1470	0.03471127242090	0.03490005914351	1473	0.02830318941479	0.02815677289956
1477	0.04840577030850	0.04836745131698	1480	0.09078823236948	0.09121393349943	1481	0.11611595748800	0.11745420013553
1485	0.04449479989720	0.04485787470676	1488	0.01607329340308	0.01609919563440	1491	0.00864187416378	0.00853682141769
1495	0.97338703109168	0.98932210182092	1498	0.93811558614037	0.95300104853184	1501	0.95145735508867	0.96715571192955
1505	0.82503439644358	0.83832983438577	1508	0.73256319760435	0.74324008236778	1511	0.77357206083738	0.78552007992821
1515	0.54940986989902	0.55769816034822	1518	0.43495028334971	0.44016076289038	1521	0.48524109405880	0.49162745378198
1525	0.25139216084576	0.25461884087082	1528	0.15847884606033	0.15952908885283	1531	0.19607431893860	0.19773482763574
1535	0.04442901344292	0.04485787470676	1538	0.00858337292818	0.00853682141769	1541	0.01617997675236	0.01609919563440
1545	0.83839350179801	0.85177670957122	1548	0.90556238127299	0.92086345838978	1549	0.88569785421839	0.90023092787438
1553	0.57418112231669	0.58216061586144	1556	0.68294964208784	0.69394532362209	1557	0.64037595617481	0.65023092787438
1561	0.27258897482204	0.27528558608466	1564	0.38166390665600	0.38707029384531	1565	0.33678588189982	0.34121393349943
1569	0.04833197811391	0.04836745131698	1572	0.11622792130129	0.11745420013553	1573	0.09040646181470	0.09121393349943
1577	0.88508665695370	0.90065277997236	1580	0.81793744398727	0.83156603115381	1583	0.82951112305299	0.84391705351846
1587	0.66258193226048	0.67373464520468	1590	0.55385514579817	0.56194993744403	1593	0.58484211056500	0.59391705351846
1597	0.36143002112248	0.36685961542790	1600	0.25238287529952	0.25507490766725	1603	0.28161967643738	0.28490005914351
1607	0.09602614622625	0.09724352171812	1610	0.02810549542909	0.02815677289956	1613	0.03489456309838	0.03490005914351
1617	0.65635267079838	0.66717453160046	1620	0.74877988510885	0.76226428361846	1621	0.73241397757643	0.74518526832117
1625	0.35935601607851	0.36409521212307	1628	0.47382854613248	0.48163260958091	1629	0.44429606895827	0.45129264217493
1633	0.08297606177439	0.08346353808552	1636	0.17614988063279	0.17855329010351	1637	0.15536636026601	0.15740001602869
1641	0.71028585686771	0.72382128880320	1644	0.61791918191712	0.62873153678521	1647	0.62654006323795	0.63806991399311
1651	0.43551824810234	0.44318961476565	1654	0.32109504085301	0.32565221730781	1657	0.33909574283249	0.34417728784688
1661	0.13792018243823	0.14011029528825	1664	0.04473188839974	0.04502054327026	1667	0.05007982740923	0.05028466170064
1671	0.41034688995027	0.41726466199836	1674	0.51894475732878	0.52904936975901	1675	0.50769484189608	0.51719568269362
1679	0.10929775494143	0.11038963222158	1682	0.21846899638585	0.22217433998223	1683	0.20475936076866	0.20817868831867

1687	0.46621915537101	0.47613712672652	1690	0.35774820828696	0.36435241896587	1693	0.36259309376253	0.36976404559162
1697	0.16601975365279	0.16926209694974	1700	0.05687791216569	0.05747738918909	1703	0.06025114905852	0.06074705121667
1707	0.12463704105198	0.12651002002600	1710	0.23879668617424	0.24404741748384	1711	0.23365010213536	0.23857938012980
1715	0.17716617541089	0.18184534524342	1718	0.06326251800604	0.06430794778558	1721	0.06412645376133	0.06526309611003
1725	0.97300345442538	0.98932210182092	1728	0.95096654046697	0.96715571192955	1731	0.93728794848522	0.95300104853184
1735	0.88477612818715	0.90065277997236	1738	0.82901027535080	0.84391705351846	1741	0.81763527394378	0.83156603115380
1745	0.70999243708722	0.72382128880320	1748	0.62600924200233	0.63806991399311	1751	0.61798226056600	0.62873153678521
1755	0.46592103120957	0.47613712672652	1758	0.36204332409295	0.36976404559162	1761	0.35805607294350	0.36435241896587
1765	0.17688758932963	0.18184534524342	1768	0.06377900868302	0.06526309611003	1771	0.06361388647389	0.06430794778558
1775	0.83764800205995	0.85177670957122	1778	0.88509390424311	0.90023092787438	1779	0.90487791951048	0.92086345838978

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
1783	0.65604570584651	0.66717453160046	1786	0.73236438048395	0.74518526832117	1787	0.74817892526272	0.76226428361846
1791	0.41033507521464	0.41726466199836	1794	0.50799324115202	0.51719568269362	1795	0.51837403170486	0.52904936975901
1799	0.12475966191614	0.12651002002600	1802	0.23414117983754	0.23857938012980	1803	0.23825534098463	0.24404741748384
1807	0.82385868143697	0.83832983438577	1810	0.77208159014525	0.78552007992821	1813	0.73142221985174	0.74324008236778
1817	0.66188261881603	0.67373464520468	1820	0.58374080417887	0.59391705351846	1823	0.55343513560535	0.56194993744403
1827	0.43507932283752	0.44318961476565	1830	0.33826282314507	0.34417728784688	1833	0.32114641982727	0.32565221730781
1837	0.16575701544912	0.16926209694974	1840	0.05984095269659	0.06074705121667	1843	0.05716687493918	0.05747738918909
1847	0.57335339889101	0.58216061586144	1850	0.63989360151120	0.65023092787438	1851	0.68184685144877	0.69394532362209
1855	0.35907512499342	0.36409521212307	1858	0.44437212720648	0.45129264217493	1859	0.47305230728508	0.48163260958091
1863	0.10934765200378	0.11038963222158	1866	0.20513772645295	0.20817868831867	1867	0.21790169162563	0.22217433998223
1871	0.54849503835245	0.55769816034822	1874	0.48384975970416	0.49162745378198	1877	0.43424966312297	0.44016076289038
1881	0.36093823623152	0.36685961542790	1884	0.28070716665863	0.28490005914351	1887	0.25225355151169	0.25507490766725
1891	0.13768315822163	0.14011029528825	1894	0.04971636807213	0.05028466170064	1897	0.04494850792711	0.04502054327026
1901	0.27216572996089	0.27528558608466	1904	0.33676011342850	0.34121393349943	1905	0.38084231585983	0.38707029384531
1909	0.08297524683275	0.08346353808552	1912	0.15564145419216	0.15740001602869	1913	0.17562806892062	0.17855329010351
1917	0.25102354877776	0.25461884087082	1920	0.19526232258410	0.19773482763574	1923	0.15827621105049	0.15952908885283
1927	0.09587757664251	0.09724352171812	1930	0.03462712597554	0.03490005914351	1933	0.02824204065362	0.02815677289956
1937	0.04830653243007	0.04836745131698	1940	0.09060613723327	0.09121393349943	1941	0.11585914521303	0.11745420013553
1945	0.04441016252778	0.04485787470676	1948	0.01604240625689	0.01609919563440	1951	0.00862613760696	0.00853682141769
1954	0.97298892461233	0.98932210182092	1957	0.93722665756930	0.95300104853184	1959	0.95113531320749	0.96715571192955
1963	0.82378200127156	0.83832983438577	1966	0.73131146114767	0.74324008236778	1969	0.77256572313822	0.78552007992821
1973	0.54838337477857	0.55769816034822	1976	0.43412824536170	0.44016076289038	1979	0.48444624370510	0.49162745378198
1983	0.25091965740453	0.25461884087082	1986	0.15818619928977	0.15952908885283	1989	0.19573780461830	0.19773482763574
1993	0.04434788553983	0.04485787470676	1996	0.00856794174998	0.00853682141769	1999	0.01615160081058	0.01609919563440
2003	0.83761979383501	0.85177670957122	2006	0.90505029714716	0.92086345838978	2007	0.88474253711801	0.90023092787438
2011	0.57335612136279	0.58216061586144	2014	0.68205965275040	0.69394532362209	2015	0.63932244415864	0.65023092787438
2019	0.27214941899623	0.27528558608466	2022	0.38106386061436	0.38707029384531	2023	0.33618623078726	0.34121393349943
2027	0.04824991950654	0.04836745131698	2030	0.11603322508603	0.11745420013553	2031	0.09024542645912	0.09121393349943
2034	0.88485712084928	0.90065277997236	2037	0.81742552980635	0.83156603115380	2039	0.82932588057632	0.84391705351846
2043	0.66189692887712	0.67373464520468	2046	0.55319527947461	0.56194993744403	2049	0.58430732239187	0.59391705351846
2053	0.36093328730045	0.36685961542790	2056	0.25201759511959	0.25507490766725	2059	0.28126398501877	0.28490005914351
2063	0.09587549772273	0.09724352171812	2066	0.02805980602024	0.02815677289956	2069	0.03484183017893	0.03490005914351
2073	0.65593161764960	0.66717453160046	2076	0.74849243448231	0.76226428361846	2077	0.73188001199110	0.74518526832117
2081	0.35896528909856	0.36409521212307	2084	0.47337762628923	0.48163260958091	2085	0.44376626140775	0.45129264217493
2089	0.08286136351539	0.08346353808552	2092	0.17592390692018	0.17855329010351	2093	0.15514171323342	0.15740001602869
2096	0.71015870262304	0.72382128880320	2099	0.61764482445799	0.62873153678521	2101	0.62644054782220	0.63806991399311
2105	0.43518058901209	0.44318961476565	2108	0.32079685964177	0.32565221730781	2111	0.33885296414335	0.34417728784688
2115	0.13776005621482	0.14011029528825	2118	0.04467447479599	0.04502054327026	2121	0.05002370809796	0.05028466170064
2125	0.41014489936254	0.41726466199836	2128	0.51879743656815	0.52904936975901	2129	0.50742197740106	0.51719568269362
2133	0.10919491670242	0.11038963222158	2136	0.21829725441003	0.22217433998223	2137	0.20455834614804	0.20817868831867
2140	0.46615596674986	0.47613712672652	2143	0.35762305178595	0.36435241896587	2145	0.36254763063804	0.36976404559162
2149	0.16590997709098	0.16926209694974	2152	0.05683075666280	0.05747738918909	2155	0.06021263983495	0.06074705121667
2159	0.12458357827927	0.12651002002600	2162	0.23874024422357	0.24404741748384	2163	0.23354572958019	0.23857938012980
2166	0.17714554848392	0.18184534524342	2169	0.06324265731456	0.06430794778558	2171	0.06411922505963	0.06526309611003

Example 3(pentagonal domain,refer section 7.5.2)

Table-5b

Mesh:number of nodes=8541,number of nine node elements= 2100

fem computed values and exact values at centroid points

```
        NODE FEM SOLUTION EXACT SOLUTION       NODE FEM SOLUTION EXACT SOLUTION       NODE FEM
SOLUTION EXACT SOLUTION
--------------------------------------------------------------------------------------------------------------------------------------------------

 2176   0.99334058161356   0.99732465707428  2180   0.98773832500749   0.99174663408490  2183   0.98420116458847
0.98811523439862
 2188   0.97077941391612   0.97482011001528  2192   0.95613342885394   0.96014371300095  2195   0.95273139590986
0.95662803121455
 2200   0.92432005015106   0.92831225660525  2204   0.90099207633806   0.90489886716863  2207   0.89781199393043
0.90158547103557
 2212   0.85510896710957   0.85894627454584  2216   0.82367417800256   0.82737240803946  2219   0.82079656240239
0.82434288436907
 2224   0.76485257811161   0.76843018452787  2228   0.72608608179689   0.72947329433276  2231   0.72358340024676
0.72680224005222
 2236   0.65577673729797   0.65899279309326  2240   0.61063497951629   0.61361212713653  2243   0.60856982639828
0.61136531246694
 2248   0.53057339974713   0.53332881202100  2252   0.48017139077361   0.48264179290830  2255   0.47859474697517
0.48087454188358
 2260   0.39233698400499   0.39453250557993  2264   0.33792254275057   0.33978721594239  2267   0.33687180965164
0.33854304414794
 2272   0.24449460611102   0.24602149947317  2276   0.18742272234921   0.18856595003087  2279   0.18691731212896
0.18787549310543
 2284   0.09074670863061   0.09145262755084  2288   0.03248568338437   0.03270156461507  2291   0.03246524833412
0.03258182389965
 2295   0.95788543152270   0.96174642343239  2298   0.97057022124839   0.97445771976688  2299   0.97593941693685
0.97993850223311
 2303   0.90793986777333   0.91169622366786  2306   0.92933828940258   0.93317442234844  2307   0.93445530736108
0.93842300923754
 2311   0.83564988130072   0.83919703713035  2314   0.86523472317840   0.86891327362344  2315   0.86997093949328
0.87380042730712
 2319   0.74279680440135   0.74603403440339  2322   0.77983898556723   0.78325659634400  2323   0.78407589857134
0.78766197887905
 2327   0.63166970875088   0.63450119780442  2330   0.67525587222792   0.67831354218297  2331   0.67888797261794
0.68212867843074
 2335   0.50501035100814   0.50734483592675  2338   0.55406478752745   0.55666815741981  2339   0.55700230971734
0.55979910606418
 2343   0.36594795267937   0.36769596040760  2346   0.41925805364900   0.42131575514528  2347   0.42142968808664
0.42368542183959
 2351   0.21792658245035   0.21899319003028  2354   0.27417068776018   0.27558916071225  2355   0.27552714403215
0.27713919639802
 2359   0.06461838220790   0.06489808051768  2362   0.12240343354141   0.12307664651450  2363   0.12292932621447
0.12376888416887
 2367   0.95401974310133   0.95782627229675  2370   0.93962381270555   0.94340572588151  2373   0.92748301893207
0.93109948672708
 2377   0.90838965218024   0.91212918069330  2380   0.88546294146503   0.88912395204085  2383   0.87404492771478
0.87752579051652
 2387   0.84040656308697   0.84397244147798  2390   0.80951360592261   0.81294899566772  2393   0.79910147970160
0.80234449700232
 2397   0.75174470101210   0.75503429976952  2400   0.71364654047794   0.71675653699822  2403   0.70449771414630
0.70740681914320
 2407   0.64458863916650   0.64750470778557  2410   0.60022439157505   0.60291515360926  2413   0.59256404168798
0.59505043764815
 2417   0.52158043895676   0.52403140095101  2420   0.47204463181774   0.47422799817777  2423   0.46605960022986
0.46804193951901
```

```
NODE     FEM SOLUTION    EXACT SOLUTION      NODE   FEM SOLUTION      EXACT SOLUTION  NODE    FEM SOLUTION      EXACT SOLUTION
----------------------------------------------------------------------------------------------------------------------------------

2427   0.38575599428770   0.38765470186453  2430   0.33227227858115   0.33386377555864  2433   0.32810503524999
0.32950869549679
 2437   0.24047202829078   0.24173265746595  2440   0.18436332120879   0.18527871875492  2443   0.18210469257135
0.18286185381476
```

2447 0.08932280900539 0.08985835277587 2450 0.03198587855304 0.03213148499064 2453 0.03164165015162 0.03171234640812
2457 0.87918855801572 0.88264418273435 2460 0.89990679964170 0.90343795880678 2461 0.91353895404213 0.91724757291113
2465 0.80921786483889 0.81245524963461 2468 0.83786090806754 0.84122455084753 2469 0.85052885511205 0.85408319411027
2473 0.71933835375618 0.72226097190448 2476 0.75520120724341 0.75829740258220 2477 0.76659150890425 0.76988845253080
2481 0.61176315034994 0.61428223199932 2484 0.65396259866028 0.65669845562050 2485 0.66379431414585 0.66673650213674
2489 0.48914268572837 0.49117782485650 2492 0.53663882734720 0.53892941322420 2493 0.54467076373270 0.54716728628849
2497 0.35449984784243 0.35597898953996 2500 0.40612159747114 0.40789014007015 2501 0.41215906883045 0.41412499590771
2505 0.21115453907683 0.21201477007442 2508 0.26562921307264 0.26680725795781 2509 0.26953079110845 0.27088557372561
2513 0.06262792778365 0.06283004333297 2516 0.11862244968087 0.11915469567202 2517 0.12030836878469 0.12097604970072
2521 0.87006895247377 0.87348645716948 2524 0.84811166353999 0.85145585438060 2527 0.82873272748207 0.83185851269382
2531 0.80497943343654 0.80821720591697 2534 0.77539422141053 0.77850808099961 2537 0.75769970979813 0.76058972529065
2541 0.72008687583216 0.72304696473567 2544 0.68360155498880 0.68639085494422 2547 0.66802779537566 0.67059269459825
2551 0.61748010820591 0.62007290762727 2554 0.57499276181545 0.57737240803946 2557 0.56192254773555 0.56408344619528
2561 0.49968518162233 0.50183059762910 2564 0.45224209682686 0.45413713625960 2567 0.44199478816842 0.44368459126136
2571 0.36960312214542 0.37123155283703 2574 0.31837276561172 0.31971950099029 2577 0.31119557285935 0.31236074918584
2581 0.23043711869592 0.23149155516722 2584 0.17667999931820 0.17742931051841 2587 0.17274090933625 0.17334554879948
2591 0.08560989058489 0.08605146713278 2594 0.03065734543324 0.03077022156744 2597 0.03001758925830 0.03006200570078
2601 0.76283068479011 0.76570811750848 2604 0.78982889614418 0.79282208776554 2605 0.81011640922291 0.81333559813481
2609 0.67813124059071 0.68070344723054 2612 0.71193392335769 0.71466641012403 2613 0.73019616718998 0.73315771736796
2617 0.57674987852778 0.57893759895656 2620 0.61652587053722 0.61891327362344 2621 0.63231310822646 0.63492706038855
2625 0.46118010231389 0.46291638561253 2628 0.50595102063749 0.50792043827084 2629 0.51887592044648 0.52106239198028
2633 0.33426457461125 0.33549663452332 2636 0.38292926456467 0.38442091603673 2637 0.39267698414699 0.39436743817451
2641 0.19912302855139 0.19981584284266 2644 0.25048571552726 0.25145567502992 2645 0.25682243274306 0.25796184921040
2649 0.05906516650497 0.05921492196044 2652 0.11187200737583 0.11229876077033 2653 0.11465200223041 0.11520438339244
2657 0.74970155463807 0.75256098042718 2660 0.72215034674627 0.72489771365710 2663 0.69760845630351 0.70010671028975
2667 0.67066275281687 0.67325581377473 2670 0.63668653038324 0.63912395204085 2673 0.61507143496223 0.61726635234273
2677 0.57512611708225 0.57737285457920 2680 0.53555978333525 0.53761283759456 2683 0.51740289519321 0.51922684821146
2687 0.46543953547128 0.46727305951329 2690 0.42125557471553 0.42286391085199 2693 0.40700167085405 0.40840225586211
2697 0.34429880338124 0.34566745093179 2700 0.29658283786209 0.29770267121937 2703 0.28657881738633 0.28752144456405
2707 0.21467953119569 0.21555036250386 2710 0.16460260612670 0.16521100380282 2713 0.15908892381969 0.15956090107186
2717 0.07976254366742 0.08012570878043 2720 0.02856375517595 0.02865129316875 2723 0.02764698935499 0.02767143864301
2727 0.62018983166664 0.62238474456056 2730 0.65110168180828 0.65343795880678 2731 0.67578660377608 0.67837420599256

2735	0.52749473385578	0.52933760084375	2738	0.56386921635523	0.56588839277437	2739	0.58522270627061	0.58748360721149
2743	0.42181874428685	0.42325640862338	2746	0.46276278267121	0.46440477643596	2747	0.48026041727337	0.48212721227457
2751	0.30575611989114	0.30675323891523	2754	0.35026469532000	0.35148597323061	2755	0.36347849353214	0.36489924528297
2759	0.18215409707036	0.18269678640934	2762	0.22913570186943	0.22991241884926	2763	0.23774482159934	0.23868624784134
2767	0.05403547322678	0.05414173268619	2770	0.10234442829512	0.10267765768026	2771	0.10614381370189	0.10659600282361
2775	0.60468857615167	0.60688687027071	2778	0.57405680314604	0.57611969630745	2781	0.54693242344979	0.54874086390296
2785	0.51857151213369	0.52045596566075	2788	0.48289968141067	0.48461545485338	2791	0.46010399864735	0.46158516200956
2795	0.41969273308363	0.42120970788190	2798	0.37985601061571	0.38117837255433	2801	0.36194853390279	0.36306370151415
2805	0.31047794371435	0.31159186918012	2808	0.26745331358831	0.26835541366451	2811	0.25487207006623	0.25560240774812
2815	0.19360490860759	0.19430160454500	2818	0.14844653827143	0.14892465386972	2821	0.14149736380452	0.14184733440758
2825	0.07193752579728	0.07222698955594	2828	0.02576181566167	0.02582687484406	2831	0.02459138844595	0.02459950892961
2835	0.46521221664358	0.46670355422739	2838	0.49728985412145	0.49892946161930	2839	0.52368373222076	0.52557435787858
2843	0.37203286579995	0.37317445414644	2846	0.40814123340315	0.40945392773414	2847	0.42977968213217	0.43132046051419
2851	0.26968534110175	0.27045656050940	2854	0.30893968109371	0.30989627924849	2855	0.32529198457781	0.32644602194140
2859	0.16067616466993	0.16107912875876	2862	0.20211540762585	0.20270795588095	2863	0.21278200786731	0.21353339889618
2867	0.04766745631641	0.04773539426710	2870	0.09028239936285	0.09052828989049	2871	0.09500554364726	0.09536287489342
2875	0.44920933795424	0.45072372357340	2878	0.41831054040702	0.41968523126722	2881	0.39143728548365	0.39257771720566
2885	0.36357434432303	0.36477477532757	2888	0.32906701689750	0.33010695766590	2891	0.30794749804801	0.30878531389558
2895	0.26897926835903	0.26984386149511	2898	0.23170821332601	0.23240035520468	2901	0.21686064597223	0.21738959135769
2905	0.16773940678389	0.16826849623220	2908	0.12861628836541	0.12897128470575	2911	0.12040379624563	0.12064101560588
2915	0.06233152885986	0.06254980214093	2918	0.02232204246482	0.02236651314621	2921	0.02092700317564	0.02092185766528
2925	0.31304993092995	0.31390370611341	2928	0.34343217214392	0.34442096443183	2929	0.36867865957778	0.36989316754540
2933	0.22694370087301	0.22750034398998	2936	0.25997450355529	0.26067591038448	2937	0.27906214584497	0.27995461412740
2941	0.13522170497335	0.13549516836715	2944	0.17009391095331	0.17051215028973	2945	0.18255452397328	0.18312264899348
2949	0.04011937257473	0.04015365201645	2952	0.07598548384765	0.07614981515745	2953	0.08151552089047	0.08178159649211
2957	0.29846608743287	0.29935787718661	2960	0.27013961756392	0.27090721391756	2963	0.24632732879816	0.24690360704919
2967	0.22082583206706	0.22145140237965	2970	0.19022926857720	0.19072282870714	2973	0.17348048851471	0.17382392175333
2977	0.13772172248926	0.13809205909104	2980	0.10560181447181	0.10584221448118	2983	0.09632798456139	0.09646411461538
2987	0.05118207573550	0.05133243100633	2990	0.01832950738564	0.01835541366451	2993	0.01674416599000	0.01672904082956
2997	0.17858261469602	0.17894231397116	3000	0.20457352936949	0.20503683547307	3001	0.22592745393499	0.22656979459748
3005	0.10641709694516	0.10657486724767	3008	0.13385909099648	0.13411776966098	3009	0.14780756966396	0.14820281172333
3013	0.03157700924775	0.03158319359083	3016	0.05980556078418	0.05989627924849	3017	0.06600664731696	0.06618658376761

3021 0.16720052603148 0.16760607478254 3024 0.14403527715194 0.14434907319406 3027 0.12579564903185 0.12597813030688
3031 0.10428897756775 0.10451533715378 3034 0.07996819354586 0.08010695766590 3037 0.06985989729566 0.06991194697701
3041 0.03876287567213 0.03885108505778 3044 0.01388221892430 0.01389234298027 3047 0.01214531439682 0.01212429948811
3051 0.07496757346626 0.07503033919484 3054 0.09429855427499 0.09442096443183 3055 0.10939369540792 0.10963372937162
3059 0.02224929565374 0.02223505212040 3062 0.04213847331718 0.04216789816007 3063 0.04885920849328 0.04896183769007
3067 0.06825813052354 0.06836510074929 3070 0.05234125770196 0.05239920169315 3073 0.04164392477659 0.04163831517961
3077 0.02537677407466 0.02541309645575 3080 0.00908858451002 0.00908719670580 3083 0.00724210075300 0.00722101765502
3087 0.01236212965521 0.01233940987285 3090 0.02341267755268 0.02340120347172 3091 0.03049075027066 0.03053148867358
3095 0.01134563678258 0.01134935307775 3098 0.00406359805619 0.00405829348976 3101 0.00215028090079 0.00213993040208
3105 0.99340016148415 0.99732465707428 3108 0.98425607534638 0.98811523439862 3111 0.98791498713410 0.99174663408490
3115 0.95407068542295 0.95782627229675 3118 0.92752856394070 0.93109948672708 3121 0.93981241829438 0.94340572588151
3125 0.87010826793930 0.87348645716948 3128 0.82876593245569 0.83185851269382 3131 0.84829626887066 0.85145585438060
3135 0.74972742543434 0.75256098042718 3138 0.69762829372471 0.70010671028975 3141 0.72231880695013 0.72489771365710
3145 0.60470123505738 0.60688687027071 3148 0.54693997671328 0.54874086390296 3151 0.57420073257832 0.57611969630745
3155 0.44921111921489 0.45072372357340 3158 0.39143556799624 0.39257771720566 3161 0.41842614667333 0.41968523126722
3165 0.29846083361278 0.29935787718661 3168 0.24632046199327 0.24690360704919 3171 0.27022680750596 0.27090721391756
3175 0.16719251236662 0.16760607478254 3178 0.12578773852043 0.12597813030688 3181 0.14409563826861 0.14434907319406
3185 0.06825101943238 0.06836510074929 3188 0.04163811896346 0.04163831517961 3191 0.05237627145581 0.05239920169315
3195 0.01134156902717 0.01134935307775 3198 0.00214659035564 0.00213993040208 3201 0.00407056736573 0.00405829348976
3205 0.95810100151452 0.96174642343239 3208 0.97617376392937 0.97993850223311 3209 0.97066280402546 0.97445771976688
3213 0.87938089257817 0.88264418273435 3216 0.91375498144432 0.91724757291113 3217 0.89996535870019 0.90343795880678
3221 0.76298750577900 0.76570811750848 3224 0.81030107795426 0.81333559813481 3225 0.78985145989764 0.79282208776554
3229 0.62030510020036 0.62238474456056 3232 0.67593140346321 0.67837420599256 3233 0.65109100947003 0.65343795880678
3237 0.46528712087175 0.46670355422739 3240 0.52378718272395 0.52557435787858 3241 0.49725361211608 0.49892946161930
3245 0.31309160222364 0.31390370611341 3248 0.36874597865702 0.36989316754540 3249 0.34338175031761 0.34442096443183
3253 0.17860081014248 0.17894231397116 3256 0.22596783574804 0.22656979459748 3257 0.20452165598241 0.20503683547307
3261 0.07497121502628 0.07503033919484 3264 0.10941665938459 0.10963372937162 3265 0.09425666637298 0.09442096443183
3269 0.01235883115460 0.01233940987285 3272 0.03050307655420 0.03053148867358 3273 0.02338912104186 0.02340120347172
3277 0.97112711430432 0.97482011001528 3280 0.95304925408245 0.95662803121455 3283 0.95659697076022 0.96014371300095
3287 0.90869748811317 0.91212918069330 3290 0.87431727277896 0.87752579051652 3293 0.88590036830300 0.88912395204085
3297 0.80522999121515 0.80821720591697 3300 0.75790989984245 0.76058972529065 3303 0.77577364293088 0.77850808099961
3307 0.67084574926376 0.67325581377473 3310 0.61521338463054 0.61726635234273 3313 0.63698952225524 0.63912395204085

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 3399  0.37214291165368  0.37317445414644  3402  0.42994770407557  0.43132046051419  3403  0.40816569616198
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 3407  0.22699158286591  0.22750034398998  3410  0.27915903506577  0.27995461412740  3411  0.25995476521958
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NODE      FEM SOLUTION    EXACT SOLUTION      NODE    FEM SOLUTION       EXACT SOLUTION   NODE     FEM SOLUTION       EXACT SOLUTION
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 3431  0.92493706025101  0.92831225660525  3434  0.89836721604963  0.90158547103557  3437  0.90171938922176
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4035 0.93652974473584 0.93993303458354 4038 0.91508084111518 0.91829927782838 4041 0.93317773124210 0.93649135662911

4045 0.87217999069080 0.87520646784491 4048 0.84247876891025 0.84527500843665 4051 0.86907184234161 0.87200179400632

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4065 0.68113358410524 0.68322629814331 4068 0.63729712699851 0.63909663833082 4071 0.67874668848994 0.68072458280648

4075 0.55912030048628 0.56069988410404 4078 0.50974626700513 0.51101933335552 4081 0.55719207473544 0.55864681398184

4085 0.42329439000941 0.42436717806191 4088 0.36959090876189 0.37035903641707 4091 0.42187431897275 0.42281330655449

4095 0.27697799146748 0.27758514370197 4098 0.22025802690581 0.22057927085086 4101 0.27609981536605 0.27656873228285

4105 0.12373911871909 0.12396804185182 4108 0.06534769478314 0.06536811157567 4111 0.12345401070192 0.12351411794341

4115 0.98094792561712 0.98461545485338 4118 0.98641291483774 0.99015336892751 4119 0.98640399313874 0.99015336892751

4123 0.94984879190468 0.95323977537202 4126 0.96428959777396 0.96781064138278 4127 0.95512408746914 0.95860121870711

4131 0.89536743235576 0.89839216899956 4134 0.91842983721456 0.92163720386783 4135 0.90032748373926 0.90344512506710

4139 0.81884054228978 0.82142316583367 4142 0.84995876735102 0.85276999965514 4143 0.82335885720284 0.82604321408547

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4179	0.18713971885122	0.18721006180255	4182	0.24390221614697	0.24425247566482	4183	0.18804499511754	0.18826301423282
4187	0.97782856562787	0.98151533055183	4190	0.97429251500323	0.97792139401507	4193	0.96515701900046	0.96871197133940
4197	0.96091016858390	0.96440477643596	4200	0.94844691669347	0.95182461554134	4203	0.94846649289968	0.95182461554134
4207	0.92034452947939	0.92354737604830	4210	0.89925968393075	0.90229075610819	4213	0.90843981090102	0.91150017878385
4217	0.85712675193463	0.85994917418431	4220	0.82793761250254	0.83053950372832	4223	0.84605924571770	0.84873158252905
4227	0.77280756881833	0.77517616964422	4230	0.73622983675918	0.73833761236408	4233	0.76285508524838	0.76506439793375
4237	0.66945392927852	0.67131575514528	4240	0.62638373711517	0.62795539858139	4243	0.66086681533574	0.66255878875816
4247	0.54959715132309	0.55092531878541	4250	0.50108856235558	0.50211083882114	4253	0.54259190667192	0.54373878329680
4257	0.41616964010575	0.41696927266073	4260	0.36340936023827	0.36390264379878	4263	0.41092228339339	0.41153012442510
4267	0.27242892131708	0.27274605920147	4270	0.21667608237209	0.21673395796246	4273	0.26909804776313	0.26918822809988
4277	0.94983701840468	0.95323977537202	4280	0.95514244224786	0.95860121870711	4281	0.96427593222055	0.96781064138278
4285	0.91973381402113	0.92286391085199	4288	0.93372535586516	0.93697046278006	4289	0.93370258332826	0.93697046278006
4293	0.86700311772884	0.86976407403706	4296	0.88933729508589	0.89226838443267	4297	0.88015264587337	0.88305896175701
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4309	0.69935751830788	0.70114980513271	4312	0.73654240230669	0.73859410389057	4313	0.70990889716920	0.71186731832090
4317	0.58855122423102	0.58978721594239	4320	0.63188153399583	0.63340587249850	4321	0.59738757033782	0.59880248232173
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4341	0.93655303469214	0.93993303458354	4344	0.93315348821213	0.93649135662911	4347	0.91509588704228	0.91829927782838
4351	0.92035632422983	0.92354737604830	4354	0.90841162921006	0.91150017878385	4357	0.89928321703429	0.90229075610819
4361	0.88152386368509	0.88442091603673	4364	0.86132757187920	0.86406484144128	4367	0.86135900604502	0.86406484144128
4371	0.82100959197258	0.82351707784760	4374	0.79305747819098	0.79535336003562	4377	0.80225561892547	0.80456278271128
4381	0.74029847383585	0.74233551610538	4384	0.70527595737138	0.70705763927942	4387	0.72342385563312	0.72524971808015
4391	0.64136889696627	0.64287519028632	4394	0.60013374668902	0.60135181285439	4397	0.62679621566288	0.62807859842407

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
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4411	0.39892342153697	0.39930419992497	4414	0.34839625100632	0.34848576036653	4417	0.39000195357232	0.39011370484219
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7355	0.31038211013594	0.31159186918012	7358	0.26737361105835	0.26835541366451	7361	0.25479010265924	0.25560240774812
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7921 0.80902047178824 0.81245524963461 7924 0.85037854977521 0.85408319411027 7925 0.83759045490975 0.84122455084753

7929 0.67791973991678 0.68070344723054 7932 0.73001212545625 0.73315771736796 7933 0.71162927611068 0.71466641012403

7937 0.52730794235396 0.52933760084375 7940 0.58505143966441 0.58748360721149 7941 0.56358167241911 0.56588839277437

7945 0.37189072389300 0.37317445414644 7948 0.42964789763743 0.43132046051419 7949 0.40790231747467 0.40945392773414

7953 0.22685083385153 0.22750034398998 7956 0.27897970286979 0.27995461412740 7957 0.25979872620324 0.26067591038448

7961 0.10636720454444 0.10657486724767 7964 0.14777171648785 0.14820281172333 7965 0.13374796616132 0.13411776966098

7969 0.02223215158963 0.02223505212040 7972 0.04885831659526 0.04896183769007 7973 0.04208456499109 0.04216789816007

7976 0.92429816748018 0.92831225660525 7979 0.89770680154709 0.90158547103557 7981 0.90099312960711 0.90489886716863

7985 0.84027090784352 0.84397244147798 7988 0.79891204683234 0.80234449700232 7991 0.80944072736129 0.81294899566772

7995 0.71990586042253 0.72304696473567 7998 0.66781338358518 0.67059269459825 8001 0.68350597289562 0.68639085494422

8005 0.57494755238461 0.57737285457920 8008 0.51720438490200 0.51922684821146 8011 0.53547706459084 0.53761283759456

8015 0.41954633861949 0.42120970788190 8018 0.36178940474056 0.36306370151415 8021 0.37980422338075 0.38117837255433

8025 0.26887905084036 0.26984386149511 8028 0.21674985671286 0.21738959135769 8031 0.23169112897478 0.23240035520468

8035 0.13766941210633 0.13809205909104 8038 0.09626352703091 0.09646411461538 8041 0.10561282981490 0.10584221448118

8045 0.03875006946801 0.03885108505778 8048 0.01212119292267 0.01212429948811 8051 0.01389781229068 0.01389234298027

8055 0.83554997686047 0.83919703713035 8058 0.86994898671378 0.87380042730712 8059 0.86509186355358 0.86891327362344

8063 0.71918631580695 0.72226097190448 8066 0.76649403340071 0.76988845253080 8067 0.75497865883709 0.75829740258220

8071 0.57659126075344 0.57893759895656 8074 0.63219419827888 0.63492706038855 8075 0.61628151301878 0.61891327362344

8079 0.42168242015385 0.42325640862338 8082 0.48015512485972 0.48212721227457 8083 0.46253727942992 0.46440477643596

8087 0.26958546470219 0.27045656050940 8090 0.32521966537506 0.32644602194140 8091 0.30875792245831 0.30989627924849

8095 0.13516077320369 0.13549516836715 8098 0.18252163151776 0.18312264899348 8099 0.16996744003065 0.17051215028973

8103 0.03155352418059 0.03158319359083 8106 0.06600940330038 0.06618658376761 8107 0.05973580186757 0.05989627924849

8110 0.85510047455317 0.85894627454584 8113 0.82070297261133 0.82434288436907 8115 0.82369138967416 0.82737240803946

8119 0.75165027724396 0.75503429976952 8122 0.70434309629989 0.70740681914320 8125 0.71361440048697 0.71675653699822

8129 0.61735492974641 0.62007290762727 8132 0.56175313953112 0.56408344619528 8135 0.57494745733007 0.57737240803946

8139 0.46532069087636 0.46727305951329 8142 0.40684898662129 0.40840225586211 8145 0.42122251027938 0.42286391085199

8149 0.31038814772085 0.31159186918012 8152 0.25475395566946 0.25560240774812 8155 0.26744434024554 0.26835541366451

8159 0.16768917171235 0.16826849623220 8162 0.12032681523608 0.12064101560588 8165 0.12863253142675 0.12897128470575

```
 8169   0.05117058476899   0.05133243100633  8172   0.01671226367370   0.01672904082956  8175   0.01835082636863
0.01835541366451
 8179   0.74271518696958   0.74603403440339  8182   0.78407580408290   0.78766197887905  8183   0.77971225016119
0.78325659634400
 8187   0.61164567097957   0.61428223199932  8190   0.66373938581085   0.66673650213674  8191   0.65377746730851
0.65669845562050
 8195   0.46106162587878   0.46291638561253  8198   0.51880875096260   0.52106239198028  8199   0.50575435927539
0.50792043827084
 8203   0.30565839883329   0.30675323891523  8206   0.36342566161479   0.36489924528297  8207   0.35008937390543
0.35148597323061
 8211   0.16060963224000   0.16107912875876  8214   0.21275790307868   0.21353339889618  8215   0.20198176709039
0.20270795588095
 8219   0.04009124403672   0.04015365201645  8222   0.08152400734626   0.08178159649211  8223   0.07590371407911
0.07614981515745
 8226   0.76485544482560   0.76843018452787  8229   0.72349793673123   0.72680224005222  8231   0.72611696196935
0.72947329433276
 8235   0.64452746733114   0.64750470778557  8238   0.59243583093585   0.59505043764815  8241   0.60022441111368
0.60291515360926
 8245   0.49960479196474   0.50183059762910  8248   0.44185998628344   0.44368459126136  8251   0.45223556179730
0.45413713625960
 8255   0.34422835372303   0.34566745093179  8258   0.28646220157008   0.28752144456405  8261   0.29658781770874
0.29770267121937
 8265   0.19356233188804   0.19430160454500  8268   0.14141307790202   0.14184733440758  8271   0.14847087999312
0.14892465386972
 8275   0.06232322407717   0.06254980214093  8278   0.02088866796780   0.02092185766528  8281   0.02234912796337
0.02236651314621
 8285   0.63160188834391   0.63450119780442  8288   0.67890598398654   0.68212867843074  8289   0.67514125897584
0.67831354218297
 8293   0.48905162016337   0.49117782485650  8296   0.54464994231534   0.54716728628849  8297   0.53648337405354
0.53892941322420
 8301   0.33417719593368   0.33549663452332  8304   0.39265121464044   0.39436743817451  8305   0.38277151307764
0.38442091603673
 8309   0.18208734356322   0.18269678640934  8312   0.23773470342324   0.23868624784134  8313   0.22900298793202
0.22991241884926
 8317   0.04763650934842   0.04773539426710  8320   0.09502169356117   0.09536287489342  8321   0.09019279531639
0.09052828989049
 8324   0.65578910592768   0.65899279309326  8327   0.60849004566024   0.61136531246694  8329   0.61067714133864
0.61361212713653
 8333   0.52154612321040   0.52403140095101  8336   0.46595174800117   0.46804193951901  8339   0.47206973464879
0.47422799817777
 8343   0.36955920522734   0.37123155283703  8346   0.31108833699221   0.31236074918584  8349   0.31839615927849
0.31971950099029
 8353   0.21464958156739   0.21555036250386  8356   0.15900251000686   0.15956090107186  8359   0.16463740240370
0.16521100380282
 8363   0.07193416802876   0.07222698955594  8366   0.02454812701611   0.02459950892961  8369   0.02579455449737
0.02582687484406
 8373   0.50495303849376   0.50734483592675  8376   0.55703507324544   0.55979910606418  8377   0.55395960318317
0.55666815741981
 8381   0.35442944833147   0.35597898953996  8384   0.41216562047774   0.41412499590771  8385   0.40599056875178
0.40789014007015
 8389   0.19906095008433   0.19981584284266  8392   0.25683040157566   0.25796184921040  8393   0.25036136316333
0.25145567502992
 8397   0.05400354484553   0.05414173268619  8400   0.10616920155098   0.10659600282361  8401   0.10225125067349
0.10267765768026
 8404   0.53059359163660   0.53332881202100  8407   0.47851917227222   0.48087454188358  8409   0.48022256473861
0.48264179290830
 8413   0.38574359025944   0.38765470186453  8416   0.32801361091950   0.32950869549679  8419   0.33231677806009
0.33386377555864
```

NODE   FEM SOLUTION   EXACT SOLUTION   NODE   FEM SOLUTION   EXACT SOLUTION   NODE   FEM SOLUTION   EXACT SOLUTION

```
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8423   0.23042374804694   0.23149155516722  8426   0.17265714251349   0.17334554879948  8429   0.17672682356054
0.17742931051841
 8433   0.07976562379289   0.08012570878043  8436   0.02760039230531   0.02767143864301  8439   0.02860184241238
0.02865129316875
```

8443	0.36589895709787	0.36769596040760	8446	0.42147423727744	0.42368542183959	8447	0.41916079785422	0.42131575514528
8451	0.21110121196948	0.21201477007442	8454	0.26955952013709	0.27088557372561	8455	0.26551953875861	0.26680725795781
8459	0.05903398096757	0.05921492196044	8462	0.11468766300961	0.11520438339244	8463	0.11177934149543	0.11229876077033
8466	0.39236349341003	0.39453250557993	8469	0.33679983527836	0.33854304414794	8471	0.33798058650729	0.33978721594239
8475	0.24047789009550	0.24173265746595	8478	0.18202765968486	0.18286185381476	8481	0.18442277031928	0.18527871875492
8485	0.08562049563395	0.08605146713278	8488	0.02996925398428	0.03006200570078	8491	0.03070025784039	0.03077022156744
8495	0.21788473641525	0.21899319003028	8498	0.27558088270819	0.27713919639802	8499	0.27408092040773	0.27558916071225
8503	0.06259899632818	0.06283004333297	8506	0.12035469227660	0.12097604970072	8507	0.11853399278333	0.11915469567202
8510	0.24452609282903	0.24602149947317	8513	0.18684915048285	0.18787549310543	8515	0.18748562863578	0.18856595003087
8519	0.08934156135770	0.08985835277587	8522	0.03159311651568	0.03171234640812	8525	0.03203287302417	0.03213148499064
8529	0.06459272938985	0.06489808051768	8532	0.12298928946282	0.12376888416887	8533	0.12232274309259	0.12307664651450
8536	0.09077924129546	0.09145262755084	8539	0.03241707042506	0.03258182389965	8541	0.03252805709123	0.03270156461507

Example 4 (square domain,refer section 7.5.2)

Table -6a(MESH NO.1- square domain,
Mesh: (number of nodes=2481,number of nine node elements= 600)
fem computed values and exact values at centroid points

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
646	0.96902996923803	0.98932210182092	650	0.94740423458267	0.96715571192955	653	0.93403439514655	0.95300104853184
658	0.88186168899164	0.90065277997236	662	0.82646073680609	0.84391705351846	665	0.81527146245855	0.83156603115381
670	0.70801250657272	0.72382128880320	674	0.62434905325806	0.63806991399311	677	0.61642736549350	0.62873153678521
682	0.46476233780122	0.47613712672652	686	0.36116992366837	0.36976404559162	689	0.35723293843446	0.36435241896587
694	0.17647165816887	0.18184534524342	698	0.06363100012132	0.06526309611003	701	0.06347405798344	0.06430794778558
705	0.83522529710944	0.85177670957122	708	0.88236347870189	0.90023092787438	709	0.90189058959931	0.92086345838978
713	0.65440092565543	0.66717453160046	716	0.73043595874941	0.74518526832117	717	0.74609495893444	0.76226428361846
721	0.40940173559384	0.41726466199836	724	0.50680019128999	0.51719568269362	725	0.51709523311369	0.52904936975901
729	0.12448981584229	0.12651002002600	732	0.23362623893643	0.23857938012980	733	0.23770586772619	0.24404741748384
737	0.82156689268265	0.83832983438577	740	0.77005160519013	0.78552007992821	743	0.72959964890093	0.74324008236778
747	0.66027879736022	0.67373464520468	750	0.58238469070505	0.59391705351846	753	0.55220750151548	0.56194993744403
757	0.43412242686107	0.44318961476565	760	0.33753808095102	0.34417728784688	763	0.32048702072224	0.32565221730781
767	0.16541010239666	0.16926209694974	770	0.05971733650313	0.06074705121667	773	0.05705418458479	0.05747738918909
777	0.57208532006535	0.58216061586144	780	0.63841881453065	0.65023092787438	781	0.68019784936268	0.69394532362209

785 0.35834424244575 0.36409521212307 788 0.44344239588810 0.45129264217493 789 0.47201948286052 0.48163260958091

793 0.10913477858500 0.11038963222158 796 0.20473224688998 0.20817868831867 797 0.21745285764904 0.22217433998223

801 0.54731399982079 0.55769816034822 804 0.48284451781850 0.49162745378198 807 0.43338017168823 0.44016076289038

811 0.36022278148629 0.36685961542790 814 0.28016324237411 0.28490005914351 817 0.25178133740352 0.25507490766725

821 0.13742175114996 0.14011029528825 824 0.04962312287989 0.05028466170064 827 0.04486743765352 0.04502054327026

831 0.27165875835472 0.27528558608466 834 0.33611753290053 0.34121393349943 835 0.38009082057636 0.38707029384531

839 0.08282677757508 0.08346353808552 842 0.15535903493286 0.15740001602869 843 0.17529862376390 0.17855329010351

847 0.25056343427960 0.25461884087082 850 0.19491159151197 0.19773482763574 853 0.15799886624346 0.15952908885283

857 0.09570853902865 0.09724352171812 860 0.03456678508065 0.03490005914351 863 0.02819428133358 0.02815677289956

867 0.04822502747283 0.04836745131698 870 0.09045123143407 0.09121393349943 871 0.11565623152721 0.11745420013553

875 0.04433538440485 0.04485787470676 878 0.01601570191411 0.01609919563440 881 0.00861208187306 0.00853682141769

885 0.96901580187087 0.98932210182092 888 0.93397311480852 0.95300104853184 891 0.94757307526631 0.96715571192955

895 0.82149020541956 0.83832983438577 898 0.72948888573366 0.74324008236778 901 0.77053572240536 0.78552007992821

905 0.54720233437172 0.55769816034822 908 0.43325875283685 0.44016076289038 911 0.48344099789862 0.49162745378198

915 0.25045954251627 0.25461884087082 918 0.15790885430320 0.15952908885283 921 0.19538707281805 0.19773482763574

925 0.04427310739175 0.04485787470676 928 0.00855388601390 0.00853682141769 931 0.01612489644984 0.01609919563440

935 0.83519705672752 0.85177670957122 938 0.90206292287087 0.92086345838978 939 0.88201209430657 0.90023092787438

943 0.57208803442553 0.58216061586144 946 0.68041063701139 0.69394532362209 947 0.63784765143169 0.65023092787438

951 0.27164244563678 0.27528558608466 954 0.38031236206917 0.38707029384531 955 0.33554364893400 0.34121393349943

959 0.04816841442641 0.04836745131698 962 0.11583031095364 0.11745420013553 963 0.09009052050843 0.09121393349943

967 0.88194258967577 0.90065277997236 970 0.81506166980062 0.83156603115380 973 0.82677626226379 0.84391705351846

977 0.66029308796868 0.67373464520468 980 0.55196763381299 0.56194993744403 983 0.58295119095369 0.59391705351846

987 0.36021782791205 0.36685961542790 990 0.25154537855326 0.25507490766725 993 0.28072005667386 0.28490005914351

997 0.09570645954372 0.09724352171812 1000 0.02801204659017 0.02815677289956 1003 0.03478148906288 0.03490005914351

1007 0.65428680822753 0.66717453160046 1010 0.74640841843486 0.76226428361846 1011 0.72995155643589 0.74518526832117

1015 0.35823439934388 0.36409521212307 1018 0.47234478940420 0.48163260958091 1019 0.44283652175676 0.45129264217493

1023 0.08271289345299 0.08346353808552 1026 0.17559445947110 0.17855329010351 1027 0.15485929257602 0.15740001602869

1031 0.70817872237942 0.72382128880320 1034 0.61608989890361 0.62873153678521 1037 0.62478032204907 0.63806991399311

1041 0.43422367968658 0.44318961476565 1044 0.32013745287289 0.32565221730781 1047 0.33812821174445 0.34417728784688

1051 0.13749864694906 0.14011029528825 1054 0.04459340398641 0.04502054327026 1057 0.04993046210185 0.05028466170064

1061 0.40921154418880 0.41726466199836 1064 0.51751861320615 0.52904936975901 1065 0.50622890714730 0.51719568269362

1069 0.10898204095685 0.11038963222158 1072 0.21784841462334 0.22217433998223 1073 0.20415286225039 0.20817868831867

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 1077  0.46499725085411  0.47613712672652  1080  0.35679990295295  0.36435241896587  1083  0.36167421405766
0.36976404559162
 1087  0.16556305913541  0.16926209694974  1090  0.05671806493676  0.05747738918909  1093  0.06008902189307
0.06074705121667
 1097  0.12431372788149  0.12651002002600  1100  0.23819076136347  0.24404741748384  1101  0.23303078035169
0.23857938012980
 1105  0.17672960993770  0.18184534524342  1108  0.06310282648947  0.06430794778558  1111  0.06397121389282
0.06526309611003
 1115  0.96902996923804  0.98932210182092  1118  0.94740423458267  0.96715571192955  1121  0.93403439514655
0.95300104853184
 1125  0.88186168899164  0.90065277997236  1128  0.82646073680610  0.84391705351846  1131  0.81527146245856
0.83156603115380
 1135  0.70801250657272  0.72382128880320  1138  0.62434905325807  0.63806991399311  1141  0.61642736549350
0.62873153678521
 1145  0.46476233780123  0.47613712672652  1148  0.36116992366837  0.36976404559162  1151  0.35723293843446
0.36435241896587
 1155  0.17647165816887  0.18184534524342  1158  0.06363100012132  0.06526309611003  1161  0.06347405798344
0.06430794778558
 1165  0.83522529710944  0.85177670957122  1168  0.88236347870190  0.90023092787438  1169  0.90189058959931
0.92086345838978
 1173  0.65440092565544  0.66717453160046  1176  0.73043595874942  0.74518526832117  1177  0.74609495893445
0.76226428361846
 1181  0.40940173559384  0.41726466199836  1184  0.50680019128999  0.51719568269362  1185  0.51709523311369
0.52904936975901
 1189  0.12448981584229  0.12651002002600  1192  0.23362623893643  0.23857938012980  1193  0.23770586772619
0.24404741748384
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NODE FEM SOLUTION EXACT SOLUTION NODE FEM SOLUTION EXACT SOLUTION NODE FEM SOLUTION EXACT SOLUTION

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FEM SOLUTION     EXACT SOLUTION
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 2415  0.43422367968658  0.44318961476565  2418  0.32013745287289  0.32565221730781  2421  0.33812821174445
0.34417728784688
 2425  0.13749864694906  0.14011029528825  2428  0.04459340398641  0.04502054327026  2431  0.04993046210185
0.05028466170064
 2435  0.40921154418880  0.41726466199836  2438  0.51751861320615  0.52904936975901  2439  0.50622890714730
0.51719568269362
 2443  0.10898204095685  0.11038963222158  2446  0.21784841462334  0.22217433998223  2447  0.20415286225039
0.20817868831867
 2450  0.46499725085411  0.47613712672652  2453  0.35679990295295  0.36435241896587  2455  0.36167421405766
0.36976404559162
 2459  0.16556305913541  0.16926209694974  2462  0.05671806493676  0.05747738918909  2465  0.06008902189307
0.06074705121667
 2469  0.12431372788149  0.12651002002600  2472  0.23819076136347  0.24404741748384  2473  0.23303078035170
0.23857938012980
 2476  0.17672960993771  0.18184534524342  2479  0.06310282648947  0.06430794778558  2481  0.06397121389282
0.06526309611003
```

Table -6b(MESH NO.2- square domain,)
Mesh:nodes=9761,number of nine node elements= 2400
fem computed values and exact values at centroid points

NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION	NODE	FEM SOLUTION	EXACT SOLUTION
2486	0.99222025782418	0.99732465707428	2490	0.98667620482132	0.99174663408490	2493	0.98309568598507	0.98811523439862
2498	0.96981113299079	0.97482011001528	2502	0.95521985947286	0.96014371300095	2505	0.95178445547329	0.95662803121455
2510	0.92349490938071	0.92831225660525	2514	0.90021827612323	0.90489886716863	2517	0.89701313985609	0.90158547103557
2522	0.85441776063309	0.85894627454584	2526	0.82303074280470	0.82737240803946	2529	0.82013464161417	0.82434288436907
2534	0.76428581390750	0.76843018452787	2538	0.72556361416198	0.72947329433276	2541	0.72304754023008	0.72680224005222
2546	0.65532539474758	0.65899279309326	2550	0.61022479835642	0.61361212713653	2553	0.60815017109950	0.61136531246694
2558	0.53022951717413	0.53332881202100	2562	0.47986604166877	0.48264179290830	2565	0.47828294938187	0.48087454188358
2570	0.39209404968079	0.39453250557993	2574	0.33771612581818	0.33978721594239	2577	0.33666132942475	0.33854304414794
2582	0.24434779669720	0.24602149947317	2586	0.18731108619539	0.18856595003087	2589	0.18680357908013	0.18787549310543
2594	0.09069302279800	0.09145262755084	2598	0.03246652968041	0.03270156461507	2601	0.03244574318887	0.03258182389965
2605	0.95688725786317	0.96174642343239	2608	0.96951084671644	0.97445771976688	2609	0.97491837166128	0.97993850223311
2613	0.90709694147438	0.91169622366786	2616	0.92843854627146	0.93317442234844	2617	0.93358441883596	0.93842300923754
2621	0.83495005301815	0.83919703713035	2624	0.86448265974883	0.86891327362344	2625	0.86924014679600	0.87380042730712
2629	0.74222817257936	0.74603403440339	2632	0.77922249405823	0.78325659634400	2633	0.78347480549538	0.78766197887905
2637	0.63122143767913	0.63450119780442	2640	0.67476364108903	0.67831354218297	2641	0.67840666624677	0.68212867843074
2645	0.50467320182057	0.50734483592675	2648	0.55368693225990	0.55666815741981	2649	0.55663199120823	0.55979910606418
2653	0.36571459674768	0.36769596040760	2656	0.41898650333195	0.42131575514528	2657	0.42116308866008	0.42368542183959
2661	0.21779176758239	0.21899319003028	2664	0.27399939874190	0.27558916071225	2665	0.27535877454550	0.27713919639802
2669	0.06457901048541	0.06489808051768	2672	0.12232849455682	0.12307664651450	2673	0.12285561409929	0.12376888416887
2677	0.95295614791020	0.95782627229675	2680	0.93862469698219	0.94340572588151	2683	0.92645515251912	0.93109948672708
2687	0.90749357193983	0.91212918069330	2690	0.88462597450644	0.88912395204085	2693	0.87318773083238	0.87752579051652
2697	0.83966359898847	0.84397244147798	2700	0.80882440132879	0.81294899566772	2703	0.79839839498352	0.80234449700232
2707	0.75114086933070	0.75503429976952	2710	0.71309153634517	0.71675653699822	2713	0.70393342397164	0.70740681914320
2717	0.64411130955465	0.64750470778557	2720	0.59979162353681	0.60291515360926	2723	0.59212523361580	0.59505043764815
2727	0.52121888683705	0.52403140095101	2730	0.47172417703671	0.47422799817777	2733	0.46573536283935	0.46804193951901
2737	0.38550169779036	0.38765470186453	2740	0.33205648248967	0.33386377555864	2743	0.32788702666642	0.32950869549679

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 2747  0.24031880460599  0.24173265746595  2750  0.18424689356003  0.18527871875492  2753  0.18198718437679
0.18286185381476
 2757  0.08926686261119  0.08985835277587  2760  0.03196592229619  0.03213148499064  2763  0.03162151840217
0.03171234640812
 2767  0.87828607331939  0.88264418273435  2770  0.89893924942841  0.90343795880678  2771  0.91259504084072
0.91724757291113
 2775  0.80847657255828  0.81245524963461  2778  0.83706121893118  0.84122455084753  2779  0.84974527502340
0.85408319411027
 2783  0.71874152387087  0.72226097190448  2786  0.75455203889612  0.75829740258220  2787  0.76595300458065
0.76988845253080
 2791  0.61129618845498  0.61428223199932  2794  0.65344848720843  0.65669845562050  2795  0.66328704900118
0.66673650213674
 2799  0.48879355991055  0.49117782485650  2802  0.53624674936785  0.53892941322420  2803  0.54428292828387
0.54716728628849
 2807  0.35425925640497  0.35597898953996  2810  0.40584121873458  0.40789014007015  2811  0.41188119364617
0.41412499590771
 2815  0.21101593878857  0.21201477007442  2818  0.26545295758449  0.26680725795781  2819  0.26935587964225
0.27088557372561
 2823  0.06258750529141  0.06283004333297  2826  0.11854548479397  0.11915469567202  2827  0.12023193402276
0.12097604970072
 2831  0.86911658023816  0.87348645716948  2834  0.84722612084224  0.85145585438060  2837  0.82783396111787
0.83185851269382
 2841  0.80419884533594  0.80821720591697  2844  0.77467293430426  0.77850808099961  2847  0.75697090071761
0.76058972529065
 2851  0.71945869563899  0.72304696473567  2854  0.68302605006921  0.68639085494422  2857  0.66744844969248
0.67059269459825
 2861  0.61698756287882  0.62007290762727  2864  0.57454736535095  0.57737240803946  2867  0.56147552358709
0.56408344619528
 2871  0.49931450255423  0.50183059762910  2874  0.45191420534220  0.45413713625960  2877  0.44166645822800
0.44368459126136
 2881  0.36934365306398  0.37123155283703  2884  0.31815288443604  0.31971950099029  2887  0.31097576160220
0.31236074918584
 2891  0.23028127722975  0.23149155516722  2894  0.17656167696692  0.17742931051841  2897  0.17262274660091
0.17334554879948
 2901  0.08555308192753  0.08605146713278  2904  0.03063708611253  0.03077022156744  2907  0.02999736787712
0.03006200570078
 2911  0.76206479419880  0.76570811750848  2914  0.78899909235611  0.79282208776554  2915  0.80929546859688
0.81333559813481
 2919  0.67752087369727  0.68070344723054  2922  0.71126763094265  0.71466641012403  2923  0.72953418496732
0.73315771736796
 2927  0.57627628239051  0.57893759895656  2930  0.61600293992088  0.61891327362344  2931  0.63179174274857
0.63492706038855
 2935  0.46082830401801  0.46291638561253  2938  0.50555506304756  0.50792043827084  2939  0.51848005724971
0.52106239198028
 2943  0.33402328472204  0.33549663452332  2946  0.38264762922417  0.38442091603673  2947  0.39239483427316
0.39436743817451
 2951  0.19898444992189  0.19981584284266  2954  0.25030931920570  0.25145567502992  2955  0.25664546503135
0.25796184921040
 2959  0.05902480884369  0.05921492196044  2962  0.11179513825895  0.11229876077033  2963  0.11457482294849
0.11520438339244
 2967  0.74890286093071  0.75256098042718  2970  0.72141555782597  0.72489771365710  2973  0.69687450801903
0.70010671028975
 2977  0.67002705843163  0.67325581377473  2980  0.63610623849959  0.63912395204085  2983  0.61449419702198
0.61726635234273
 2987  0.57463213857907  0.57737285457920  2990  0.53511436483774  0.53761283759456  2993  0.51696126796860
0.51922684821146
 2997  0.46507037664575  0.46727305951329  3000  0.42092972803036  0.42286391085199  3003  0.40667939714304
0.40840225586211
 3007  0.34404172811010  0.34566745093179  3010  0.29636530729526  0.29770267121937  3013  0.28636404858138
0.28752144456405
 3017  0.21452565348082  0.21555036250386  3020  0.16448587377678  0.16521100380282  3023  0.15897379832128
0.15956090107186
 3027  0.07970654884309  0.08012570878043  3030  0.02854379079365  0.02865129316875  3033  0.02762731078678
0.02767143864301
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3037 0.61958470100313 0.62238474456056 3040 0.65043845998146 0.65343795880678 3041 0.67511955118286 0.67837420599256
3045 0.52702945583474 0.52933760084375 3048 0.56335384630963 0.56588839277437 3049 0.58470236662992 0.58748360721149
3053 0.42147552840622 0.42325640862338 3056 0.46237556455209 0.46440477643596 3057 0.47986830092227 0.48212721227457
3061 0.30552189633557 0.30675323891523 3064 0.34999085208029 0.35148597323061 3065 0.36320057824556 0.36489924528297
3069 0.18202000904966 0.18269678640934 3072 0.22896485030310 0.22991241884926 3073 0.23757117298980 0.23868624784134
3077 0.05399648258722 0.05414173268619 3080 0.10227013497208 0.10267765768026 3081 0.10606824163226 0.10659600282361
3085 0.60406665127214 0.60688687027071 3088 0.57349132716021 0.57611969630745 3091 0.54637833908859 0.54874086390296
3095 0.51809295350119 0.52045596566075 3098 0.48246949206155 0.48461545485338 3101 0.45968394870908 0.46158516200956
3105 0.41933777595133 0.42120970788190 3108 0.37954341072045 0.38117837255433 3111 0.36164409253974 0.36306370151415
3115 0.31023210029099 0.31159186918012 3118 0.26724560879435 0.26835541366451 3121 0.25467015070025 0.25560240774812
3125 0.19345827715838 0.19430160454500 3128 0.14833540079574 0.14892465386972 3131 0.14138944121077 0.14184733440758
3135 0.07188426447786 0.07222698955594 3138 0.02574283049908 0.02582687484406 3141 0.02457296315122 0.02459950892961
3145 0.46477295496442 0.46670355422739 3148 0.49680165733630 0.49892946161930 3149 0.52318285444849 0.52557435787858
3153 0.37171121504260 0.37317445414644 3156 0.40777744026914 0.40945392773414 3157 0.42940526813645 0.43132046051419
3161 0.26946697392493 0.27045656050940 3164 0.30868393771694 0.30989627924849 3165 0.32502817933085 0.32644602194140
3169 0.16055156526690 0.16107912875876 3172 0.20195648366331 0.20270795588095 3173 0.21261782751313 0.21353339889618
3177 0.04763128069916 0.04773539426710 3180 0.09021344349983 0.09052828989049 3181 0.09493424757558 0.09536287489342
3185 0.44876583229496 0.45072372357340 3188 0.41791314814037 0.41968523126722 3191 0.39105707308292 0.39257771720566
3195 0.36324795903845 0.36477477532757 3198 0.32878025044228 0.33010695766590 3201 0.30767384491554 0.30878531389558
3205 0.26875447009083 0.26984386149511 3208 0.23151858612986 0.23240035520468 3211 0.21668001201626 0.21738959135769
3215 0.16760580881077 0.16826849623220 3218 0.12851511866418 0.12897128470575 3221 0.12030752830845 0.12064101560588
3225 0.06228308973104 0.06254980214093 3228 0.02230478035931 0.02236651314621 3231 0.02091058695089 0.02092185766528
3235 0.31276401907905 0.31390370611341 3238 0.34310798749630 0.34442096443183 3239 0.36833766406521 0.36989316754540
3243 0.22675059951287 0.22750034398998 3246 0.25974796627684 0.26067591038448 3247 0.27882332914463 0.27995461412740
3251 0.13511187768744 0.13549516836715 3254 0.16995368844841 0.17051215028973 3255 0.18240648507284 0.18312264899348
3259 0.04008753369117 0.04015365201645 3262 0.07592477194276 0.07614981515745 3263 0.08145137322014 0.08178159649211
3267 0.29818361339199 0.29935787718661 3270 0.26989200073845 0.27090721391756 3273 0.24609781036530 0.24690360704919
3277 0.22063233319767 0.22145140237965 3280 0.19006628979064 0.19072282870714 3283 0.17332967030747 0.17382392175333
3287 0.13760712216228 0.13809205909104 3290 0.10551510385168 0.10584221448118 3293 0.09624782255263 0.09646411461538
3297 0.05114059633334 0.05133243100633 3300 0.01831472885771 0.01835541366451 3303 0.01673051109242 0.01672904082956
3307 0.17842417210075 0.17894231397116 3310 0.20438735908536 0.20503683547307 3311 0.22572487192832 0.22656979459748
3315 0.10632724956390 0.10657486724767 3318 0.13374427299829 0.13411776966098 3319 0.14768246561211 0.14820281172333

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 3323   0.03155099819062   0.03158319359083  3326   0.05975594500328   0.05989627924849  3327   0.06595254776156   0.06618658376761
 3331   0.16704823403916   0.16760607478254  3334   0.14390717803154   0.14434907319406  3337   0.12568254611556   0.12597813030688
 3341   0.10419905860160   0.10451533715378  3344   0.07990020783705   0.08010695766590  3347   0.06979991882288   0.06991194697701
 3351   0.03873037907639   0.03885108505778  3354   0.01387064311684   0.01389234298027  3357   0.01213510692484   0.01212429948811
 3361   0.07490237908617   0.07503033919484  3364   0.09421517806205   0.09442096443183  3365   0.10929788916778   0.10963372937162
 3369   0.02223044301073   0.02223505212040  3372   0.04210250228809   0.04216789816007  3373   0.04881785099964   0.04896183769007
 3377   0.06819774708013   0.06836510074929  3380   0.05229562904974   0.05239920169315  3383   0.04160741578786   0.04163831517961
 3387   0.02535497731832   0.02541309645575  3390   0.00908082127859   0.00908719670580  3393   0.00723589149981   0.00722101765502
 3397   0.01235146661909   0.01233940987285  3400   0.02339232870776   0.02340120347172  3401   0.03046435728973   0.03053148867358
 3405   0.01133575997113   0.01134935307775  3408   0.00406008048682   0.00405829348976  3411   0.00214842152390   0.00213993040208
 3415   0.99221932836816   0.99732465707428  3418   0.98309167568237   0.98811523439862  3421   0.98668693308122   0.99174663408490
 3425   0.95295082955850   0.95782627229675  3428   0.92644687608543   0.93109948672708  3431   0.93865886796193   0.94340572588151
 3435   0.86910723316961   0.87348645716948  3438   0.82782214597077   0.83185851269382  3441   0.84727935655760   0.85145585438060
 3445   0.74889041785613   0.75256098042718  3448   0.69686027711868   0.70010671028975  3451   0.72148262747026   0.72489771365710
 3455   0.60405235055409   0.60688687027071  3458   0.54636304990749   0.54874086390296  3461   0.57356564439101   0.57611969630745
 3465   0.44875109079872   0.45072372357340  3468   0.39104218247676   0.39257771720566  3471   0.41798740668539   0.41968523126722
 3475   0.29816988491876   0.29935787718661  3478   0.24608472812864   0.24690360704919  3481   0.26995888050896   0.27090721391756
 3485   0.16703685993886   0.16760607478254  3488   0.12567248600502   0.12597813030688  3491   0.14396003773267   0.14434907319406
 3495   0.06818979867454   0.06836510074929  3498   0.04160121633675   0.04163831517961  3501   0.05232905300201   0.05239920169315
 3505   0.01133163644087   0.01134935307775  3508   0.00214472612857   0.00213993040208  3511   0.00406700992112   0.00405829348976
 3515   0.95688601997811   0.96174642343239  3518   0.97492942164403   0.97993850223311  3519   0.96948764090913   0.97445771976688
 3523   0.87828891312704   0.88264418273435  3526   0.91260974042798   0.91724757291113  3527   0.89889497049777   0.90343795880678
 3531   0.76207062915696   0.76570811750848  3534   0.80931376299857   0.81333559813481  3535   0.78893814519631   0.79282208776554
 3539   0.61959224685032   0.62238474456056  3542   0.67514031687660   0.67837420599256  3543   0.65036679802795   0.65343795880678
 3547   0.46478074872185   0.46670355422739  3550   0.52320471360022   0.52557435787858  3551   0.49672627706633   0.49892946161930
 3555   0.31277055203343   0.31390370611341  3558   0.36835911472255   0.36989316754540  3559   0.34303623920935   0.34442096443183
 3563   0.17842801481912   0.17894231397116  3566   0.22574441922746   0.22656979459748  3567   0.20432621724276   0.20503683547307
 3571   0.07490225452255   0.07503033919484  3574   0.10931414416260   0.10963372937162  3575   0.09417052714843   0.09442096443183
 3579   0.01234789532233   0.01233940987285  3582   0.03047569521821   0.03053148867358  3583   0.02336843703049   0.02340120347172
 3587   0.96981613431525   0.97482011001528  3590   0.95177066914879   0.95662803121455  3593   0.95524069436949   0.96014371300095
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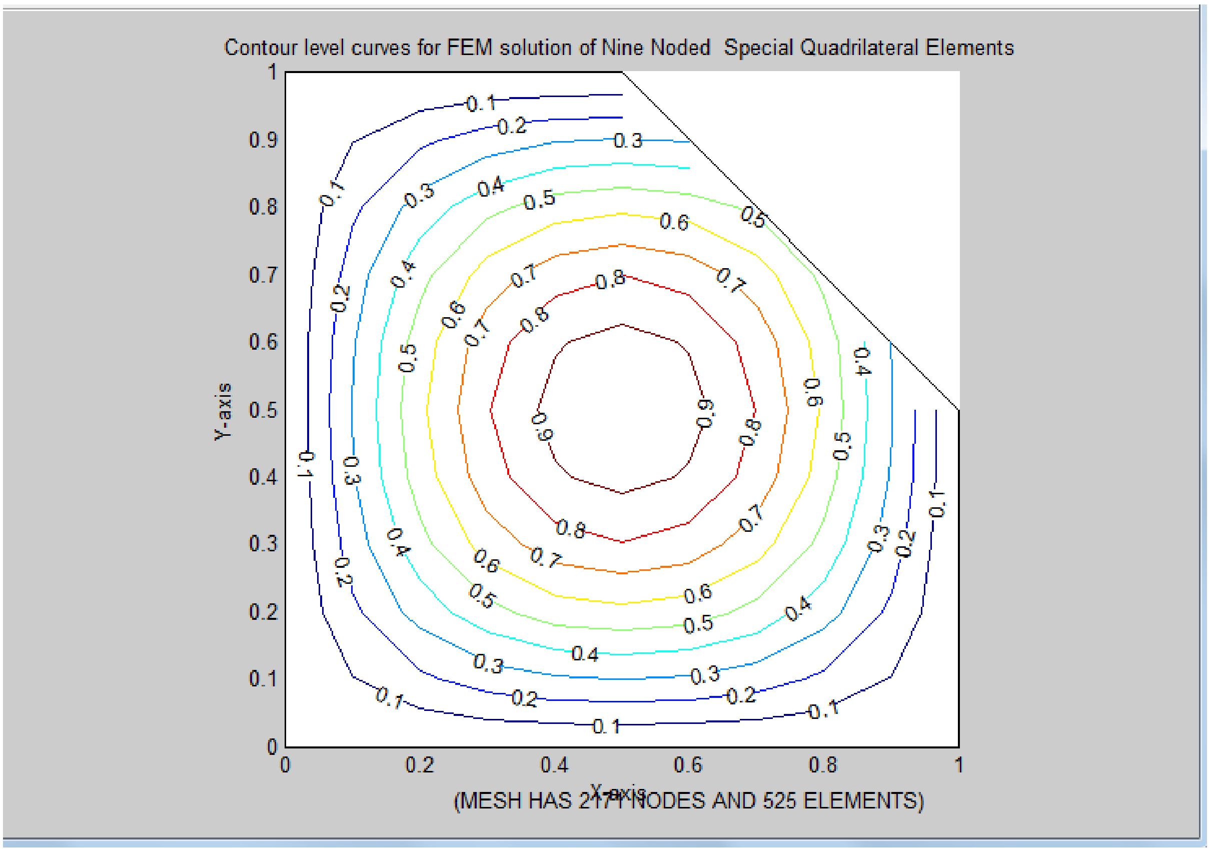
 9679  0.05900454397586  0.05921492196044  9682  0.11462913222152  0.11520438339244  9683  0.11172336699699
0.11229876077033
 9686  0.39212679841847  0.39453250557993  9689  0.33660385233233  0.33854304414794  9691  0.33777936927105
0.33978721594239
 9695  0.24034514765918  0.24173265746595  9698  0.18193000997821  0.18286185381476  9701  0.18432173608361
0.18527871875492
 9705  0.08557637636180  0.08605146713278  9708  0.02995416385750  0.03006200570078  9711  0.03068451029534
0.03077022156744
```

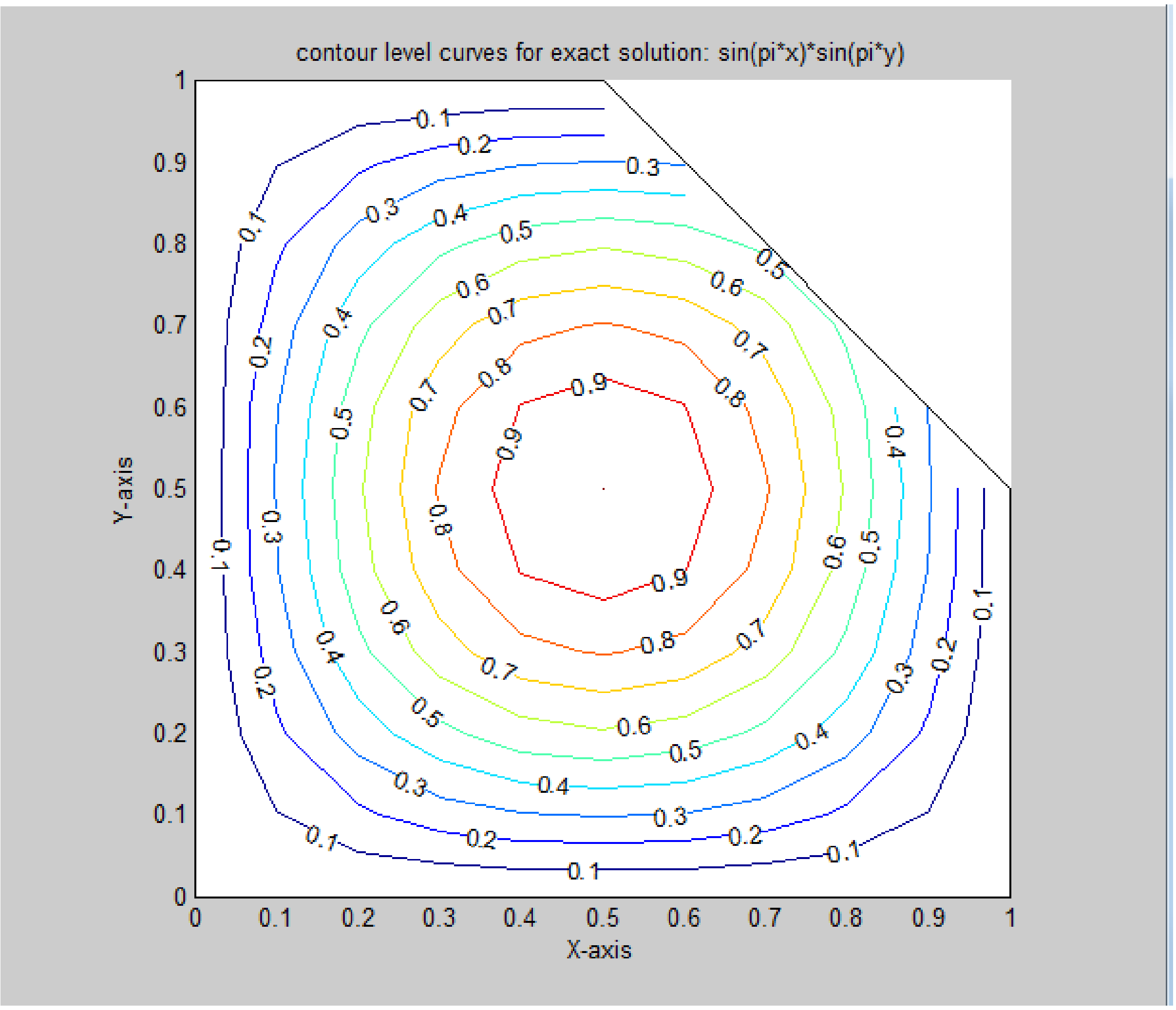
```
 9715   0.21776198616359   0.21899319003028  9718   0.27542073940148   0.27713919639802  9719   0.27392515737276
0.27558916071225
 9723   0.06256609634919   0.06283004333297  9726   0.12028984428072   0.12097604970072  9727   0.11847141871391
0.11915469567202
 9730   0.24438288932732   0.24602149947317  9733   0.18674297133615   0.18787549310543  9735   0.18737670210338
0.18856595003087
 9739   0.08929292576657   0.08985835277587  9742   0.03157633687452   0.03171234640812  9745   0.03201551627851
0.03213148499064
 9749   0.06455681222077   0.06489808051768  9752   0.12291907904851   0.12376888416887  9753   0.12225441238539
0.12307664651450
 9756   0.09072684297265   0.09145262755084  9759   0.03239884060445   0.03258182389965  9761   0.03250936103486
0.03270156461507
--------------------------------------------------------------------------------------------------------------------------------------------------------------
---------------------------------------
```

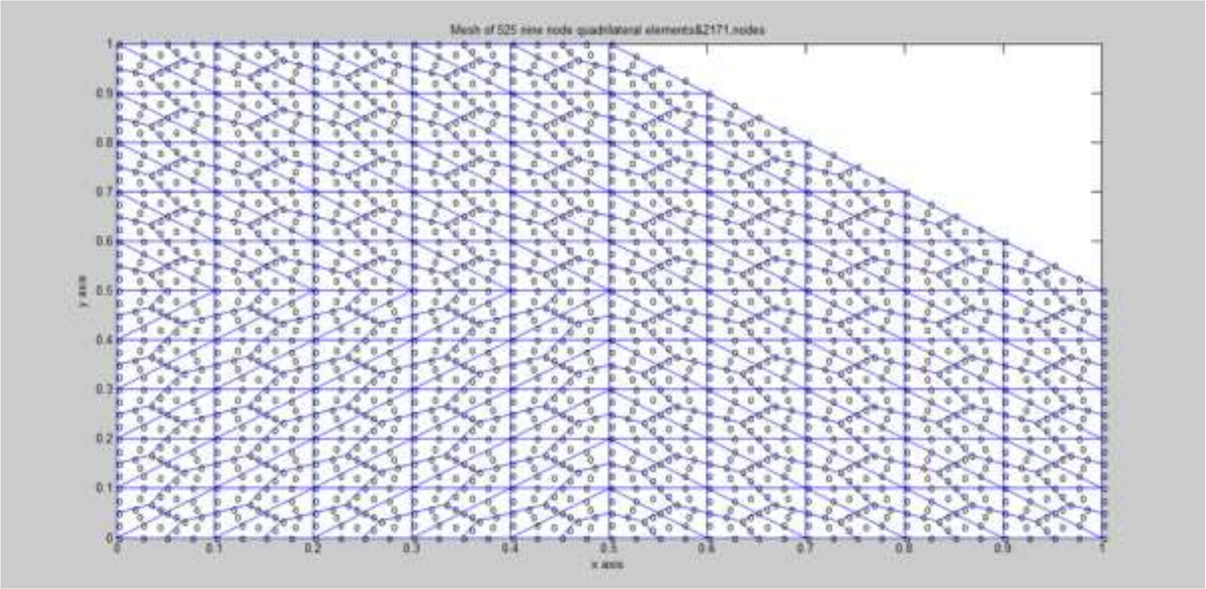
MESHES AND CONTOUR LEVEL CURVES FOR A PENTAGONAL DOMAIN WITH 9-NO0DED QUADRILATERAL ELEMENTS

Mesh No.1

Mesh with 525 nine noded quadrilateral elements & no. of nodes = 2171

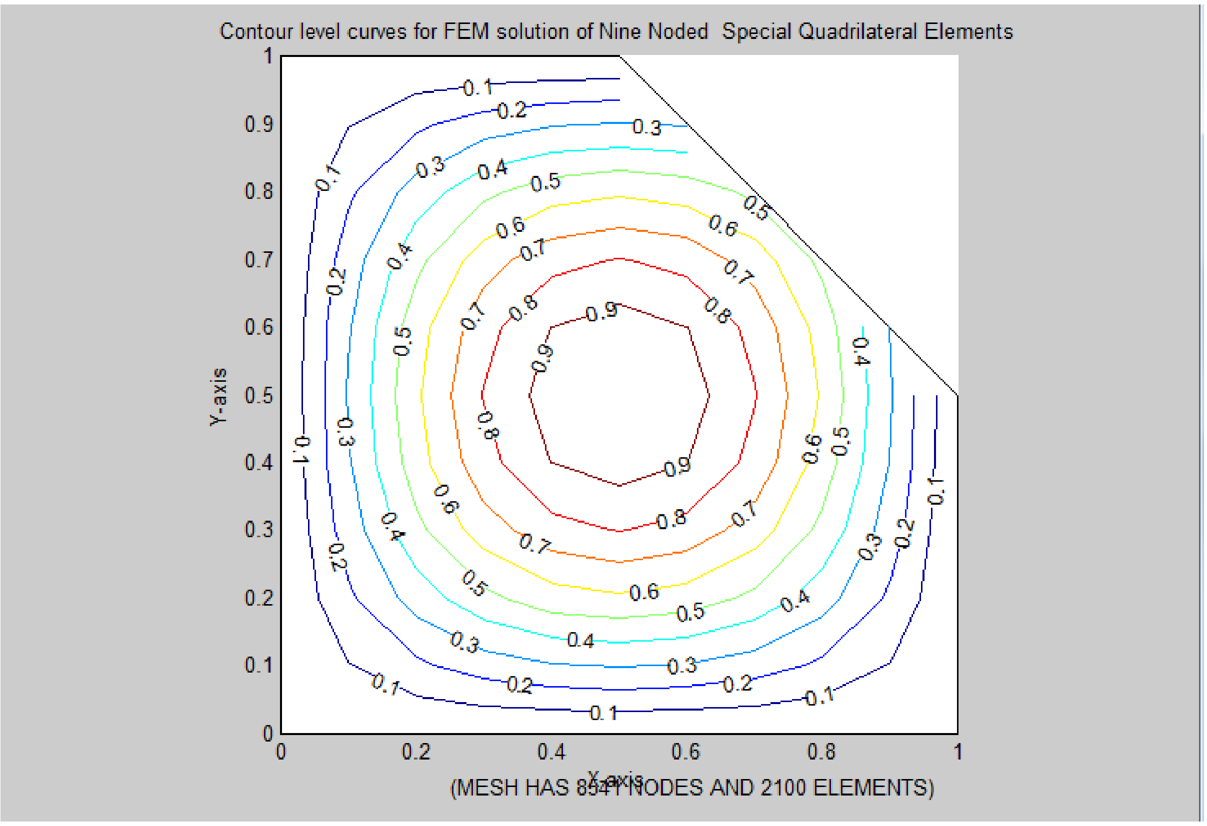

Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
Y-axis
X-axis
(MESH HAS 217 NODES AND 525 ELEMENTS)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

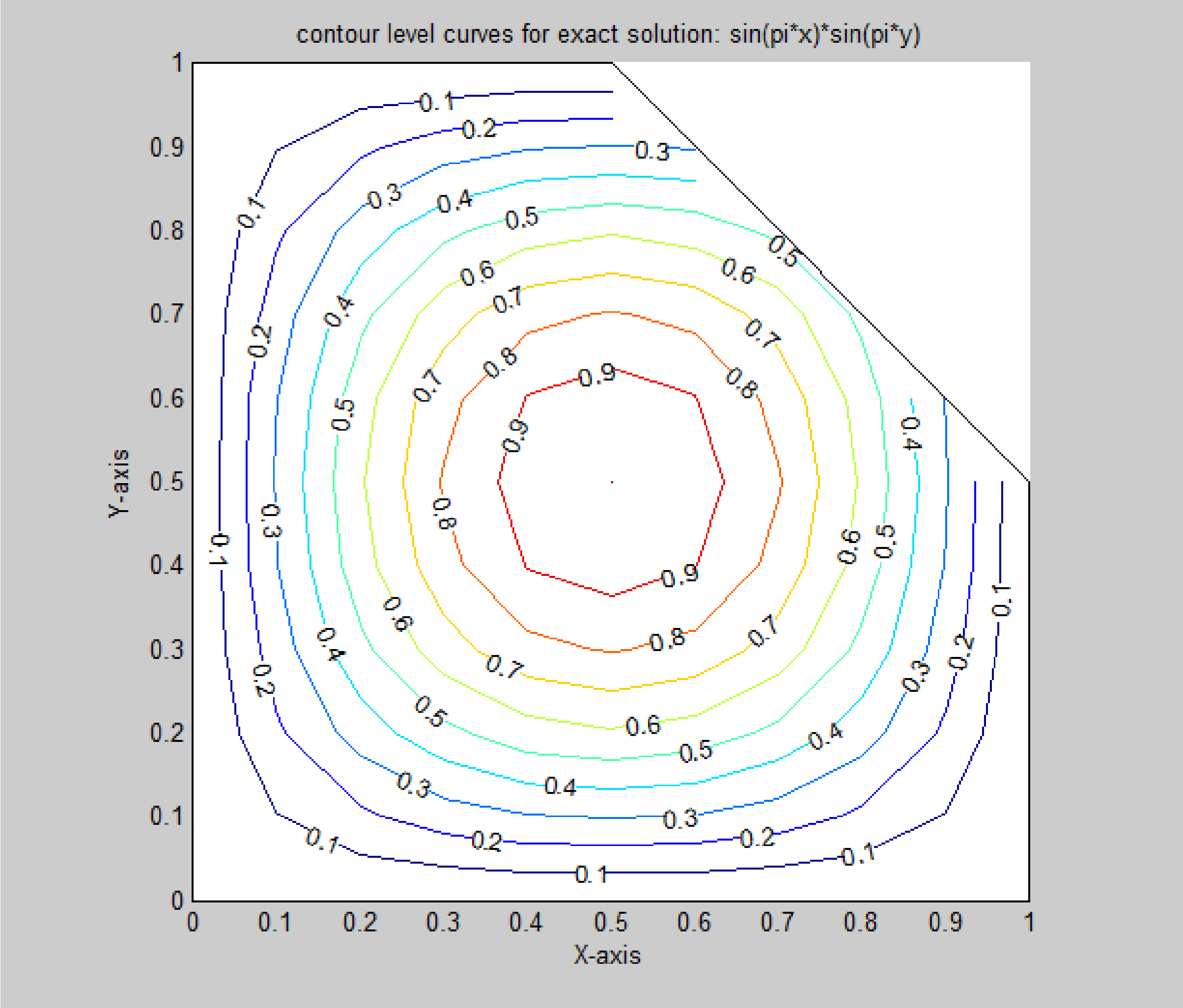

contour level curves for exact solution: sin(pi*x)*sin(pi*y)
Y-axis
X-axis
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

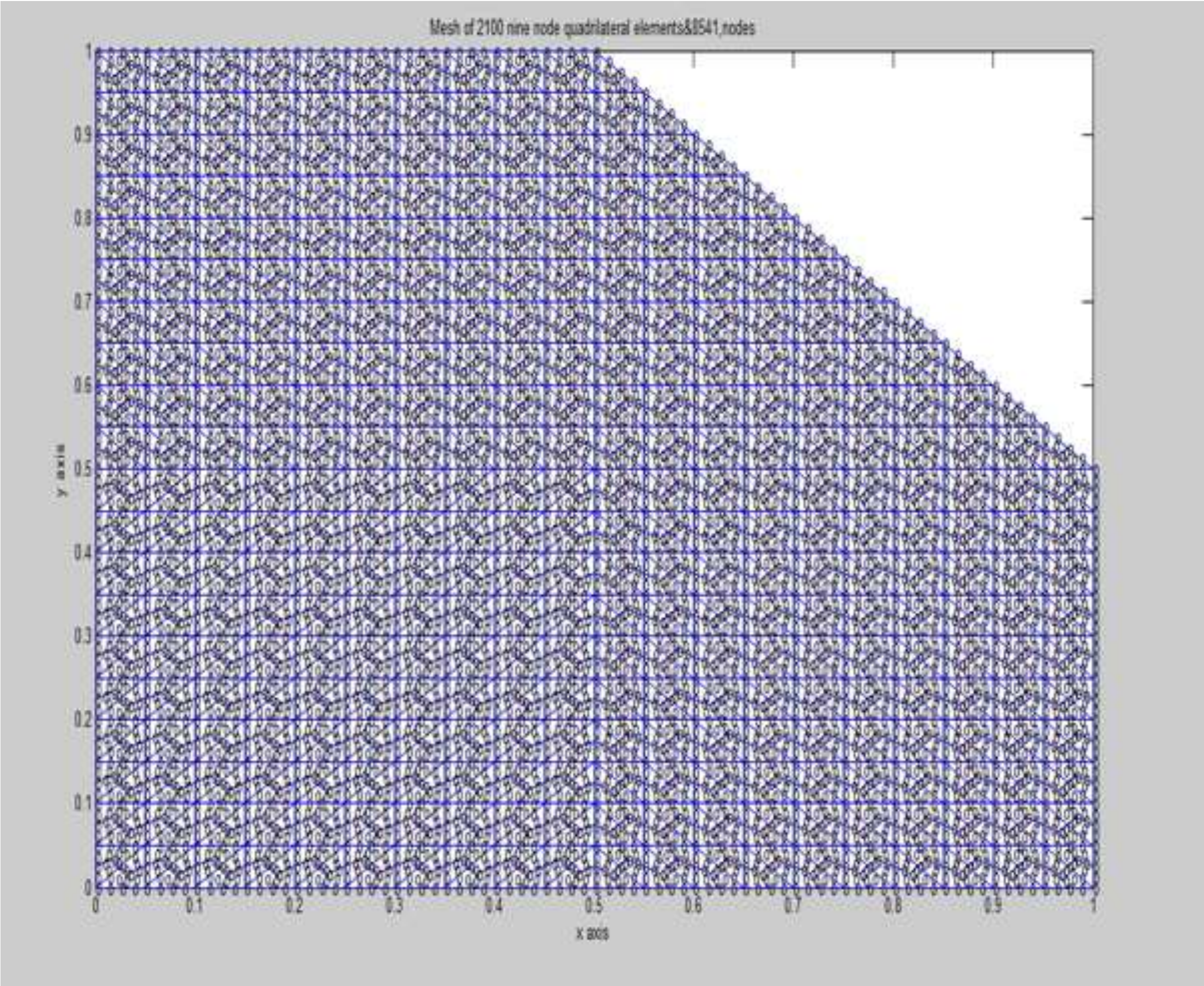


Mesh No.2

Mesh with 2100 nine noded quadrilateral elements & no. of nodes = 8541


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
Y-axis
X-axis
(MESH HAS 8541 NODES AND 2100 ELEMENTS)
0
0.2
0.4
0.6
0.8
1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

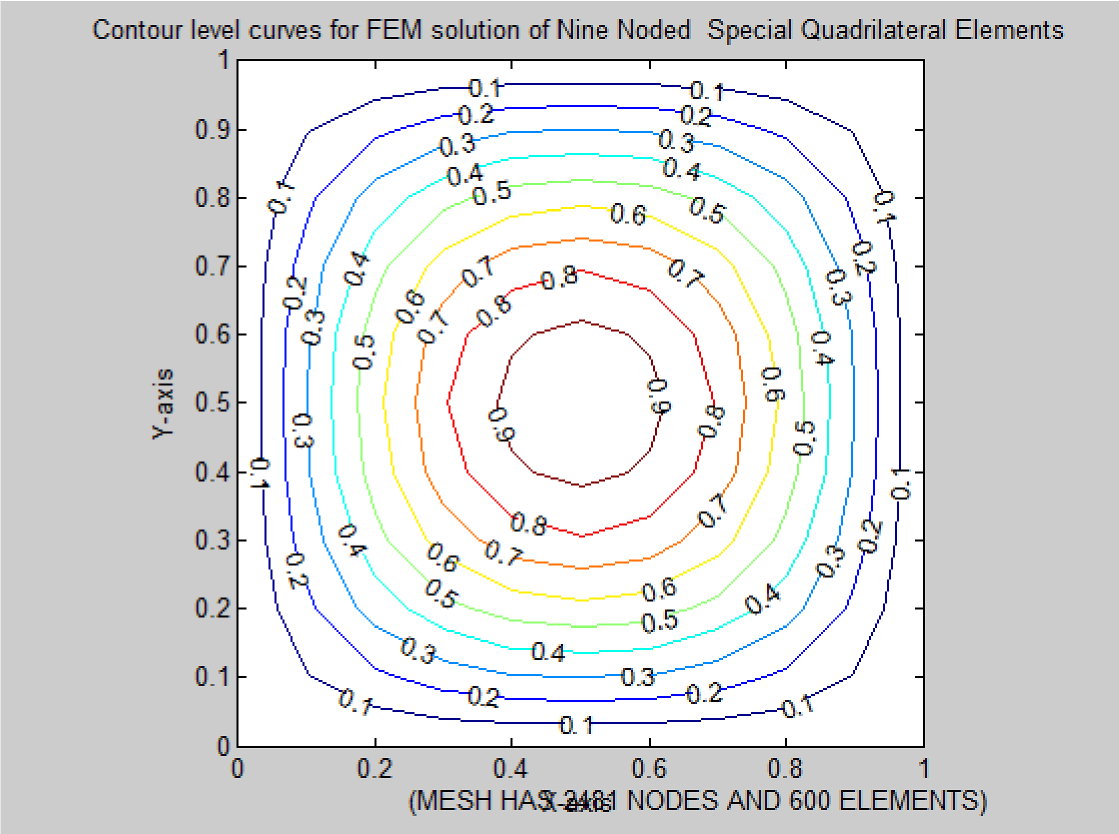

contour level curves for exact solution: sin(pi*x)*sin(pi*y)
Y-axis
X-axis
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

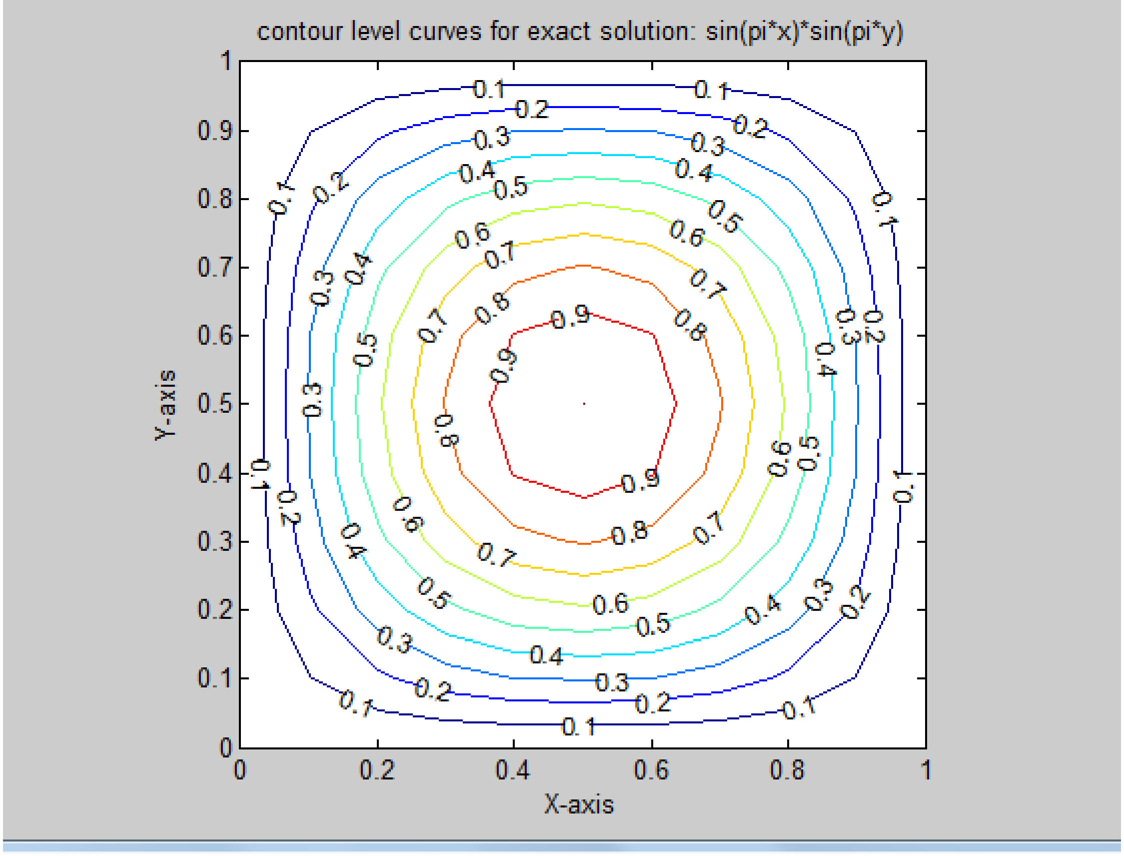


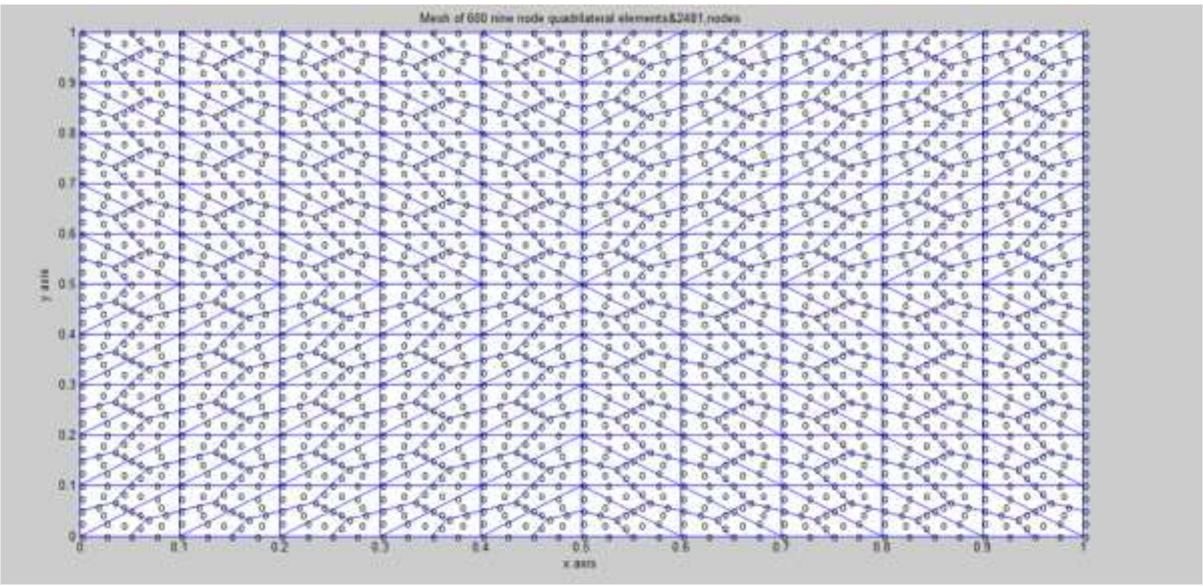
MESHES AND CONTOUR LEVEL CURVES FOR A SQUARE DOMAIN WITH 9-NO0DED QUADRILATERAL ELEMENTS

Mesh No.1

Mesh with 600 nine noded quadrilateral elements & no. of nodes = 2481

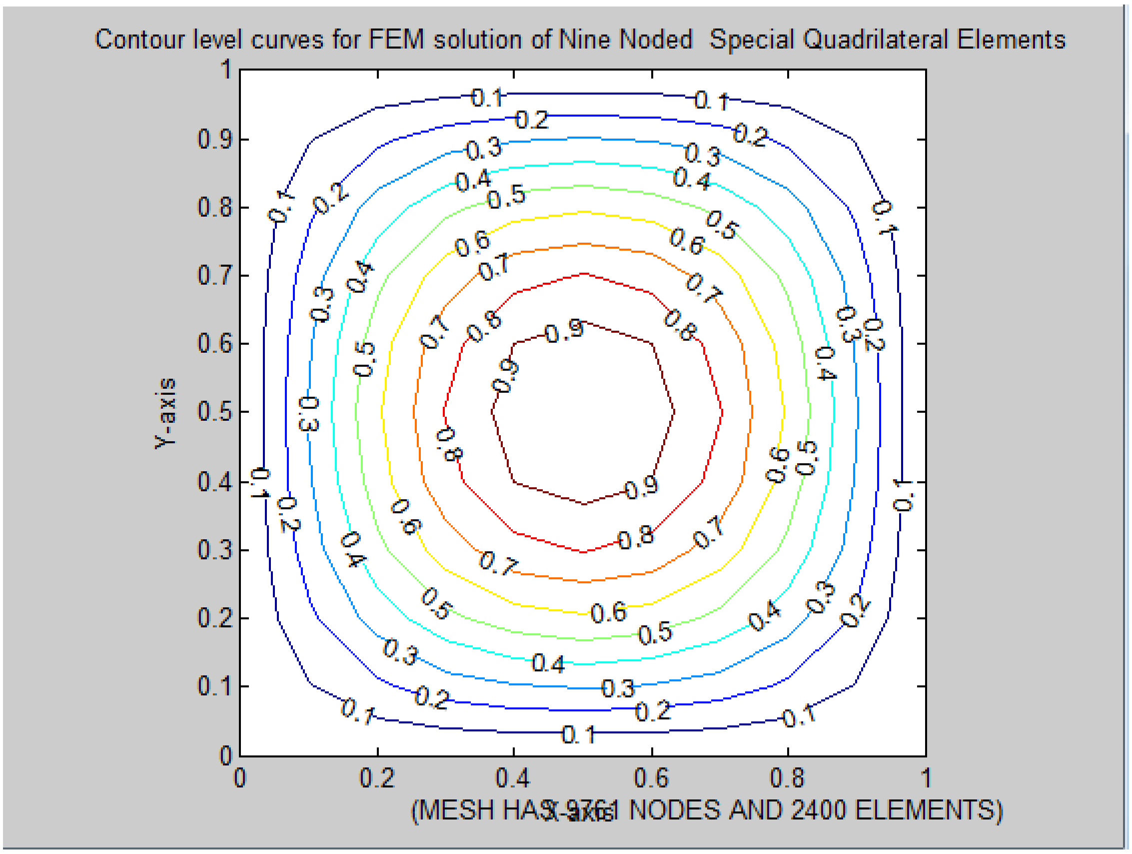

Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
Y-axis
X-axis
(MESH HAS 2481 NODES AND 600 ELEMENTS)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

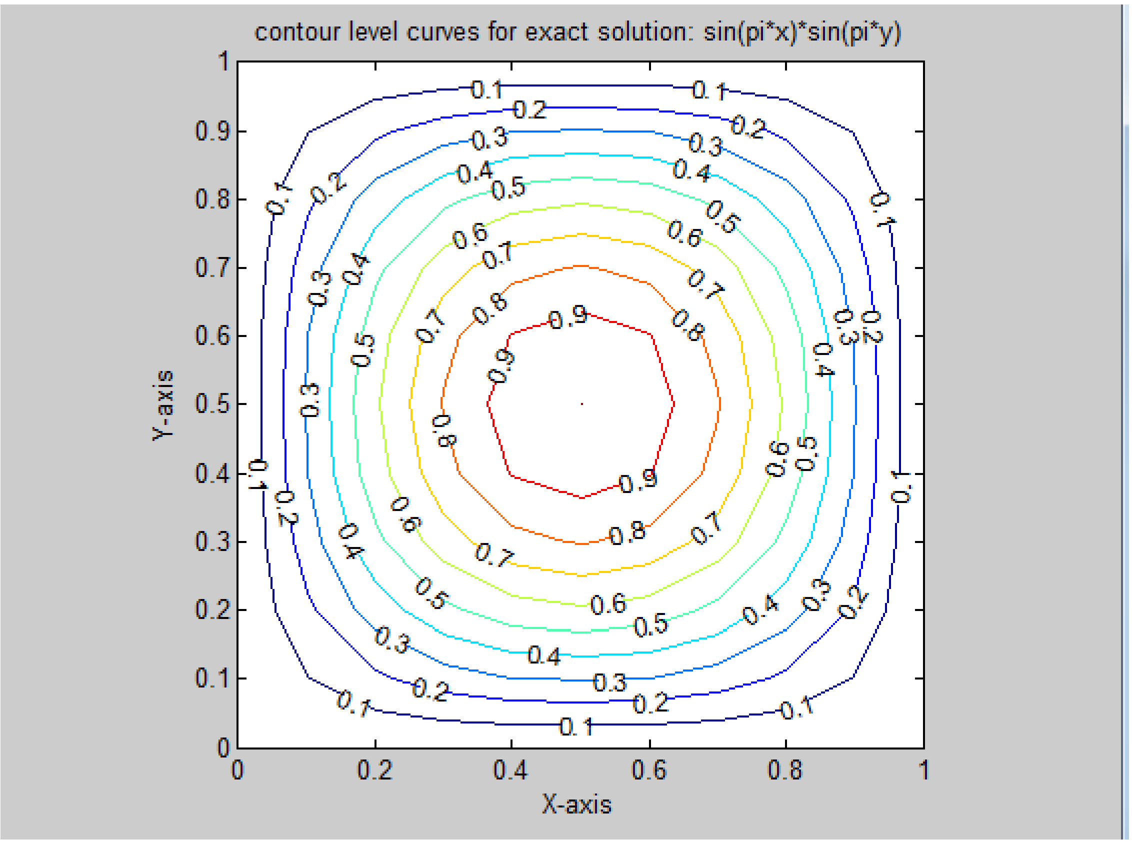

contour level curves for exact solution: sin(pi*x)*sin(pi*y)
Y-axis
X-axis
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

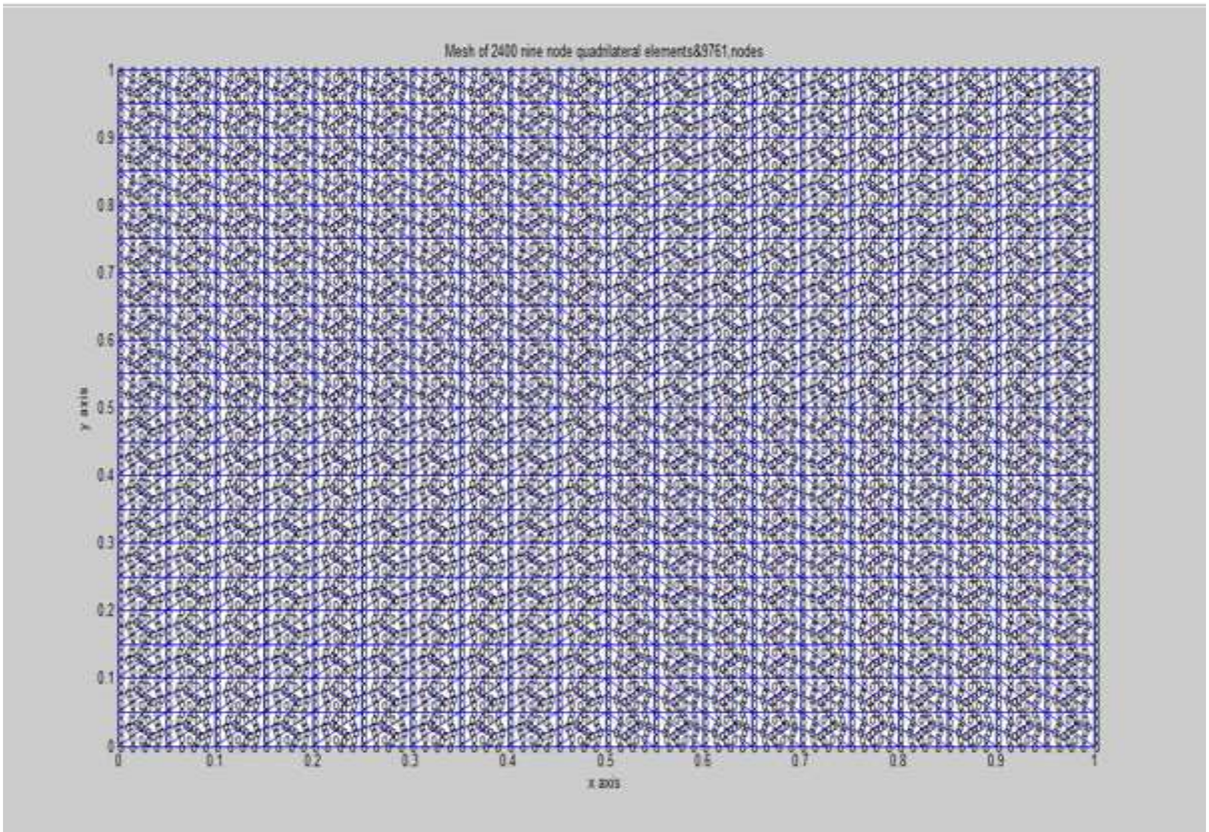


Mesh No.2

Mesh with 2400 nine noded quadrilateral elements & no. of nodes = 9761


Contour level curves for FEM solution of Nine Noded Special Quadrilateral Elements
Y-axis
X-axis
(MESH HAS 9761 NODES AND 2400 ELEMENTS)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1


contour level curves for exact solution: sin(pi*x)*sin(pi*y)
Y-axis
X-axis
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1



MATLAB Codes-I

(1)quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen.m
(2)D2LaplaceEquationQ9Ex3automeshgenNew.m
(3)nodaladdresses_special_convex_quadrilaterals_2nd_orderLG.m
(4)coordinate_arbitrarytriangle_2ndorderLAGR.m
(5)coordinate_special_quadrilaterals_in_stdtriangle_2nd_orderLAGR.m
(6)nodaladdresses4Lagrangespecial_convex_quadrilaterals_2nd_order.m
(7)D2LaplaceEquationQ9Ex3automeshgenNewPolygon.m

%(1)quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen.m

```
function[]=quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen(mmesh,nmesh,tri)
%quadrilateralmesh_over_arbitrarytriangle_q8automeshgen(mmesh,nmesh,tri)
%quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen(1,1,1)
%quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen(1,1,2)
%quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen(1,1,3)
%quadrilateralmesh_over_arbitrarytriangle_q9LGautomeshgen(1,1,4)
clf
switch tri
case 1%standard triangle
xx=sym([0;1;0])
yy=sym([0;0;1])
case 2
 xx=sym([0;1/2;1/2])
 yy=sym([0;0;1/2])

case 3%equilateral triangle

xx=sym([0;1;1/2])
yy=sym([0;0;sqrt(3)/2])
    case 4%equilateral triangle
 xx=sym([-sqrt(3);sqrt(3); 0])
 yy=sym([     -1;      -1; 2])


end
for mesh=mmesh:nmesh
    figure(mesh)
```

```
    ndiv=2*mesh;

[eln,nodetel,nodes,nnode]=nodaladdresses_special_convex_quadrilaterals_2nd_orderLG(ndiv
);

%[coord,gcoord]=coordinate_rtisoscelestriangle00_h0_hh_2ndorder(ndiv);
[coord,gcoord]=coordinate_arbitrarytriangle_2ndorderLAGR(xx,yy,ndiv)

[nel,nnel]=size(nodes)

for i=1:nel
NN(i,1)=i;
end

table1=[NN nodes]

[nnode,dimension]=size(gcoord)
%plot the mesh for the generated data
%x and y coordinates
xcoord(1:nnode,1)=gcoord(1:nnode,1);
ycoord(1:nnode,1)=gcoord(1:nnode,2);
%extract coordinates for each element

for i=1:nel
for j=1:nnel
x(1,j)=xcoord(nodes(i,j),1);
y(1,j)=ycoord(nodes(i,j),1);
end;%j loop
xvec(1,1:5)=[x(1,1),x(1,2),x(1,3),x(1,4),x(1,1)];
yvec(1,1:5)=[y(1,1),y(1,2),y(1,3),y(1,4),y(1,1)];
%axis equal
switch tri

case 1
axis tight
xmin=0;xmax=1;ymin=0;ymax=1;
axis([xmin,xmax,ymin,ymax]);
%place element number
midx=mean(xvec(1,1:4));
midy=mean(yvec(1,1:4));
case 2
axis tight
xmin=0;xmax=1/2;ymin=0;ymax=1/2;
axis([xmin,xmax,ymin,ymax]);
%place element number
midx=mean(xvec(1,1:4));
midy=mean(yvec(1,1:4));
case 3
axis tight
xmin=0;xmax=1;ymin=0;ymax=1;
axis([xmin,xmax,ymin,ymax]);
end
plot(xvec,yvec);%plot element
hold on;
%place element number
midx=mean(xvec(1,1:4));
midy=mean(yvec(1,1:4));
if (mesh<=5)&(tri~=2)
text(midx-.1,midy+.05,['\bf(',num2str(i),'\bf)']);
end
if (mesh<=2)&(tri==2)
text(midx-.005,midy+.02,['\bf(',num2str(i),'\bf)']);
end
end;%i loop
```

```
xlabel('\bfx axis')
ylabel('\bfy axis')

switch tri
case 1
st1='\bfstandard triangle  ';
st2=' using  ';
st3='9-node ';
st4='quadriateral';
st5=' elements'
title([st1,st2,st3,st4,st5])
text(.6,.9,['\bfMESH NO.=',num2str(mesh)])
text(.6,.8,['\bfnumber of elements=',num2str(nel)])
text(.6,.7,['\bfnumber of nodes=',num2str(nnode)])

case 2
st1='\bfone eigth (1/8)square cross section ';
st2=' using  ';
st3='9-node parabolic ';
st4='quadriateral';
st5=' elements'
title([st1,st2,st3,st4,st5])
text(0.1,0.4,['\bfMESH NO.=',num2str(mesh)])
text(0.1,0.38,['\bfnumber of elements=',num2str(nel)])
text(0.1,0.36,['\bfnumber of nodes=',num2str(nnode)])
case 3
st1='\bfequilateral triangle  ';
st2=' using  ';
st3='9-node parabolic ';
st4='quadriateral';
st5=' elements'
title([st1,st2,st3,st4,st5])
text(0.6,0.8,['\bfMESH NO.=',num2str(mesh)])
text(0.6,0.75,['\bfnumber of elements=',num2str(nel)])
text(0.6,0.70,['\bfnumber of nodes=',num2str(nnode)])

    case 4
st1='\bfequilateral triangle  ';
st2=' using  ';
st3='9-node parabolic ';
st4='quadriateral';
st5=' elements'
title([st1,st2,st3,st4,st5])
text(1,1.8,['\bfMESH NO.=',num2str(mesh)])
text(1,1.6,['\bfnumber of elements=',num2str(nel)])
text(1,1.4,['\bfnumber of nodes=',num2str(nnode)])




end

%put node numbers
for jj=1:nnode
if mesh<=2
text(gcoord(jj,1),gcoord(jj,2),['\bfo',num2str(jj)]);
else
text(gcoord(jj,1),gcoord(jj,2),['\bfo']);
end
end
hold on
figure(mesh),scatter(gcoord(:,1),gcoord(:,2),'MarkerFaceColor','g')
%axis off
end%for nmesh-the number of meshes
```

% (2)D2LaplaceEquationQ9Ex3automeshgenNew.m

```
function[]=D2LaplaceEquationQ9Ex3automeshgenNew(n1,n2,n3,numtri,ndiv,mesh)
%
%D2LaplaceEquationQ9Ex3automeshgenNew(1,2,3,1,2,1)
%*************************************************************************
syms coord
syms x y
ndof=1;


switch mesh
    case 1
    x=sym([0;1/2;1/2])
    y=sym([0;0;1/2])
    case 2  %isoscles triangle(torsion of an equilateral triangle,each side=2*sqrt(3))
 x=sym([-sqrt(3);sqrt(3); 0])
 y=sym([     -1;      -1; 2])


end
syms ui vi wi xi yi
%[ui,vi,wi]=coordinate_special_quadrilaterals_in_stdtriangle_2nd_order(ndiv);
[ui,vi,wi]=coordinate_special_quadrilaterals_in_stdtriangle_2nd_orderLAGR(ndiv)
%disp([ui vi wi])
N=length(ui);
    NN=(1:N)';
    x
    y
    x1=x(n1,1);x2=x(n2,1);x3=x(n3,1);y1=y(n1,1);y2=y(n2,1);y3=y(n3,1);
for i=1:N
    xxi(i,1)=x1+(x2-x1)*ui(i,1)+(x3-x1)*vi(i,1);
    yyi(i,1)=y1+(y2-y1)*ui(i,1)+(y3-y1)*vi(i,1);
end
%disp('_________________________________________________')
%disp('NN   xi    yi')
%disp([NN xi yi])
%disp('_________________________________________________')
coord(:,1)=(xxi(:,1));
coord(:,2)=(yyi(:,1));
gcoord(:,1)=double(xxi(:,1));
gcoord(:,2)=double(yyi(:,1));
%disp(gcoord);
%[eln,nodetel,nodes,nnode]=nodaladdresses_special_convex_quadrilaterals_2nd_order(ndiv)
;
[eln,nodetel,nodes,nnode]=nodaladdresses4Lagrangespecial_convex_quadrilaterals_2nd_orde
r(ndiv);




%*********************************************************************
%syms coord
%ndof=1;

%[eln,nodetel,nodes,nnode]=nodaladdresses4Lagrangespecial_convex_quadrilaterals_2nd_ord
er(ndiv);
%[coord,gcoord]=coordinate_rtisoscelestriangle00_h0_hh_2ndorderLAGR(ndiv);

    %[coord,gcoord]=coordinate_rtisoscelestriangle00_h0_hh(ndiv);
    %[nodetel,nodes]=nodaladdresses4special_convex_quadrilaterals(ndiv)
[nel,nnel]=size(nodes);
%disp([nel nnode nnel ndof])
format long g
for i=1:nel
N(i,1)=i;
```

```
end
for i=1:nel
NN(i,1)=i;
end




sdof=nnode*ndof;
ff=(zeros(sdof,1));ss=(zeros(sdof,sdof));

format long g
for i=1:nel
N(i,1)=i;
end
%radius of the hole=1.25cm
%input data for nodal coordinate values
%gcoord(i,j),where i->node no. and j->x or y

%
table1=[N nodes]
[nel,nnel]=size(nodes);
%************************************
switch mesh
    case 1
    nnn=0;
    for nn=1:nnode
        if gcoord(nn,1)==(1/2)
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
        end
    end
    format long g
k1 =double(0.1405770149551555103784039602O329);
 xi=(zeros(nnode,1));
a0=8/pi^3;
for m=1:nnode
    gx=(gcoord(m,1));gy=(gcoord(m,2));rr=(0);
for n=1:2:99
rr=rr+(-1)^((n-1)/2)*(1-(cosh(n*pi*gy)/cosh(n*pi/2)))*cos(n*pi*gx)/n^3;
end
xi(m,1)=(a0*rr);
end
 mm=length(bcdof);

    case 2%torsion of an equilateral triangle

 nnn=0;
 %boundary conditions on side 1
  for nn=1:nnode
      xnn=gcoord(nn,1);ynn=gcoord(nn,2);
        if ((ynn+1)<1.e-5)
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
  end
%boundary conditions on side 2
  for nn=1:nnode
      xnn=gcoord(nn,1);ynn=gcoord(nn,2);
        if ((-(sqrt(3))*xnn-ynn+2)<1.e-5)
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
```

```
  end
%boundary conditions on side 3
  for nn=1:nnode
      xnn=gcoord(nn,1);ynn=gcoord(nn,2);
        if (((sqrt(3))*xnn-ynn+2)<1.e-5)
            nnn=nnn+1
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
  end
  bcdof
  bcval
  mm=length(bcdof);
 for m=1:nnode
    gx=(gcoord(m,1));gy=(gcoord(m,2));
    xi(m,1)=((gy+1)*((sqrt(3))*gx-gy+2)*(-(sqrt(3))*gx-gy+2))/12;
  end
xi=double(xi);
format long g
k1 =9*sqrt(3)/5;

end%switch

%***********************************


 for L=1:nel
   for M=1:3
      LM=nodetel(L,M);
      xx(L,M)=gcoord(LM,1);
      yy(L,M)=gcoord(LM,2);
    end
  end
%_________________________________________________________________


%
table2=[N xx yy];
%disp([xx yy])
```

```
intJdn1dn1uvrs =[vpa(sym('  .688003696999291704075192889')) vpa(sym(' .630268637329029196546930317 48'));...
vpa(sym(' .63026863732902919654693031748')) vpa(sym('  .688003696999291704075192889'))];

intJdn1dn2uvrs =[vpa(sym('  .4946976667946543132242312849e-1')) vpa(sym(' -.9249249979013890567437258873e-1'));...
vpa(sym('  .7417416687652776099229407793e-1')) vpa(sym(' -.5804150224125689851696121830e-1'))];

intJdn1dn3uvrs =[vpa(sym(' -.2763918707186419647824461270e-1')) vpa(sym(' -.3640622770513571608466053928e-1'));...
vpa(sym(' -.3640622770513571608466053928e-1')) vpa(sym(' -.2763918707186419647824461270e-1'))];

intJdn1dn4uvrs =[vpa(sym(' -.5804150224125689851696121830e-1')) vpa(sym('  .7417416687652776099229407793e-1'));...
vpa(sym(' -.9249249979013890567437258873e-1')) vpa(sym('  .4946976667946543132242312849e-1'))];

intJdn1dn5uvrs =[vpa(sym(' -.420340201552488982561479599 3')) vpa(sym('  .238355345065568618361178470 3'));...
vpa(sym(' -.4283113216010980483054881964')) vpa(sym('  .1020378129397056233087080320'))];

intJdn1dn6uvrs =[vpa(sym(' .1262687000516458834074677285 3')) vpa(sym(' .1686516911320240413733489215 6'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .15224655673422927054938212414'))];

intJdn1dn7uvrs =[vpa(sym(' .15224655673422927054938212414')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .12626870005164588340746772853'))];

intJdn1dn8uvrs =[vpa(sym('  .1020378129397056233087080320')) vpa(sym(' -.4283113216010980483054881964'));...
vpa(sym('  .2383553450655686183611784703')) vpa(sym(' -.4203402015524889825614795993'))];

intJdn1dn9uvrs =[vpa(sym(' -.6120056425393653014388148716')) vpa(sym(' -.7228914824388009885825793847'));...
vpa(sym(' -.7228914824388009885825793847')) vpa(sym(' -.6120056425393653014388148716'))];

intJdn2dn1uvrs =[vpa(sym('  .4946976667946543132242312849e-1')) vpa(sym('  .7417416687652776099229407793e-1'));...
vpa(sym(' -.9249249979013890567437258873e-1')) vpa(sym(' -.5804150224125689851696121830e-1'))];

intJdn2dn2uvrs =[vpa(sym('  .2482224272357035985941591016 5')) vpa(sym(' -.13943577781062161645243966287'));...
vpa(sym(' -.13943577781062161645243966287')) vpa(sym('  .22525137401606776447455758469'))];

intJdn2dn3uvrs =[vpa(sym('  -.9368329064821876846205153869e-1')) vpa(sym(' -.8285228368179732406596827984 6e-1'));...
vpa(sym('  .8381438298486934260069838682 0e-1')) vpa(sym('   .3790480089812643647503258264e-1'))];

intJdn2dn4uvrs =[vpa(sym('   .6292570002530214045175196 0e-3')) vpa(sym(' .2610005204712297985144922346e-1'));...
vpa(sym(' .2610005204712297985144922346e-1')) vpa(sym('   .6292570002530214045175196 0e-3'))];

intJdn2dn5uvrs =[vpa(sym('   -.4686821675279013233513591962')) vpa(sym('  -.3464619445210953430278280635 8'));...
vpa(sym('   .3202047221455713236388386030 9')) vpa(sym(' .8583672920954138128722924288e-1'))];

intJdn2dn6uvrs =[vpa(sym('   .2048671103020189876458458532')) vpa(sym('  .2571877444243557310522163472 2'));...
vpa(sym(' -.4094789222423109356144503194 5')) vpa(sym(' -.2807386267765162146339380350 2'))];
```

```
intJdn2dn7uvrs =[vpa(sym(' .508859176419664984628888 6644e-1')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' -.17609016019421764322513358 0e-2'))];
intJdn2dn8uvrs =[vpa(sym(' -.41736856619169902011341267 58e-1')) vpa(sym(' -.94765689307712237483661393 07e-1'));...
vpa(sym(' -.9476568930771223748366139307e-1')) vpa(sym('  .33881015844374607003400451 61e-1'))];
intJdn2dn9uvrs =[vpa(sym('   .5002783593588245639491753 32e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym(' -.4296214634864792106158679226e-1'))];
intJdn3dn1uvrs =[vpa(sym(' -.2763918707186419647824461270e-1')) vpa(sym(' -.3640622770513571608466053928e-1'));...
vpa(sym(' -.3640622770513571608466053928e-1')) vpa(sym(' -.2763918707186419647824461270e-1'))];
intJdn3dn2uvrs =[vpa(sym('  -.9368329064821876846205153869e-1')) vpa(sym('  .8381438298486934260069838682 0e-1'));...
vpa(sym(' -.8285228368179732406596827984 6e-1')) vpa(sym('   .3790480089812643647503258264e-1'))];
intJdn3dn3uvrs =[vpa(sym('  .33203856905137126709551224250')) vpa(sym(' .29812794600013651335576030277 1'));...
vpa(sym(' .29812794600013651335576030277 1')) vpa(sym('  .33203856905137126709551224250'))];
intJdn3dn4uvrs =[vpa(sym('   .3790480089812643647503258264e-1')) vpa(sym(' -.8285228368179732406596827984 6e-1'));...
vpa(sym('  .8381438298486934260069838682 0e-1')) vpa(sym('  -.9368329064821876846205153869e-1'))];
intJdn3dn5uvrs =[vpa(sym('   .15710844821544667058002407565')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym(' .4305964422926219650773122697e-1'))];
intJdn3dn6uvrs =[vpa(sym('  .14159217838663875388665351332')) vpa(sym(' -.35360191630747700837476451283'));...
vpa(sym('  .31306475035918965829190215383')) vpa(sym(' -.34086853405320051524855521401'))];
intJdn3dn7uvrs =[vpa(sym(' -.34086853405320051524855521401')) vpa(sym('  .31306475035918965829190215383'));...
vpa(sym(' -.35360191630747700837476451283')) vpa(sym('  .14159217838663875388665351332'))];
intJdn3dn8uvrs =[vpa(sym(' .4305964422926219650773122697e-1')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym('   .15710844821544667058002407565'))];
intJdn3dn9uvrs =[vpa(sym('  -.24951262900756184435610227 57')) vpa(sym(' -.48105764260800696388913166479'));...
vpa(sym(' -.48105764260800696388913166479')) vpa(sym('  -.24951262900756184435610227 57'))];
intJdn4dn1uvrs =[vpa(sym(' -.5804150224125689851696121830e-1')) vpa(sym(' -.9249249979013890567437258873e-1'));...
vpa(sym('  .7417416687652776099229407793e-1')) vpa(sym('  .4946976667946543132242312849e-1'))];
intJdn4dn2uvrs =[vpa(sym('   .6292570002530214045175196 0e-3')) vpa(sym(' .2610005204712297985144922346e-1'));...
vpa(sym(' .2610005204712297985144922346e-1')) vpa(sym('   .6292570002530214045175196 0e-3'))];
intJdn4dn3uvrs =[vpa(sym('   .3790480089812643647503258264e-1')) vpa(sym('  .8381438298486934260069838682 0e-1'));...
vpa(sym(' -.8285228368179732406596827984 6e-1')) vpa(sym('  -.9368329064821876846205153869e-1'))];
intJdn4dn4uvrs =[vpa(sym('  .22525137401606776447455758469')) vpa(sym(' -.13943577781062161645243966287'));...
vpa(sym(' -.13943577781062161645243966287')) vpa(sym('  .24822242723570359859415910165'))];
intJdn4dn5uvrs =[vpa(sym('  .33881015844374607003400451 61e-1')) vpa(sym(' -.94765689307712237483661393 07e-1'));...
vpa(sym(' -.9476568930771223748366139307e-1')) vpa(sym(' -.41736856619169902011341267 58e-1'))];
intJdn4dn6uvrs =[vpa(sym(' -.17609016019421764322513358 0e-2')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' .508859176419664984628888 6644e-1'))];
intJdn4dn7uvrs =[vpa(sym(' -.28073862677651621463393803502')) vpa(sym(' -.40947892224231093561445031945'));...
vpa(sym('  .25718774442435573105221634722')) vpa(sym('   .20486711030201898764584585 32'))];
intJdn4dn8uvrs =[vpa(sym(' .8583672920954138128722924288e-1')) vpa(sym('   .32020472214557132363883860309'));...
vpa(sym('  -.34646194452109534302782806358')) vpa(sym('   -.46868216752790132335135919 62'))];
intJdn4dn9uvrs =[vpa(sym(' -.4296214634864792106158679226e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym('   .5002783593588245639491753 32e-1'))];
intJdn5dn1uvrs =[vpa(sym(' -.42034020155248898256147959 93')) vpa(sym(' -.42831132160109804830548819 64'));...
vpa(sym('  .23835534506556861836117847 03')) vpa(sym('  .10203781293970562330870803 20'))];
intJdn5dn2uvrs =[vpa(sym('   -.46868216752790132335135919 62')) vpa(sym('   .32020472214557132363883860309'));...
vpa(sym('  -.34646194452109534302782806358')) vpa(sym(' .8583672920954138128722924288e-1'))];
intJdn5dn3uvrs =[vpa(sym('   .15710844821544667058002407565')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym(' .4305964422926219650773122697e-1'))];
intJdn5dn4uvrs =[vpa(sym('  .33881015844374607003400451 61e-1')) vpa(sym(' -.94765689307712237483661393 07e-1'));...
vpa(sym(' -.9476568930771223748366139307e-1')) vpa(sym(' -.41736856619169902011341267 58e-1'))];
intJdn5dn5uvrs =[vpa(sym(' 1.06633848333445648567503766 51')) vpa(sym('  .56556859863119073241392390 19'));...
vpa(sym('  .56556859863119073241392390 19')) vpa(sym(' 1.12113332953297566519504242 78'))];
intJdn5dn6uvrs =[vpa(sym('  -.58306188548401601151324941 69')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym(' -.22579554943131879593427777753'))];
intJdn5dn7uvrs =[vpa(sym('   -.16767681050348731002396257 42')) vpa(sym(' -.1709647811328463063306097 92e-2'));...
vpa(sym(' -.1709647811328463063306097 92e-2')) vpa(sym('   .17132457681637017904834081102'))];
intJdn5dn8uvrs =[vpa(sym(' .9027148605382036792067597 12e-1')) vpa(sym('   .41024643803333829631989611 84'));...
vpa(sym('   .41024643803333829631989611 84')) vpa(sym(' .9027148605382036792067597 12e-1'))];
intJdn5dn9uvrs =[vpa(sym('   .29216163161979549627091262 30')) vpa(sym('  -.35816040869776877564539558 36'));...
vpa(sym('  -.35816040869776877564539558 36')) vpa(sym(' -1.34613117273118671532210866 66'))];
intJdn6dn1uvrs =[vpa(sym(' .12626870005164588340746772853')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .15224655673422927054938212414'))];
intJdn6dn2uvrs =[vpa(sym('   .20486711030201898764584585 32')) vpa(sym(' -.40947892224231093561445031945'));...
vpa(sym('  .25718774442435573105221634722')) vpa(sym(' -.28073862677651621463393803502'))];
intJdn6dn3uvrs =[vpa(sym('  .14159217838663875388665351332')) vpa(sym('  .31306475035918965829190215383'));...
vpa(sym(' -.35360191630747700837476451283')) vpa(sym(' -.34086853405320051524855521401'))];
intJdn6dn4uvrs =[vpa(sym(' -.17609016019421764322513358 0e-2')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' .508859176419664984628888 6644e-1'))];
intJdn6dn5uvrs =[vpa(sym('  -.58306188548401601151324941 69')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym(' -.22579554943131879593427777753'))];
intJdn6dn6uvrs =[vpa(sym(' 1.59985108580953085463759282 28')) vpa(sym(' .24469327225090109904911663174'));...
vpa(sym(' .24469327225090109904911663174')) vpa(sym('  .64675514174853057100283264 58'))];
intJdn6dn7uvrs =[vpa(sym(' -.7461268601626634110113636 05e-1')) vpa(sym('   .42496611860422613701397774221'));...
vpa(sym('   .42496611860422613701397774221')) vpa(sym(' -.7461268601626634110113636 05e-1'))];
intJdn6dn8uvrs =[vpa(sym('   .17132457681637017904834081102')) vpa(sym(' -.1709647811328463063306097 92e-2'));...
vpa(sym(' -.1709647811328463063306097 92e-2')) vpa(sym('   -.16767681050348731002396257 42'))];
intJdn6dn9uvrs =[vpa(sym('   -1.58446817826398012957926361 56')) vpa(sym(' -.9430112713910361521519372896e-1'));...
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vpa(sym(' -.943011271391036152151937289 6e-1')) vpa(sym('    .23980459065606283692676632 48'))];
```

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vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('    .2398045906560628369267663248'))];
intJdn7dn1uvrs =[vpa(sym(' .1522465567342292705493821241 4')) vpa(sym(' .1686516911320240413733489215 6'));...
vpa(sym(' .1686516911320240413733489215 6')) vpa(sym(' .1262687000516458834074677285 3'))];
intJdn7dn2uvrs =[vpa(sym(' .5088591764196649846288886644e-1')) vpa(sym('  -.1033579482822943448775058737 1'));...
vpa(sym('  -.1033579482822943448775058737 1')) vpa(sym(' -.1760901601942176432251335 80e-2'))];
intJdn7dn3uvrs =[vpa(sym(' -.3408685340532005152485552140 1')) vpa(sym(' -.3536019163074770083747645128 3'));...
vpa(sym('  .3130647503591896582919021538 3')) vpa(sym('  .1415921783866387538866535133 2'))];
intJdn7dn4uvrs =[vpa(sym(' -.2807386267765162146339380350 2')) vpa(sym('  .2571877444243557310522163472 2'));...
vpa(sym(' -.4094789222423109356144503194 5')) vpa(sym('   .2048671103020189876458458532'))];
intJdn7dn5uvrs =[vpa(sym('   -.1676768105034873100239625742')) vpa(sym(' -.1709647811328463063306097 92e-2'));...
vpa(sym(' -.1709647811328463063306097 92e-2')) vpa(sym('   .1713245768163701790483408110 2'))];
intJdn7dn6uvrs =[vpa(sym(' -.7461268601626634110113636 05e-1')) vpa(sym('   .4249661186042261370139777422 1'));...
vpa(sym('   .4249661186042261370139777422 1')) vpa(sym(' -.7461268601626634110113636 05e-1'))];
intJdn7dn7uvrs =[vpa(sym('  .6467551417485305710028326458')) vpa(sym(' .2446932722509010990491166317 4'));...
vpa(sym(' .2446932722509010990491166317 4')) vpa(sym(' 1.599851085809530854637592822 8'))];
intJdn7dn8uvrs =[vpa(sym(' -.2257955494313187959342777775 3')) vpa(sym(' -.5425281868713035769578894293 2'));...
vpa(sym(' -.5425281868713035769578894293 2')) vpa(sym('  -.5830618854840160115132494169'))];
intJdn7dn9uvrs =[vpa(sym('    .2398045906560628369267663248')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('   -1.584468178263980129579263615 6'))];
intJdn8dn1uvrs =[vpa(sym('  .1020378129397056233087080320')) vpa(sym('  .2383553450655686183611784703'));...
vpa(sym(' -.4283113216010980483054881964')) vpa(sym(' -.4203402015524889825614795993'))];
intJdn8dn2uvrs =[vpa(sym(' -.4173685661916990201134126758e-1')) vpa(sym(' -.9476568930771223748366139307e-1'));...
vpa(sym(' -.9476568930771223748366139307e-1')) vpa(sym('  .3388101584437460700340045161e-1'))];
intJdn8dn3uvrs =[vpa(sym(' .4305964422926219650773122697e-1')) vpa(sym('   .1294554954791107490830820766 8'));...
vpa(sym('   .1294554954791107490830820766 8')) vpa(sym('   .1571084482154466705800240756 5'))];
intJdn8dn4uvrs =[vpa(sym(' .8583672920954138128722924288e-1')) vpa(sym('  -.3464619445210953430278280635 8'));...
vpa(sym('   .3202047221455713236388386030 9')) vpa(sym('   -.4686821675279013233513591962'))];
intJdn8dn5uvrs =[vpa(sym(' .9027148605382036792067597 12e-1')) vpa(sym('   .4102464380333382963198961184'));...
vpa(sym('   .4102464380333382963198961184')) vpa(sym(' .9027148605382036792067597 12e-1'))];
intJdn8dn6uvrs =[vpa(sym('   .1713245768163701790483408110 2')) vpa(sym(' -.1709647811328463063306097 92e-2'));...
vpa(sym(' -.1709647811328463063306097 92e-2')) vpa(sym('   -.1676768105034873100239625742'))];
intJdn8dn7uvrs =[vpa(sym(' -.2257955494313187959342777775 3')) vpa(sym(' -.5425281868713035769578894293 2'));...
vpa(sym(' -.5425281868713035769578894293 2')) vpa(sym('  -.5830618854840160115132494169'))];
intJdn8dn8uvrs =[vpa(sym(' 1.121133329532975665195042427 8')) vpa(sym('  .5655685986311907324139239019'));...
vpa(sym('  .5655685986311907324139239019')) vpa(sym(' 1.066338483334456485675037665 1'))];
intJdn8dn9uvrs =[vpa(sym(' -1.346131172731186715322108666 6')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym('   .2921616316197954962709126230'))];
intJdn9dn1uvrs =[vpa(sym(' -.6120056425393653014388148716')) vpa(sym(' -.7228914824388009885825793847'));...
vpa(sym(' -.7228914824388009885825793847')) vpa(sym(' -.6120056425393653014388148716'))];
intJdn9dn2uvrs =[vpa(sym('   .5002783593588245639491753 32e-1')) vpa(sym('    .4094116802555143940114436244 5'));...
vpa(sym('    .4094116802555143940114436244 5')) vpa(sym(' -.4296214634864792106158679226e-1'))];
intJdn9dn3uvrs =[vpa(sym('  -.2495126290075618443561022757')) vpa(sym(' -.4810576426080069638891316647 9'));...
vpa(sym(' -.4810576426080069638891316647 9')) vpa(sym('  -.2495126290075618443561022757'))];
intJdn9dn4uvrs =[vpa(sym(' -.4296214634864792106158679226e-1')) vpa(sym('    .4094116802555143940114436244 5'));...
vpa(sym('    .4094116802555143940114436244 5')) vpa(sym('   .5002783593588245639491753 32e-1'))];
intJdn9dn5uvrs =[vpa(sym('   .2921616316197954962709126230')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym(' -1.346131172731186715322108666 6'))];
intJdn9dn6uvrs =[vpa(sym('   -1.584468178263980129579263615 6')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('    .2398045906560628369267663248'))];
intJdn9dn7uvrs =[vpa(sym('    .2398045906560628369267663248')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('   -1.584468178263980129579263615 6'))];
intJdn9dn8uvrs =[vpa(sym(' -1.346131172731186715322108666 6')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym('   .2921616316197954962709126230'))];
intJdn9dn9uvrs =[vpa(sym(' 3.253085710679001122165279740 6')) vpa(sym(' 1.290048836209523946170002425 7'));...
vpa(sym(' 1.290048836209523946170002425 7')) vpa(sym(' 3.253085710679001122165279740 6'))];
%integrals of products of gl0bal derivatives
intJdndn=[intJdn1dn1uvrs intJdn1dn2uvrs intJdn1dn3uvrs intJdn1dn4uvrs intJdn1dn5uvrs intJdn1dn6uvrs intJdn1dn7uvrs intJdn1dn8uvrs intJdn1dn9uvrs ;...
          intJdn2dn1uvrs intJdn2dn2uvrs intJdn2dn3uvrs intJdn2dn4uvrs intJdn2dn5uvrs intJdn2dn6uvrs intJdn2dn7uvrs intJdn2dn8uvrs intJdn2dn9uvrs;...
          intJdn3dn1uvrs intJdn3dn2uvrs intJdn3dn3uvrs intJdn3dn4uvrs intJdn3dn5uvrs intJdn3dn6uvrs intJdn3dn7uvrs intJdn3dn8uvrs intJdn3dn9uvrs;...
          intJdn4dn1uvrs intJdn4dn2uvrs intJdn4dn3uvrs intJdn4dn4uvrs intJdn4dn5uvrs intJdn4dn6uvrs intJdn4dn7uvrs intJdn4dn8uvrs intJdn4dn9uvrs;...
          intJdn5dn1uvrs intJdn5dn2uvrs intJdn5dn3uvrs intJdn5dn4uvrs intJdn5dn5uvrs intJdn5dn6uvrs intJdn5dn7uvrs intJdn5dn8uvrs intJdn5dn9uvrs;...
          intJdn6dn1uvrs intJdn6dn2uvrs intJdn6dn3uvrs intJdn6dn4uvrs intJdn6dn5uvrs intJdn6dn6uvrs intJdn6dn7uvrs intJdn6dn8uvrs intJdn6dn9uvrs;...
          intJdn7dn1uvrs intJdn7dn2uvrs intJdn7dn3uvrs intJdn7dn4uvrs intJdn7dn5uvrs intJdn7dn6uvrs intJdn7dn7uvrs intJdn7dn8uvrs intJdn7dn9uvrs;...
          intJdn8dn1uvrs intJdn8dn2uvrs intJdn8dn3uvrs intJdn8dn4uvrs intJdn8dn5uvrs intJdn8dn6uvrs intJdn8dn7uvrs intJdn8dn8uvrs intJdn8dn9uvrs;...
          intJdn9dn1uvrs intJdn9dn2uvrs intJdn9dn3uvrs intJdn9dn4uvrs intJdn9dn5uvrs intJdn9dn6uvrs intJdn9dn7uvrs intJdn9dn8uvrs intJdn9dn9uvrs];


intJdndn=double(intJdndn);
%
for iel=1:nel
index=zeros(nnel*ndof,1);

X=xx(iel,1:3);
Y=yy(iel,1:3);
%disp([X Y])
xa=X(1,1);
xb=X(1,2);
xc=X(1,3);
```

```
ya=Y(1,1);
yb=Y(1,2);
yc=Y(1,3);
bta=yb-yc;btb=yc-ya;
gma=xc-xb;gmb=xa-xc;
delabc=gmb*bta-gma*btb;
G=[bta btb;gma gmb]/delabc;
GT=[bta gma;btb gmb]/delabc;
Q=GT*G;

sk(1:9,1:9)=(zeros(9,9));
for i=1:9
 for j=i:9
 sk(i,j)=(delabc*sum(sum(Q.*(intJdndn(2*i-1:2*i,2*j-1:2*j)))));
 sk(j,i)=sk(i,j);
 end
end
%f =[5/144;1/24;7/144;1/24]*(2*delabc);
%f=[ -7/432;  -1/72; -5/432;  -1/72; 11/216; 13/216; 13/216; 11/216]*(2*delabc)
f=[ 1/432; 1/216; 1/144; 1/216;  1/72; 5/216; 5/216;  1/72;  2/27]*(2*delabc);

%______________________________________
 edof=nnel*ndof;
 k=0;
 for i=1:nnel
     nd(i,1)=nodes(iel,i);
     start=(nd(i,1)-1)*ndof;
     for j=1:ndof
         k=k+1;
         index(k,1)=start+j;
     end
 end
 %-------------------------------------------------------------------------
for i=1:edof
    ii=index(i,1);
    ff(ii,1)=ff(ii,1)+f(i,1);
 for j=1:edof
     jj=index(j,1);
     ss(ii,jj)=ss(ii,jj)+sk(i,j);
 end
end
end%for iel
 %-------------------------------------------------------------------------
%bcdof=[13;37;35;33;31;29;27;25;23;21;19;17;15];
for ii=1:mm
    kk=bcdof(ii,1);
    ss(kk,1:nnode)=zeros(1,nnode);
    ss(1:nnode,kk)=zeros(nnode,1);
    ff(kk,1)=0;
end
for ii=1:mm
    kk=bcdof(ii,1);
    ss(kk,kk)=1;
end
phi=ss\ff;
%
phi=double(phi);
if mesh==2
    phi=phi/2;
end
[phi xi]
for I=1:nnode
NN(I,1)=I;
phi_xi(I,1)=phi(I,1)-xi(I,1);
end
```

```
MAXPHI_XI=max(abs(phi_xi));


%disp('__________________________________________________________________')
%disp('number of nodes,elements & nodes per element')
%[nnode nel nnel ndof]
%disp('element number    nodal connectivity for quadrilateral element')
%table1
%disp('________________________________________________________________________________
_____')
%disp('element number   coordinates of the triangle spanning the quadrilateral
element')
%table2
%disp('________________________________________________________________________________
_____')
%disp('node number         Prandtl Stress Values')
%disp('                         fem-computed values             anlytical(theoretical)-
values           ')

disp([NN phi xi])
t=0;
 for iii=1:nnode
    t=t+phi(iii,1)*ff(iii,1);
 end
t=0;
 for iii=1:nnode
    t=t+phi(iii,1)*ff(iii,1);
 end
switch mesh
    case 1
T=8*t;
    case 2
        T=2*t;
end

disp('---------------------------------------------------------------------------------
------------')
disp('number of nodes,elements & nodes per element')
disp([nnode nel nnel ])
disp('torisonal constants(fem=phi&exact=xi)  error(max(abs(phi_xi))')
%disp('--------------------------------------------------------------------------------
-------------')
%disp([nnode nel nnel ])
disp([T k1 MAXPHI_XI ])
disp('---------------------------------------------------------------------------------
------------')

%###################################################################################
if (mesh==2)

[x,y]=meshgrid(-sqrt(3):(1/15)*sqrt(3):sqrt(3),-1:(0.1):2);
 z=(zeros(31,31));
for i=1:31
    for j=1:31
      for iel=1:nel
 %node numbers of quadrilateral
       nd1=nodes(iel,1);nd2=nodes(iel,2);nd3=nodes(iel,3);nd4=nodes(iel,4);

nd5=nodes(iel,5);nd6=nodes(iel,6);nd7=nodes(iel,7);nd8=nodes(iel,8);nd9=nodes(iel,9);
 %coordinates of quadrilateral(u,v)

u(1,1)=gcoord(nd1,1);u(2,1)=gcoord(nd2,1);u(3,1)=gcoord(nd3,1);u(4,1)=gcoord(nd4,1);

v(1,1)=gcoord(nd1,2);v(2,1)=gcoord(nd2,2);v(3,1)=gcoord(nd3,2);v(4,1)=gcoord(nd4,2);
 %coordinates of the grid(x,y)
```

```
          in=inpolygon(x(i,j),y(i,j),u,v);
          if  (in==1)
           X=x(i,j);Y=y(i,j);
           [t]=convexquadrilateral_coordinates(u,v,X,Y);
           r=t(1,1);
           s=t(2,1);
%======================================================================================
========================================
%
%======================================================================================
==================================================
 shn1=(r^2-r)*(s^2-s)/4;
           shn2=(r^2+r)*(s^2-s)/4;
           shn3=(r^2+r)*(s^2+s)/4;
           shn4=(r^2-r)*(s^2+s)/4;
           shn5=(1-r^2)*(s^2-s)/2;
           shn6=(r^2+r)*(1-s^2)/2;
           shn7=(1-r^2)*(s^2+s)/2;
           shn8=(r^2-r)*(1-s^2)/2;
           shn9=(1-r^2)*(1-s^2);

PHI(i,j)=shn1*phi(nd1,1)+shn2*phi(nd2,1)+shn3*phi(nd3,1)+shn4*phi(nd4,1)+shn5*phi(nd5,1
)+shn6*phi(nd6,1)+shn7*phi(nd7,1)+shn8*phi(nd8,1)+shn9*phi(nd9,1);
%======================================================================================
===============================================

%             PHI(i,j)=(1-r)*(1-s)*phi(nd1,1)/4+(1+r)*(1-
%s)*phi(nd2,1)/4+(1+r)*(1+s)*phi(nd3,1)/4+(1-r)*(1+s)*phi(nd4,1)/4;
             z(i,j)=((Y+1)*((sqrt(3))*X-Y+2)*(-(sqrt(3))*X-Y+2))/12;;
              break
           end%if (in==1)
        end%for iel
   %THE PROGRAM EXECUTION JUMPS TO HERE if (in==1)
    end%for j
end%for i
 % z=sin(pi*x).*sin(pi*y);
  %z=(zeros(31,31));

%for ii=1:31
 %   for jj=1:31
  %  xx=(x(ii,jj));yy=(y(ii,jj));
%z(ii,jj)=((yy+1/2)*((sqrt(3))*xx-yy+1)*(-(sqrt(3))*xx-yy+1))/6;;
%end %ii
%end%jj

for i=1:31
    for j=1:31
        if (abs(PHI(i,j))<=1e-5)
            PHI(i,j)=0;
        end
        if (abs(z(i,j))<=1e-5)
            z(i,j)=0;
        end

    end
end


end%(mesh==2)

switch mesh
    case 2
  clf
```

```
figure(1)
 x=[-sqrt(3);sqrt(3);0];
 y=[      -1;      -1;2];
 patch(x,y,'w')
hold on
%[x,y]=meshgrid(0:.1:1,0:0.1:1)
[x,y]=meshgrid(-sqrt(3):(1/15)*sqrt(3):sqrt(3),-1:(0.1):2);
%y((y>1/2)&(y<=1)&(x>1/2)&(x<=1)&(x+y>3/2))=NaN;
%%y((y>-1/2)&(y<=1)&(x>0)&(x<=(sqrt(3)/2))&((-sqrt(3)*x-y+1)<0))=NaN;
%%y((y>-1/2)&(y<=1)&(x>(-sqrt(3)/2))&(x<=0)&((sqrt(3)*x-y+1)<0))=NaN;
%[c,h]=contour(x,y,PHI)
contour(x,y,PHI,20)
xlabel('X-axis');
ylabel('Y-axis');
%clabel(c,h);
axis square
st1='Contour level curves for ';
st2='FEM solution of ';
st3='Nine Noded ';
st4='Special Quadrilateral';
st5=' Elements'
title([st1,st2,st3,st4,st5])
sst1='(MESH HAS '
sst2=num2str(nnode)
sst3=' NODES'
sst4=' AND '
sst5=num2str(nel)
sst6=' ELEMENTS)'
text(0.6,1.8,[sst1 sst2])
text(0.6,1.6,[sst3 sst4])
text(0.6,1.4,[sst5 sst6])
%text(0.25,-.08,[sst1 sst2 sst3 sst4 sst5 sst6])
%
figure(2)
 %x=[0.0 1.0 1.0 0.5 0.0];
 %y=[0.0 0.0 0.5 1.0 1.0];
 x=[-sqrt(3);sqrt(3);0];
 y=[      -1;      -1;2];
 patch(x,y,'w')
hold on
%[x,y]=meshgrid(0:.1:1,0:0.1:1)
%y((y>1/2)&(y<=1)&(x>1/2)&(x<=1)&(x+y>3/2))=NaN;
%[c,h]=contour(x,y,z)
[x,y]=meshgrid(-sqrt(3):(1/15)*sqrt(3):sqrt(3),-1:(0.1):2);

contour(x,y,z,20)
xlabel('X-axis');
ylabel('Y-axis');
%clabel(c,h);
axis square
title('contour level curves for exact solution: ')
hold off

figure(3)
% x=[0.0 1.0 1.0 0.5 0.0];
 %y=[0.0 0.0 0.5 1.0 1.0];
x=[-sqrt(3);sqrt(3);0];
y=[      -1;      -1;2];
 patch(x,y,'w')
hold on
[x,y]=meshgrid(-sqrt(3):(1/15)*sqrt(3):sqrt(3),-1:(0.1):2);
%[x,y]=meshgrid(0:.1:1,0:0.1:1)
%y((y>1/2)&(y<=1)&(x>1/2)&(x<=1)&(x+y>3/2))=NaN;
%%y((y>-1/2)&(y<=1)&(x>0)&(x<=(sqrt(3)/2))&((-sqrt(3)*x-y+1)<0))=NaN;
%%y((y>-1/2)&(y<=1)&(x>(-sqrt(3)/2))&(x<=0)&((sqrt(3)*x-y+1)<0))=NaN;
```

```
contour(x,y,PHI,'r-')

xlabel('X-axis');
ylabel('Y-axis');
%clabel(c,h);
axis square
st1='Contour level curves for ';
st2='FEM solution of ';
st3='Nine Noded ';
st4='Special Quadrilateral';
st5=' Elements'
title([st1,st2,st3,st4,st5])
sst1=' NODES='
sst2=num2str(nnode)
sst3=' ELEMENTS='
sst4=num2str(nel)
text(0.6,1.1,[sst1 sst2])
text(0.6,.9,[sst3 sst4])

hold on
%[x,y]=meshgrid(0:.1:1,0:0.1:1)
%[c,h]=contour(x,y,z,'g-')
contour(x,y,z,'b-')
%xlabel('X-axis');
%ylabel('Y-axis');
%clabel(c,h);
axis square
text(0.6,1.9,'{ SUPERPOSITION OF }')
text(0.6,1.7,'{ FEM/EXACT SOLUTIONS}')
text(0.6,1.5,'--(red)FEM ')
text(0.6,1.3,'--(blue)EXACT')

 mm=0;
 for i=1:31
  for j=1:31
      mm=mm+1;
      femsoln(mm,1)=PHI(i,j);
      exactsoln(mm,1)=z(i,j);
  end
end
end
% [femsoln exactsoln]

disp('-----------------------------------------------------------------------------------------
-------------')
disp('number of nodes,elements & nodes per element')
disp([nnode nel nnel ])
disp('torisonal constants(fem=phi&exact=xi)  error(max(abs(phi_xi))')
%disp('---------------------------------------------------------------------------------------
--------------')
%disp([nnode nel nnel ])
disp([T k1 MAXPHI_XI ])
disp('-----------------------------------------------------------------------------------------
-------------')
```

%(3)nodaladdresses_special_convex_quadrilaterals_2nd_orderLG.m

```
function[eln,nodetel,nodes,nnode]=nodaladdresses_special_convex_quadrilaterals_2nd_orde
rLG(n)
%division of a standard triangle(right isoscles triangle)
%into eight node special_convex_quadrilaterals
for nelm=1:3*(n/2)^2
    spqd(nelm,1:9)=0;
end
%disp('vertex nodes of triangle')
```

```
elm(1,1)=1;
elm(n+1,1)=2;
elm((n+1)*(n+2)/2,1)=3;
%disp('vertex nodes of triangle')
kk=3;
for k=2:n
    kk=kk+1;
    elm(k,1)=kk;
end
%disp('left edge nodes')
nni=1;
for i=0:(n-2)
    nni=nni+(n-i)+1;
    elm(nni,1)=3*n-i;
end
%disp('right edge nodes')
nni=n+1;
for i=0:(n-2)
    nni=nni+(n-i);
    elm(nni,1)=(n+3)+i;
end

%disp('interior nodes')
nni=1;jj=0;
for i=0:(n-3)
    nni=nni+(n-i)+1;
    for j=1:(n-2-i)
        jj=jj+1;
        nnj=nni+j;
        elm(nnj,1)=3*n+jj;
    end
end
%disp(elm)
%disp(length(elm))

jj=0;kk=0;
for j=0:n-1
    jj=j+1;
for k=1:(n+1)-j
    kk=kk+1;
    row_nodes(jj,k)=elm(kk,1);
end
end
row_nodes(n+1,1)=3;
%for jj=(n+1):-1:1
 %   disp(row_nodes(jj,:))
 %end
[row_nodes]
rr=row_nodes;
rrr(:,:,1)=rr;
%rr
%disp('element computations')
if rem(n,2)==0
ne=0;N=n+1;

for k=1:2:n-1
N=N-2;
i=k;
for j=1:2:N
    ne=ne+1;
eln(ne,1)=rr(i,j);
eln(ne,2)=rr(i,j+2);
eln(ne,3)=rr(i+2,j);
eln(ne,4)=rr(i,j+1);
eln(ne,5)=rr(i+1,j+1);
```

```
eln(ne,6)=rr(i+1,j);
end%i
%me=ne;
%N-2
if (N-2)>0
for jj=1:2:N-2
ne=ne+1;
eln(ne,1)=rr(i+2,jj+2);
eln(ne,2)=rr(i+2,jj);
eln(ne,3)=rr(i,jj+2);
eln(ne,4)=rr(i+2,jj+1);
eln(ne,5)=rr(i+1,jj+1);
eln(ne,6)=rr(i+1,jj+2);
end%jj
end
end%k
end
%ne
%for kk=1:ne
%[eln(kk,1:6)];
%end
%add node numbers for element centroids

nnd=(n+1)*(n+2)/2;
for kkk=1:ne
    nnd=nnd+1;
    eln(kkk,7)=nnd;
end
%for kk=1:ne
%[eln(kk,1:7)]
%end
%to generate special quadrilaterals
mm=0;
for iel=1:ne
    for jel=1:3
    mm=mm+1;
        switch jel
         case 1
          nodes(mm,1:4)=[eln(iel,7) eln(iel,6) eln(iel,1) eln(iel,4)];
          nodetel(mm,1:3)=[eln(iel,2) eln(iel,3) eln(iel,1)];
          case 2
          nodes(mm,1:4)=[eln(iel,7) eln(iel,4) eln(iel,2) eln(iel,5)];
          nodetel(mm,1:3)=[eln(iel,3) eln(iel,1) eln(iel,2)];
          case 3
          nodes(mm,1:4)=[eln(iel,7) eln(iel,5) eln(iel,3) eln(iel,6)];
          nodetel(mm,1:3)=[eln(iel,1) eln(iel,2) eln(iel,3)];
        end
     end
end


%for mmm=1:mm
    %spqd(:,1:4)
 %end
%mesh generation of eight node special quadrilaterals


%*********************************************************************
for inum=1:nnd
    for jnum=1:nnd
        mdpt(inum,jnum)=0;
    end
end
nd=nnd;
for mmm=1:mm
```

```
    mmm1=nodes(mmm,1);
    mmm2=nodes(mmm,2);
    mmm3=nodes(mmm,3);
    mmm4=nodes(mmm,4);
%midpoint side-1 of 4-node special quadrilateral
if((mdpt(mmm1,mmm2)==0)&(mdpt(mmm2,mmm1)==0))
    nd=nd+1;
   mdpt(mmm1,mmm2)=nd;
   mdpt(mmm2,mmm1)=nd;
end%if
%midpoint side-2 of 4-node special quadrilateral
if((mdpt(mmm2,mmm3)==0)&(mdpt(mmm3,mmm2)==0))
    nd=nd+1;
   mdpt(mmm2,mmm3)=nd;
   mdpt(mmm3,mmm2)=nd;
end%if
%midpoint side-3 of 4-node special quadrilateral
if((mdpt(mmm3,mmm4)==0)&(mdpt(mmm4,mmm3)==0))
    nd=nd+1;
   mdpt(mmm3,mmm4)=nd;
   mdpt(mmm4,mmm3)=nd;
end%if
%midpoint side-4 of 4-node special quadrilateral
if((mdpt(mmm4,mmm1)==0)&(mdpt(mmm1,mmm4)==0))
    nd=nd+1;
   mdpt(mmm4,mmm1)=nd;
   mdpt(mmm1,mmm4)=nd;
end
   nodes(mmm,5)=mdpt(mmm1,mmm2);
   nodes(mmm,6)=mdpt(mmm2,mmm3);
   nodes(mmm,7)=mdpt(mmm3,mmm4);
   nodes(mmm,8)=mdpt(mmm4,mmm1);
   nd=nd+1;
   nodes(mmm,9)=nd;
end
nnode=nd;
nel=mm;
spqd=nodes;

%***************************************************************************%(4)coordinate_
arbitrarytriangle_2ndorderLAGR.m

function[coord,gcoord]=coordinate_arbitrarytriangle_2ndorderLAGR(x,y,n)
syms ui vi wi xi yi
x1=x(1,1);x2=x(2,1);x3=x(3,1);
y1=y(1,1);y2=y(2,1);y3=y(3,1);
[ui,vi,wi]=coordinate_special_quadrilaterals_in_stdtriangle_2nd_orderLAGR(n)
%disp([ui vi wi])
N=length(ui);
    NN=(1:N)';
for i=1:N
    xi(i,1)=x1*wi(i,1)+x2*ui(i,1)+x3*vi(i,1);
    yi(i,1)=y1*wi(i,1)+y2*ui(i,1)+y3*vi(i,1);
end
%disp('_________________________________________________')
%disp('NN    xi      yi')
%disp([NN xi yi])
%disp('_________________________________________________')
coord(:,1)=(xi(:,1));
coord(:,2)=(yi(:,1));
gcoord(:,1)=double(xi(:,1));
gcoord(:,2)=double(yi(:,1));
%disp(gcoord);
%(5)coordinate_special_quadrilaterals_in_stdtriangle_2nd_orderLAGR.m
```

```
function[ui,vi,wi]=coordinate_special_quadrilaterals_in_stdtriangle_2nd_orderLAGR(n)
%n must be even:n=2,4,6,......
syms ui vi wi
ui(1:3,1)=[0;1;0];
vi(1:3,1)=[0;0;1];
wi(1:3,1)=[1;0;0];
if (n-1)>0
 kk=3;
 for i=1:n-1
    kk=kk+1;
    ui(kk,1)=sym(i/n);
    vi(kk,1)=sym(0);
    wi(kk,1)=sym(1-ui(kk,1)-vi(kk,1));
 end
 kkk=kk;
 for ii=1:n-1
    kkk=kkk+1;
    ui(kkk,1)=sym((n-ii)/n);
    vi(kkk,1)=sym(1-(n-ii)/n);
    wi(kkk,1)=0;
 end;
 kkkk=kkk;
  for iii=1:n-1
    kkkk=kkkk+1;
    ui(kkkk,1)=0;
    vi(kkkk,1)=sym(1-iii/n);
    wi(kkkk,1)=sym(iii/n);
  end
end%if (n-1)>0
if (n-2)>0
  kkkkk=kkkk;
  for iiii=1:(n-2)
    for jjjj=1:(n-1)-iiii
         kkkkk=kkkkk+1;
         ui(kkkkk,1)=sym(jjjj/n);
         vi(kkkkk,1)=sym(iiii/n);
         wi(kkkkk,1)=sym(1-ui(kkkkk,1)-vi(kkkkk,1));
     end
  end
end%if (n-2)>0
if n==2

    num=(1:6)';
else
  num=(1:kkkkk)';

end
%disp([ui'])
%disp([vi'])
%disp([wi'])
%length(ui)
%length(vi)
%length(wi)

%disp([num ui vi wi])
[eln,nodetel,spqd,nnode]=nodaladdresses4Lagrangespecial_convex_quadrilaterals_2nd_order
(n)
%n=number of divisions along sides;n must be even i.e n=2,4,6,.....
%qq=number of nodes on the triangle
%nc=number of six node triangles,we can insert a centroid in each 6-node triangle
qq=(n+1)*(n+2)/2;
    nc=(n/2)^2;
for pp=1:nc
 qq=qq+1;
 q1=eln(pp,1);
```

```
 q2=eln(pp,2);
 q3=eln(pp,3);
 ui(qq,1)=(ui(q1,1)+ui(q2,1)+ui(q3,1))/3;
 vi(qq,1)=(vi(q1,1)+vi(q2,1)+vi(q3,1))/3;
 wi(qq,1)=1-ui(qq,1)-vi(qq,1);
end
 %disp([ui vi wi])
%length(ui)
%length(vi)
%length(wi)

 num=(1:qq)';
%disp([num ui vi wi])
qqq=qq;
for ppp=1:3*nc
qq1=spqd(ppp,1);
qq2=spqd(ppp,2);
qq3=spqd(ppp,3);
qq4=spqd(ppp,4);
%midside nodes-1,2
 qqq=spqd(ppp,5);
 ui(qqq,1)=(ui(qq1,1)+ui(qq2,1))/2;
 vi(qqq,1)=(vi(qq1,1)+vi(qq2,1))/2;
 wi(qqq,1)=1-ui(qqq,1)-vi(qqq,1);
%midside nodes-2,3
 qqq=spqd(ppp,6);
 ui(qqq,1)=(ui(qq2,1)+ui(qq3,1))/2;
 vi(qqq,1)=(vi(qq2,1)+vi(qq3,1))/2;
 wi(qqq,1)=1-ui(qqq,1)-vi(qqq,1);
 %midside nodes-3,4
 qqq=spqd(ppp,7);
 ui(qqq,1)=(ui(qq3,1)+ui(qq4,1))/2;
 vi(qqq,1)=(vi(qq3,1)+vi(qq4,1))/2;
 wi(qqq,1)=1-ui(qqq,1)-vi(qqq,1);
%midside nodes-4,1
 qqq=spqd(ppp,8);
 ui(qqq,1)=(ui(qq1,1)+ui(qq4,1))/2;
 vi(qqq,1)=(vi(qq1,1)+vi(qq4,1))/2;
 wi(qqq,1)=1-ui(qqq,1)-vi(qqq,1);
 %centre node
 qqq=spqd(ppp,9);
 ui(qqq,1)=(ui(qq1,1)+ui(qq2,1)+ui(qq3,1)+ui(qq4,1))/4;
 vi(qqq,1)=(vi(qq1,1)+vi(qq2,1)+vi(qq3,1)+vi(qq4,1))/4;
 wi(qqq,1)=1-ui(qqq,1)-vi(qqq,1);

end
maxnode=max(max(spqd(1:3*nc,1:9)));
 num=(1:maxnode)';
 %disp(['maximum value of node number=',num2str(maxnode)])
%disp(' node    ui      vi      wi')
%disp([num ui vi wi])
```

% (6)nodaladdresses4Lagrangespecial_convex_quadrilaterals_2nd_order.m

```
function[eln,nodetel,nodes,nnode]=nodaladdresses4Lagrangespecial_convex_quadrilaterals_
2nd_order(n)
%division of a standard triangle(right isoscles triangle)
%into eight node special_convex_quadrilaterals
for nelm=1:3*(n/2)^2
    spqd(nelm,1:9)=0;
end
syms mst_tri x
%disp('vertex nodes of triangle')
elm(1,1)=1;
elm(n+1,1)=2;
```

```
elm((n+1)*(n+2)/2,1)=3;
%disp('vertex nodes of triangle')
kk=3;
for k=2:n
    kk=kk+1;
    elm(k,1)=kk;
end
%disp('left edge nodes')
nni=1;
for i=0:(n-2)
    nni=nni+(n-i)+1;
    elm(nni,1)=3*n-i;
end
%disp('right edge nodes')
nni=n+1;
for i=0:(n-2)
    nni=nni+(n-i);
    elm(nni,1)=(n+3)+i;
end

%disp('interior nodes')
nni=1;jj=0;
for i=0:(n-3)
    nni=nni+(n-i)+1;
    for j=1:(n-2-i)
        jj=jj+1;
        nnj=nni+j;
        elm(nnj,1)=3*n+jj;
    end
end
%disp(elm)
%disp(length(elm))

jj=0;kk=0;
for j=0:n-1
    jj=j+1;
for k=1:(n+1)-j
    kk=kk+1;
    row_nodes(jj,k)=elm(kk,1);
end
end
row_nodes(n+1,1)=3;
%for jj=(n+1):-1:1
 %   disp(row_nodes(jj,:))
 %end
%[row_nodes]
rr=row_nodes;
%rr
%disp('element computations')
if rem(n,2)==0
ne=0;N=n+1;

for k=1:2:n-1
N=N-2;
i=k;
for j=1:2:N
    ne=ne+1;
eln(ne,1)=rr(i,j);
eln(ne,2)=rr(i,j+2);
eln(ne,3)=rr(i+2,j);
eln(ne,4)=rr(i,j+1);
eln(ne,5)=rr(i+1,j+1);
eln(ne,6)=rr(i+1,j);
end%i
%me=ne;
```

```
%N-2;
if (N-2)>0
for jj=1:2:N-2
ne=ne+1;
eln(ne,1)=rr(i+2,jj+2);
eln(ne,2)=rr(i+2,jj);
eln(ne,3)=rr(i,jj+2);
eln(ne,4)=rr(i+2,jj+1);
eln(ne,5)=rr(i+1,jj+1);
eln(ne,6)=rr(i+1,jj+2);
end%jj
end
end%k
end
%ne
%for kk=1:ne
%[eln(kk,1:6)]
%end
%add node numbers for element centroids

nnd=(n+1)*(n+2)/2;
for kkk=1:ne
    nnd=nnd+1;
    eln(kkk,7)=nnd;
end
%for kk=1:ne
%[eln(kk,1:7)]
%end
%to generate special quadrilaterals
mm=0;
for iel=1:ne
    for jel=1:3
    mm=mm+1;
        switch jel
         case 1
          nodes(mm,1:4)=[eln(iel,7) eln(iel,6) eln(iel,1) eln(iel,4)];
          nodetel(mm,1:3)=[eln(iel,2) eln(iel,3) eln(iel,1)];
          case 2
          nodes(mm,1:4)=[eln(iel,7) eln(iel,4) eln(iel,2) eln(iel,5)];
          nodetel(mm,1:3)=[eln(iel,3) eln(iel,1) eln(iel,2)];
          case 3
          nodes(mm,1:4)=[eln(iel,7) eln(iel,5) eln(iel,3) eln(iel,6)];
          nodetel(mm,1:3)=[eln(iel,1) eln(iel,2) eln(iel,3)];
        end
     end
end


%for mmm=1:mm
    %spqd(:,1:4)
 %end
%mesh generation of eight node special quadrilaterals



for inum=1:nnd
    for jnum=1:nnd
        mdpt(inum,jnum)=0;
    end
end
nd=nnd;
for mmm=1:mm
    mmm1=nodes(mmm,1);
    mmm2=nodes(mmm,2);
    mmm3=nodes(mmm,3);
```

```
    mmm4=nodes(mmm,4);
%midpoint side-1 of 4-node special quadrilateral
if((mdpt(mmm1,mmm2)==0)&(mdpt(mmm2,mmm1)==0))
    nd=nd+1;
   mdpt(mmm1,mmm2)=nd;
   mdpt(mmm2,mmm1)=nd;
end
%midpoint side-2 of 4-node special quadrilateral
if((mdpt(mmm2,mmm3)==0)&(mdpt(mmm3,mmm2)==0))
    nd=nd+1;
   mdpt(mmm2,mmm3)=nd;
   mdpt(mmm3,mmm2)=nd;
end
%midpoint side-3 of 4-node special quadrilateral
if((mdpt(mmm3,mmm4)==0)&(mdpt(mmm4,mmm3)==0))
    nd=nd+1;
   mdpt(mmm3,mmm4)=nd;
   mdpt(mmm4,mmm3)=nd;
end
%midpoint side-4 of 4-node special quadrilateral
if((mdpt(mmm4,mmm1)==0)&(mdpt(mmm1,mmm4)==0))
    nd=nd+1;
   mdpt(mmm4,mmm1)=nd;
   mdpt(mmm1,mmm4)=nd;
end
nd=nd+1;
   nodes(mmm,5)=mdpt(mmm1,mmm2);
   nodes(mmm,6)=mdpt(mmm2,mmm3);
   nodes(mmm,7)=mdpt(mmm3,mmm4);
   nodes(mmm,8)=mdpt(mmm4,mmm1);
   nodes(mmm,9)=nd;
end
nnode=nd;
nel=mm;
```

%(7)D2LaplaceEquationQ9Ex3automeshgenNewPolygon.m

```
function[]=D2LaplaceEquationQ9Ex3automeshgenNewPolygon(n1,n2,n3,nmax,numtri,ndiv,mesh)
%ndiv=2,4,6,8,......
%polygonal_domain_coordinates([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,1,2)
%polygonal_domain_coordinates([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,4,4)
%D2LaplaceEquationQ4MoinExautomeshgen(n1,n2,n3,nmax,numtri,ndiv)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8
,1,2,1)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8
,4,4,1)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;
9;2],9,1,2,2)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;
9;2],9,4,4,2)
%quadrilateral_mesh4MOINEX_q4(n1,n2,n3,nmax,numtri,ndiv,mesh,xlength,ylength)([1;1;1;1;
1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,1,2,2,1,1)
%D2POISSONEQUATION_NODALINTERPOLATION_VALUES(n1,n2,n3,nmax,numtri,ndiv,mesh)([1;1;1;1;1
;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,1,2,2)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;
9;2],9,100,20,2)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,4,4,1)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,9,6,1)
```

```
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,16,8,1)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,25,10,1)
%D2PoissonEquationQ8MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2)
%D2PoissonEquationQ8MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;
8;2],8,1,2,1,1,10)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2,1,10)
%D2LaplaceEquationQ9Ex3automeshgenNewPolygon([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5
;6;7;8;9;2],9,1,2,4)
syms coord
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates(n1,n2,n3,nmax,numt
ri,ndiv,mesh)
%nnel=4;
[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_orderLG(n1,n2,n
3,nmax,numtri,ndiv,mesh)
nnel=9;
ndof=1;
%nc=(ndiv/2)^2;
%nnode=(ndiv+1)*(ndiv+2)/2+nc;
%nel=3*nc;
sdof=nnode*ndof;
ff=(zeros(sdof,1));ss=(zeros(sdof,sdof));

%nnode=17,nel=12,nnel=4,ndof=1
%>>LaplaceEquationQuad4twodimension(12,17,4,1)
%
%Ex1:nnode=41,nel=36,,nnel=4,nodf=1
%>>LaplaceEquationQuad4twodimensionEx1(36,41,4,1)
%>>improvedLaplaceEquationQuad4twodimensionEx1_explicit(36,41,4,1)
%Ex2:nnode=83,nel=69,,nnel=4,nodf=1
%>>improvedLaplaceEquationQuad4twodimensionEx2_explicit(69,83,4,1)#
%>>improvedLaplaceEquationQuad4twodimensionEx2_explicitfnmesh(69,83,4,1)#
%improvedLaplaceEquationQuad4twodimensionEx2_explicitvfnmesh(72,87,4,1)#new
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=3,nnode=7,nnel=4,ndof=1)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel,nnode,nnel=4,ndof=1,quadt
ype=0/3,mesh=1,2,3...)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=12,nnode=19,nnel=4,ndof=1
,quadtype=0/3,mesh=3)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=27,nnode=37,nnel=4,ndof=1
,quadtype=0/3,mesh=4)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=48,nnode=61,nnel=4,ndof=1
,quadtype=0/3,mesh=5)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=75,nnode=91,nnel=4,ndof=1
,quadtype=0/3,mesh=6)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(108,127,4,1,3,7)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(147,169,4,1,3,8)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(192,217,4,1,3,9)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(243,271,4,1,3,10)
disp([nel nnode nnel ndof])
format long g
for i=1:nel
N(i,1)=i;
end
for i=1:nel
NN(i,1)=i;
```

```
end
    %[coord,gcoord]=coordinate_rtisoscelestriangle00_h0_hh(ndiv);
    %[nodetel,nodes]=nodaladdresses4special_convex_quadrilaterals(ndiv)
    %
    %bcdof=[2;5;3]
   %boundary conditions-1
 switch mesh


case 4
    %boundary conditions-2
    nnn=0;
    for nn=1:nnode
        xnn=coord(nn,1);ynn=gcoord(nn,2);
        if (xnn==-1/2)&((ynn>=-1/2)&(ynn<=1/2))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
    %boundary conditions-2
     for nn=1:nnode
        xnn=gcoord(nn,1);ynn=coord(nn,2);
        if (ynn==-1/2)&((xnn>=-1/2)&(xnn<=1/2))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
    %boundary conditions-3
    for nn=1:nnode
        xnn=gcoord(nn,1);ynn=coord(nn,2);
        if (ynn==1/2)&((xnn>=-1/2)&(xnn<=1/2))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
   %boundary conditions-4
   for nn=1:nnode
      xnn=coord(nn,1);ynn=gcoord(nn,2);
        if (xnn==1/2)&((ynn>=-1/2)&(ynn<=1/2))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end


%end
    bcdof
    mm=length(bcdof);


   format long g
k1 =double(0.14057701495515551037840396020329);
 xi=(zeros(nnode,1));
a0=8/pi^3;
for m=1:nnode
    gx=(gcoord(m,1));gy=(gcoord(m,2));rr=(0);
for n=1:2:99
rr=rr+(-1)^((n-1)/2)*(1-(cosh(n*pi*gy)/cosh(n*pi/2)))*cos(n*pi*gx)/n^3;
end
xi(m,1)=(a0*rr);
end
```

```
   end %switch
 for L=1:nel
   for M=1:3
      LM=nodetel(L,M);
      xx(L,M)=gcoord(LM,1);
      yy(L,M)=gcoord(LM,2);
    end
  end
%__________________________________________________________________
%ng=10

table2=[N xx yy];
%disp([xx yy])
%_______________________________________________________________________
```

```
intJdn1dn1uvrs =[vpa(sym('  .68800369699992917040751928889')) vpa(sym(' .63026863732902919654693031748'));...
vpa(sym(' .63026863732902919654693031748')) vpa(sym('  .68800369699992917040751928889'))];
intJdn1dn2uvrs =[vpa(sym('  .49469766679465431322423128849e-1')) vpa(sym(' -.92492499790138905674372588873e-1'));...
vpa(sym('  .74174166876527760992294077793e-1')) vpa(sym(' -.58041502241256898516961218830e-1'))];
intJdn1dn3uvrs =[vpa(sym(' -.27639187071864196478244612770e-1')) vpa(sym(' -.36406227705135716084660539928e-1'));...
vpa(sym(' -.36406227705135716084660539928e-1')) vpa(sym(' -.27639187071864196478244612770e-1'))];
intJdn1dn4uvrs =[vpa(sym(' -.58041502241256898516961218830e-1')) vpa(sym('  .74174166876527760992294077793e-1'));...
vpa(sym(' -.92492499790138905674372588873e-1')) vpa(sym('  .49469766679465431322423128849e-1'))];
intJdn1dn5uvrs =[vpa(sym(' -.42034020155248898256147959993')) vpa(sym('  .23835534506556861836117847703'));...
vpa(sym(' -.42831132160109804830548819964')) vpa(sym('  .10203781293970562330870800320'))];
intJdn1dn6uvrs =[vpa(sym(' .12626870005164588340746772853')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .15224655673422927054938212414'))];
intJdn1dn7uvrs =[vpa(sym(' .15224655673422927054938212414')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .12626870005164588340746772853'))];
intJdn1dn8uvrs =[vpa(sym('  .10203781293970562330870800320')) vpa(sym(' -.42831132160109804830548819964'));...
vpa(sym('  .23835534506556861836117847703')) vpa(sym(' -.42034020155248898256147959993'))];
intJdn1dn9uvrs =[vpa(sym(' -.61200564253936530143881487716')) vpa(sym(' -.72289148243880098858257938847'));...
vpa(sym(' -.72289148243880098858257938847')) vpa(sym(' -.61200564253936530143881487716'))];
intJdn2dn1uvrs =[vpa(sym('  .49469766679465431322423128849e-1')) vpa(sym('  .74174166876527760992294077793e-1'));...
vpa(sym(' -.92492499790138905674372588873e-1')) vpa(sym(' -.58041502241256898516961218830e-1'))];
intJdn2dn2uvrs =[vpa(sym('  .24822242723570359859415910165')) vpa(sym(' -.13943577781062161645243966287'));...
vpa(sym(' -.13943577781062161645243966287')) vpa(sym('  .22525137401606776447455758469'))];
intJdn2dn3uvrs =[vpa(sym('  -.93683290648218768462051538869e-1')) vpa(sym(' -.82852283681797324065968279846e-1'));...
vpa(sym('  .83814382984869342600698386820e-1')) vpa(sym('   .37904800898126436475032582264e-1'))];
intJdn2dn4uvrs =[vpa(sym('   .62925700025302140451751960e-3')) vpa(sym(' .26100052047122979851449223346e-1'));...
vpa(sym(' .26100052047122979851449223346e-1')) vpa(sym('   .62925700025302140451751960e-3'))];
intJdn2dn5uvrs =[vpa(sym('   -.46868216752790132335135919662')) vpa(sym('  -.34646194452109534302782806358'));...
vpa(sym('   .32020472214557132363883860309')) vpa(sym(' .85836729209541381287229242888e-1'))];
intJdn2dn6uvrs =[vpa(sym('   .20486711030201898764584585532')) vpa(sym('  .25718774442435573105221634722'));...
vpa(sym(' -.40947892224231093561445031945')) vpa(sym(' -.28073862677651621463393803502'))];
intJdn2dn7uvrs =[vpa(sym(' .50885917641966498462888866644e-1')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' -.17609016019421764322513358580e-2'))];
intJdn2dn8uvrs =[vpa(sym(' -.41736856619169902011341267758e-1')) vpa(sym(' -.94765689307712237483661393307e-1'));...
vpa(sym(' -.94765689307712237483661393307e-1')) vpa(sym('  .33881015844374607003400451161e-1'))];
intJdn2dn9uvrs =[vpa(sym('   .50027835935882456394917533332e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym(' -.42962146348647921061586792226e-1'))];
intJdn3dn1uvrs =[vpa(sym(' -.27639187071864196478244612770e-1')) vpa(sym(' -.36406227705135716084660539928e-1'));...
vpa(sym(' -.36406227705135716084660539928e-1')) vpa(sym(' -.27639187071864196478244612770e-1'))];
intJdn3dn2uvrs =[vpa(sym('  -.93683290648218768462051538869e-1')) vpa(sym('  .83814382984869342600698386820e-1'));...
vpa(sym(' -.82852283681797324065968279846e-1')) vpa(sym('   .37904800898126436475032582264e-1'))];
intJdn3dn3uvrs =[vpa(sym('  .33203856905137126709551224250')) vpa(sym(' .29812794600013651335576030271'));...
vpa(sym(' .29812794600013651335576030271')) vpa(sym('  .33203856905137126709551224250'))];
intJdn3dn4uvrs =[vpa(sym('   .37904800898126436475032582264e-1')) vpa(sym(' -.82852283681797324065968279846e-1'));...
vpa(sym('  .83814382984869342600698386820e-1')) vpa(sym('  -.93683290648218768462051538869e-1'))];
intJdn3dn5uvrs =[vpa(sym('   .15710844821544667058002407565')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym(' .43059644229262196507731226697e-1'))];
intJdn3dn6uvrs =[vpa(sym('  .14159217838663875388665351332')) vpa(sym(' -.35360191630747700837476451283'));...
vpa(sym('  .31306475035918965829190215383')) vpa(sym(' -.34086853405320051524855521401'))];
intJdn3dn7uvrs =[vpa(sym(' -.34086853405320051524855521401')) vpa(sym('  .31306475035918965829190215383'));...
vpa(sym(' -.35360191630747700837476451283')) vpa(sym('  .14159217838663875388665351332'))];
```

```
intJdn3dn8uvrs =[vpa(sym(' .4305964422926219650773122697e-1')) vpa(sym('   .1294554954791107490830820766 8'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym('   .15710844821544667058002407565'))];
intJdn3dn9uvrs =[vpa(sym('  -.24951262900756184435610227 57')) vpa(sym(' -.48105764260800696388913166479'));...
vpa(sym(' -.48105764260800696388913166479')) vpa(sym('  -.24951262900756184435610227 57'))];
intJdn4dn1uvrs =[vpa(sym(' -.5804150224125689851696121830e-1')) vpa(sym(' -.9249249979013890567437258873e-1'));...
vpa(sym('  .7417416687652776099229407793e-1')) vpa(sym('  .4946976667946543132242312849e-1'))];
intJdn4dn2uvrs =[vpa(sym('   .6292570002530214045175196 0e-3')) vpa(sym(' .2610005204712297985144922346e-1'));...
vpa(sym(' .2610005204712297985144922346e-1')) vpa(sym('   .62925700025302140451751960e-3'))];
intJdn4dn3uvrs =[vpa(sym('   .3790480089812643647503258264e-1')) vpa(sym('  .8381438298486934260069838682 0e-1'));...
vpa(sym(' -.8285228368179732406596827984 6e-1')) vpa(sym('  -.9368329064821876846205153869e-1'))];
intJdn4dn4uvrs =[vpa(sym('  .22525137401606776447455758469')) vpa(sym(' -.13943577781062161645243966287'));...
vpa(sym(' -.13943577781062161645243966287')) vpa(sym('  .24822242723570359859415910165'))];
intJdn4dn5uvrs =[vpa(sym('  .3388101584437460700340045161e-1')) vpa(sym(' -.9476568930771223748366139307e-1'));...
vpa(sym(' -.9476568930771223748366139307e-1')) vpa(sym(' -.4173685661916990201134126758e-1'))];
intJdn4dn6uvrs =[vpa(sym(' -.1760901601942176432251335 80e-2')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' .5088591764196649846288886644e-1'))];
intJdn4dn7uvrs =[vpa(sym(' -.28073862677651621463393803502')) vpa(sym(' -.40947892224231093561445031945'));...
vpa(sym('  .25718774442435573105221634722')) vpa(sym('   .20486711030201898764584585 32'))];
intJdn4dn8uvrs =[vpa(sym(' .8583672920954138128722924288e-1')) vpa(sym('   .32020472214557132363883860309'));...
vpa(sym('  -.34646194452109534302782806358')) vpa(sym('   -.46868216752790132335135919 62'))];
intJdn4dn9uvrs =[vpa(sym(' -.4296214634864792106158679226e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym('   .5002783593588245639491753 32e-1'))];
intJdn5dn1uvrs =[vpa(sym(' -.42034020155248898256147959 93')) vpa(sym(' -.42831132160109804830548819 64'));...
vpa(sym('  .23835534506556861836117847 03')) vpa(sym('  .10203781293970562330870803 20'))];
intJdn5dn2uvrs =[vpa(sym('   -.46868216752790132335135919 62')) vpa(sym('   .32020472214557132363883860309'));...
vpa(sym('  -.34646194452109534302782806358')) vpa(sym(' .8583672920954138128722924288e-1'))];
intJdn5dn3uvrs =[vpa(sym('   .15710844821544667058002407565')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym(' .4305964422926219650773122697e-1'))];
intJdn5dn4uvrs =[vpa(sym('  .3388101584437460700340045161e-1')) vpa(sym(' -.9476568930771223748366139307e-1'));...
vpa(sym(' -.9476568930771223748366139307e-1')) vpa(sym(' -.4173685661916990201134126758e-1'))];
intJdn5dn5uvrs =[vpa(sym(' 1.0663384833344564856750376651')) vpa(sym('  .56556859863119073241392390 19'));...
vpa(sym('  .56556859863119073241392390 19')) vpa(sym(' 1.1211333295329756651950424278'))];
intJdn5dn6uvrs =[vpa(sym('  -.58306188548401601151324941 69')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym(' -.22579554943131879593427777753'))];
intJdn5dn7uvrs =[vpa(sym('   -.16767681050348731002396257 42')) vpa(sym(' -.1709647811328463063306097 92e-2'));...
vpa(sym(' -.1709647811328463063306097 92e-2')) vpa(sym('   .17132457681637017904834081102'))];
intJdn5dn8uvrs =[vpa(sym(' .9027148605382036792067597 12e-1')) vpa(sym('   .41024643803333829631989611 84'));...
vpa(sym('   .41024643803333829631989611 84')) vpa(sym(' .9027148605382036792067597 12e-1'))];
intJdn5dn9uvrs =[vpa(sym('   .29216163161979549627091262 30')) vpa(sym('  -.35816040869776877564539558 36'));...
vpa(sym('  -.35816040869776877564539558 36')) vpa(sym(' -1.3461311727311867153221086666'))];
intJdn6dn1uvrs =[vpa(sym(' .12626870005164588340746772853')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .15224655673422927054938212414'))];
intJdn6dn2uvrs =[vpa(sym('   .20486711030201898764584585 32')) vpa(sym(' -.40947892224231093561445031945'));...
vpa(sym('  .25718774442435573105221634722')) vpa(sym(' -.28073862677651621463393803502'))];
intJdn6dn3uvrs =[vpa(sym('  .14159217838663875388665351332')) vpa(sym('  .31306475035918965829190215383'));...
vpa(sym(' -.35360191630747700837476451283')) vpa(sym(' -.34086853405320051524855521401'))];
intJdn6dn4uvrs =[vpa(sym(' -.1760901601942176432251335 80e-2')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' .5088591764196649846288886644e-1'))];
intJdn6dn5uvrs =[vpa(sym('  -.58306188548401601151324941 69')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym(' -.22579554943131879593427777753'))];
intJdn6dn6uvrs =[vpa(sym(' 1.5998510858095308546375928228')) vpa(sym(' .24469327225090109904911663174'));...
vpa(sym(' .24469327225090109904911663174')) vpa(sym('  .64675514174853057100283264 58'))];
intJdn6dn7uvrs =[vpa(sym(' -.7461268601626634110113636 05e-1')) vpa(sym('   .42496611860422613701397774221'));...
vpa(sym('   .42496611860422613701397774221')) vpa(sym(' -.7461268601626634110113636 05e-1'))];
intJdn6dn8uvrs =[vpa(sym('   .17132457681637017904834081102')) vpa(sym(' -.1709647811328463063306097 92e-2'));...
vpa(sym(' -.1709647811328463063306097 92e-2')) vpa(sym('   -.16767681050348731002396257 42'))];
intJdn6dn9uvrs =[vpa(sym('   -1.5844681782639801295792636156')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('    .23980459065606283692676632 48'))];
intJdn7dn1uvrs =[vpa(sym(' .15224655673422927054938212414')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .12626870005164588340746772853'))];
intJdn7dn2uvrs =[vpa(sym(' .5088591764196649846288886644e-1')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' -.1760901601942176432251335 80e-2'))];
intJdn7dn3uvrs =[vpa(sym(' -.34086853405320051524855521401')) vpa(sym(' -.35360191630747700837476451283'));...
vpa(sym('  .31306475035918965829190215383')) vpa(sym('  .14159217838663875388665351332'))];
intJdn7dn4uvrs =[vpa(sym(' -.28073862677651621463393803502')) vpa(sym('  .25718774442435573105221634722'));...
vpa(sym(' -.40947892224231093561445031945')) vpa(sym('   .20486711030201898764584585 32'))];
intJdn7dn5uvrs =[vpa(sym('   -.16767681050348731002396257 42')) vpa(sym(' -.1709647811328463063306097 92e-2'));...
vpa(sym(' -.1709647811328463063306097 92e-2')) vpa(sym('   .17132457681637017904834081102'))];
```

```
intJdn7dn6uvrs =[vpa(sym(' -.746126860162663411011363605e-1')) vpa(sym('   .42496611860422613701397774221'));...
vpa(sym('   .42496611860422613701397774221')) vpa(sym(' -.746126860162663411011363605e-1'))];
intJdn7dn7uvrs =[vpa(sym('  .6467551417485305710028326458')) vpa(sym(' .24469327225090109904911663174'));...
vpa(sym(' .24469327225090109904911663174')) vpa(sym(' 1.5998510858095308546375928228'))];
intJdn7dn8uvrs =[vpa(sym(' -.22579554943131879593427777753')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym('  -.5830618854840160115132494169'))];
intJdn7dn9uvrs =[vpa(sym('    .2398045906560628369267663248')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('   -1.5844681782639801295792636156'))];
intJdn8dn1uvrs =[vpa(sym('  .1020378129397056233087080320')) vpa(sym('  .2383553450655686183611784703'));...
vpa(sym(' -.4283113216010980483054881964')) vpa(sym(' -.4203402015524889825614795993'))];
intJdn8dn2uvrs =[vpa(sym(' -.4173685661916990201134126758e-1')) vpa(sym(' -.9476568930771223748366139307e-1'));...
vpa(sym(' -.9476568930771223748366139307e-1')) vpa(sym('  .3388101584437460700340045161e-1'))];
intJdn8dn3uvrs =[vpa(sym(' .4305964422926219650773122697e-1')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym('   .15710844821544667058002407565'))];
intJdn8dn4uvrs =[vpa(sym(' .8583672920954138128722924288e-1')) vpa(sym('  -.34646194452109534302782806358'));...
vpa(sym('   .32020472214557132363883860309')) vpa(sym('   -.4686821675279013233513591962'))];
intJdn8dn5uvrs =[vpa(sym(' .902714860538203679206759712e-1')) vpa(sym('   .4102464380333382963198961184'));...
vpa(sym('   .4102464380333382963198961184')) vpa(sym(' .902714860538203679206759712e-1'))];
intJdn8dn6uvrs =[vpa(sym('   .17132457681637017904834081102')) vpa(sym(' -.170964781132846306330609792e-2'));...
vpa(sym(' -.170964781132846306330609792e-2')) vpa(sym('   -.1676768105034873100239625742'))];
intJdn8dn7uvrs =[vpa(sym(' -.22579554943131879593427777753')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym('  -.5830618854840160115132494169'))];
intJdn8dn8uvrs =[vpa(sym(' 1.1211333295329756651950424278')) vpa(sym('  .5655685986311907324139239019'));...
vpa(sym('  .5655685986311907324139239019')) vpa(sym(' 1.0663384833344564856750376651'))];
intJdn8dn9uvrs =[vpa(sym(' -1.3461311727311867153221086666')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym('   .2921616316197954962709126230'))];
intJdn9dn1uvrs =[vpa(sym(' -.6120056425393653014388148716')) vpa(sym(' -.7228914824388009885825793847'));...
vpa(sym(' -.7228914824388009885825793847')) vpa(sym(' -.6120056425393653014388148716'))];
intJdn9dn2uvrs =[vpa(sym('   .500278359358824563949175332e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym(' -.4296214634864792106158679226e-1'))];
intJdn9dn3uvrs =[vpa(sym('  -.2495126290075618443561022757')) vpa(sym(' -.48105764260800696388913166479'));...
vpa(sym(' -.48105764260800696388913166479')) vpa(sym('  -.2495126290075618443561022757'))];
intJdn9dn4uvrs =[vpa(sym(' -.4296214634864792106158679226e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym('   .500278359358824563949175332e-1'))];
intJdn9dn5uvrs =[vpa(sym('   .2921616316197954962709126230')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym(' -1.3461311727311867153221086666'))];
intJdn9dn6uvrs =[vpa(sym('   -1.5844681782639801295792636156')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('    .2398045906560628369267663248'))];
intJdn9dn7uvrs =[vpa(sym('    .2398045906560628369267663248')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('   -1.5844681782639801295792636156'))];
intJdn9dn8uvrs =[vpa(sym(' -1.3461311727311867153221086666')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym('   .2921616316197954962709126230'))];
intJdn9dn9uvrs =[vpa(sym(' 3.2530857106790011221652797406')) vpa(sym(' 1.2900488362095239461700024257'));...
vpa(sym(' 1.2900488362095239461700024257')) vpa(sym(' 3.2530857106790011221652797406'))];
%integrals of products of glbal derivatives
intJdndn=double([intJdn1dn1uvrs intJdn1dn2uvrs intJdn1dn3uvrs intJdn1dn4uvrs intJdn1dn5uvrs intJdn1dn6uvrs intJdn1dn7uvrs intJdn1dn8uvrs intJdn1dn9uvrs ;...
      intJdn2dn1uvrs intJdn2dn2uvrs intJdn2dn3uvrs intJdn2dn4uvrs intJdn2dn5uvrs intJdn2dn6uvrs intJdn2dn7uvrs intJdn2dn8uvrs intJdn2dn9uvrs;...
      intJdn3dn1uvrs intJdn3dn2uvrs intJdn3dn3uvrs intJdn3dn4uvrs intJdn3dn5uvrs intJdn3dn6uvrs intJdn3dn7uvrs intJdn3dn8uvrs intJdn3dn9uvrs;...
      intJdn4dn1uvrs intJdn4dn2uvrs intJdn4dn3uvrs intJdn4dn4uvrs intJdn4dn5uvrs intJdn4dn6uvrs intJdn4dn7uvrs intJdn4dn8uvrs intJdn4dn9uvrs;...
      intJdn5dn1uvrs intJdn5dn2uvrs intJdn5dn3uvrs intJdn5dn4uvrs intJdn5dn5uvrs intJdn5dn6uvrs intJdn5dn7uvrs intJdn5dn8uvrs intJdn5dn9uvrs;...
      intJdn6dn1uvrs intJdn6dn2uvrs intJdn6dn3uvrs intJdn6dn4uvrs intJdn6dn5uvrs intJdn6dn6uvrs intJdn6dn7uvrs intJdn6dn8uvrs intJdn6dn9uvrs;...
      intJdn7dn1uvrs intJdn7dn2uvrs intJdn7dn3uvrs intJdn7dn4uvrs intJdn7dn5uvrs intJdn7dn6uvrs intJdn7dn7uvrs intJdn7dn8uvrs intJdn7dn9uvrs;...
      intJdn8dn1uvrs intJdn8dn2uvrs intJdn8dn3uvrs intJdn8dn4uvrs intJdn8dn5uvrs intJdn8dn6uvrs intJdn8dn7uvrs intJdn8dn8uvrs intJdn8dn9uvrs;...
      intJdn9dn1uvrs intJdn9dn2uvrs intJdn9dn3uvrs intJdn9dn4uvrs intJdn9dn5uvrs intJdn9dn6uvrs intJdn9dn7uvrs intJdn9dn8uvrs intJdn9dn9uvrs]);
%____________________________________________________________________________________________

%
for iel=1:nel
index=zeros(nnel*ndof,1);

X=xx(iel,1:3);
Y=yy(iel,1:3);
%disp([X Y])
xa=X(1,1);
xb=X(1,2);
xc=X(1,3);
ya=Y(1,1);
yb=Y(1,2);
yc=Y(1,3);
```

```
bta=yb-yc;btb=yc-ya;
gma=xc-xb;gmb=xa-xc;
delabc=gmb*bta-gma*btb;
G=[bta btb;gma gmb]/delabc;
GT=[bta gma;btb gmb]/delabc;
Q=GT*G;
sk(1:9,1:9)=(zeros(9,9));
for i=1:9
 for j=i:9
 sk(i,j)=(delabc*sum(sum(Q.*(intJdndn(2*i-1:2*i,2*j-1:2*j)))));
 sk(j,i)=sk(i,j);
 end
end
%f =[5/144;1/24;7/144;1/24]*(2*delabc);

 xe(1,1)=(xa+xb+xc)/3;
 xe(2,1)=(xa+xc)/2;
 xe(3,1)=xa;
 xe(4,1)=(xa+xb)/2;
 %
 ye(1,1)=(ya+yb+yc)/3;
 ye(2,1)=(ya+yc)/2;
 ye(3,1)=ya;
 ye(4,1)=(ya+yb)/2;
 %


 %for j=1:4
 %    qe(j,1)=(2*pi^2)*sin(pi*xe(j,1))*sin(pi*ye(j,1));
 %end
 %II =([  1/72, 7/864, 1/216, 7/864;...
 %     7/864,  1/54,  1/96, 1/216;...
  %    1/216,  1/96, 5/216,  1/96;...
   %   7/864, 1/216,  1/96,  1/54]);
 %f=(2*delabc)*(II*qe);
 %+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++



 f=[ 1/432; 1/216; 1/144; 1/216;  1/72; 5/216; 5/216;  1/72;  2/27]*(2*delabc);



 %+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
%_______________________________________
 edof=nnel*ndof;
 k=0;
 for i=1:nnel
     nd(i,1)=nodes(iel,i);
     start=(nd(i,1)-1)*ndof;
     for j=1:ndof
         k=k+1;
         index(k,1)=start+j;
     end
 end
 %-------------------------------------------------------------------------
for i=1:edof
    ii=index(i,1);
    ff(ii,1)=ff(ii,1)+f(i,1);
 for j=1:edof
     jj=index(j,1);
     ss(ii,jj)=ss(ii,jj)+sk(i,j);
 end
end
end%for iel
 %-------------------------------------------------------------------------
%bcdof=[13;37;35;33;31;29;27;25;23;21;19;17;15];
%apply boundary conditions
```

```
%
mm=length(bcdof);
sdof=size(ss);
%
for i=1:mm
c=bcdof(i,1);
for j=1:sdof
ss(c,j)=0;
end
%
ss(c,c)=1;
ff(c,1)=bcval(i,1);
end
%solve the equations

phi=ss\ff;
for I=1:nnode
NN(I,1)=I;
end

disp('__________________________________________________________________')
disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
disp('_________________________________________________________________________________
____')
disp('                         fem-computed values             anlytical(theoretical)-
values           ')

disp([NN phi xi])
disp('_________________________________________________________________________________
____')

disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
nodes
gcoord
[x,y]=meshgrid(-1/2:0.1:1/2,-1/2:0.1:1/2);

a0=8/pi^3;


for i=1:11
    for j=1:11
      for iel=1:nel
 %node numbers of quadrilateral
       nd1=nodes(iel,1);nd2=nodes(iel,2);nd3=nodes(iel,3);nd4=nodes(iel,4);
       nd5=nodes(iel,5);nd6=nodes(iel,6);nd7=nodes(iel,7);nd8=nodes(iel,8);
       nd9=nodes(iel,9);
 %coordinates of quadrilateral(u,v)

u(1,1)=gcoord(nd1,1);u(2,1)=gcoord(nd2,1);u(3,1)=gcoord(nd3,1);u(4,1)=gcoord(nd4,1);

v(1,1)=gcoord(nd1,2);v(2,1)=gcoord(nd2,2);v(3,1)=gcoord(nd3,2);v(4,1)=gcoord(nd4,2);
 %coordinates of the grid(x,y)

          in=inpolygon(x(i,j),y(i,j),u,v);
          if (in==1)
           X=x(i,j);Y=y(i,j);
           [t]=convexquadrilateral_coordinates(u,v,X,Y);
           r=t(1,1);
           s=t(2,1);
           shn1=(r^2-r)*(s^2-s)/4;
           shn2=(r^2+r)*(s^2-s)/4;
```

```
           shn3=(r^2+r)*(s^2+s)/4;
           shn4=(r^2-r)*(s^2+s)/4;
           shn5=(1-r^2)*(s^2-s)/2;
           shn6=(r^2+r)*(1-s^2)/2;
           shn7=(1-r^2)*(s^2+s)/2;
           shn8=(r^2-r)*(1-s^2)/2;
           shn9=(1-r^2)*(1-s^2);
%


%
PHI(i,j)=shn1*phi(nd1,1)+shn2*phi(nd2,1)+shn3*phi(nd3,1)+shn4*phi(nd4,1)+shn5*phi(nd5,1
)+shn6*phi(nd6,1)+shn7*phi(nd7,1)+shn8*phi(nd8,1)+shn9*phi(nd9,1);
              break
           end%if (in==1)
        end%for iel
   %THE PROGRAM EXECUTION JUMPS TO HERE if (in==1)
    end%for j
end%for i

a0=8/pi^3;
for ii=1:11
    for jj=1:11
    xx=(x(ii,jj));yy=(y(ii,jj));rr=(0);

for n=1:2:99
rr=rr+(-1)^((n-1)/2)*(1-(cosh(n*pi*yy)/cosh(n*pi/2)))*cos(n*pi*xx)/n^3;
end
z(ii,jj)=(a0*rr);
end %ii
end%jj

 % z=sin(pi*x).*sin(pi*y);

for i=1:11
    for j=1:11
        if (abs(PHI(i,j))<=1e-5)
            PHI(i,j)=0;
        end
        if (abs(z(i,j))<=1e-5)
            z(i,j)=0;
        end

    end
end
switch mesh

 case 4

    hold off
 clf
figure(1)
[x,y]=meshgrid(-1/2:.1:1/2,-1/2:0.1:1/2)
[c,h]=contour(x,y,PHI)
xlabel('X-axis');
ylabel('Y-axis');
clabel(c,h);
axis square
st1='Contour level curves for ';
st2='FEM solution of ';
st3='Nine Noded  ';
st4='Special Quadrilateral';
st5=' Elements'
title([st1,st2,st3,st4,st5])
sst1='(MESH HAS '
```

```
sst2=num2str(nnode)
sst3=' NODES'
sst4=' AND '
sst5=num2str(nel)
sst6=' ELEMENTS)'
text(0.25,-.08,[sst1 sst2 sst3 sst4 sst5 sst6])

figure(2)
%[x,y]=meshgrid(0:.1:1,0:0.1:1)
[x,y]=meshgrid(-1/2:.1:1/2,-1/2:0.1:1/2)
[c,h]=contour(x,y,z)
xlabel('X-axis');
ylabel('Y-axis');
clabel(c,h);
axis square
title('contour level curves for exact solution: in aseries')
 mm=0;
 for i=1:11
  for j=1:11
      mm=mm+1;
      femsoln(mm,1)=PHI(i,j);
      exactsoln(mm,1)=z(i,j);
  end
end
%+++++++++
hold off

figure(3)
[x,y]=meshgrid(-1/2:.1:1/2,-1/2:0.1:1/2)
%[x,y]=meshgrid(-sqrt(2)/2:(0.1)*sqrt(2)/2:sqrt(2)/2,-
sqrt(2)/2:(0.1)*sqrt(2)/2:sqrt(2)/2);
%[c,h]=contour(x,y,PHI,'r-')
contour(x,y,PHI,'r-')
xlabel('X-axis');
ylabel('Y-axis');
%clabel(c,h);
axis square
st1='Contour level curves for ';
st2='FEM(red)&exact(green) ';
st3='Nine Noded ';
st4='Special Quadrilateral';
st5=' Elements'
title([st1,st2,st3,st4,st5])
sst1='(MESH HAS '
sst2=num2str(nnode)
sst3=' NODES'
sst4=' AND '
sst5=num2str(nel)
sst6=' ELEMENTS)'
text(-1/2,-1/2,[sst1 sst2 sst3 sst4 sst5 sst6])
hold on
%[x,y]=meshgrid(0:.1:1,0:0.1:1)
%[c,h]=contour(x,y,z,'g-')
contour(x,y,z,'g-')
%xlabel('X-axis');
%ylabel('Y-axis');
%clabel(c,h);
axis square


end%switch mesh
%[femsoln exactsoln]

 disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
```

```
[1 phi(1,1) xi(1,1)]
disp('__________________________________________________________________')
disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
disp('_________________________________________________________________________________
____')
disp('                         fem-computed values             anlytical(theoretical)-
values           ')

disp([NN phi xi])
disp('_________________________________________________________________________________
____')

disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]

if mesh==4
%[phi xi]
for I=1:nnode
NN(I,1)=I;
phi_xi(I,1)=phi(I,1)-xi(I,1);
end
MAXPHI_XI=max(abs(double(phi_xi)));

t=0;
 for iii=1:nnode
    t=t+phi(iii,1)*ff(iii,1);
 end
 T=t;
disp('---------------------------------------------------------------------------------
------------')
disp('number of nodes,elements & nodes per element')
disp([nnode nel nnel ])
disp('torisonal constants(fem=phi&exact=xi)  error(max(abs(phi_xi))')
%disp('--------------------------------------------------------------------------------
-------------')
%disp([nnode nel nnel ])
disp([T k1 MAXPHI_XI ])
disp('---------------------------------------------------------------------------------
------------')
end
disp('_________________________________________________________________________________
____')

disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
for ncpt=1:nel
    centrpt=nodes(ncpt,9)
    elcentr(ncpt,1)=centrpt;
    phicpt(ncpt,1)=phi(centrpt,1);
    xicpt(ncpt,1)=xi(centrpt,1);
end
for I=1:nel
elnumm(I,1)=I;
end
disp('_________________________________________________________________________________
____')
disp(' serial no        center point                fem-computed values
anlytical(theoretical)-values           ')

disp([elnumm  elcentr phicpt xicpt])
disp('_________________________________________________________________________________
____')
```

```
format compact
%======================================================================================
===========
disp('__________________________________________________________________')
disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
disp('_________________________________________________________________________________
_____________________________________________________')
disp('        NODE FEM SOLUTION EXACT SOLUTION       NODE FEM SOLUTION EXACT SOLUTION
NODE FEM SOLUTION EXACT SOLUTION          ')

disp('---------------------------------------------------------------------------------
------------------------------------------------------')
for I=1:3:nel
   % A=[elcentr(I) phicpt(I) xicpt(I)];B=[elcentr(I+1) phicpt(I+1)
xicpt(I+1)];C=[elcentr(I+2) phicpt(I+2) xicpt(I+2)];
    %disp([elcentr(I) phicpt(I) xicpt(I) elcentr(I+1) phicpt(I+1) xicpt(I+1)])
    %disp([elcentr(I+2) phicpt(I+2) xicpt(I+2)])
    %disp([A B C])
    fprintf('\n%5d %18.14f %18.14f %5d %18.14f %18.14f %5d %18.14f %18.14f',elcentr(I),
phicpt(I), xicpt(I), elcentr(I+1), phicpt(I+1), xicpt(I+1), elcentr(I+2), phicpt(I+2),
xicpt(I+2));
end
 fprintf('\n')
disp('---------------------------------------------------------------------------------
------------------------------------------------------')
```

MATLAB Codes-II

(1)quadrilateral_mesh4MOINEX_q9LG.m
(2)polygonal_domain_coordinates_2nd_orderLG.m
(3)nodaladdresses_special_convex_quadrilaterals_trial_2nd_orderLG.m
(4)generate_area_coordinate_over_the_standard_triangle.m
(5)glsampleptsweights.m
(6)D2PoissonEquationQ9MoinEx_MeshgridContour.m

%(1)quadrilateral_mesh4MOINEX_q9LG.m

```
function[]=quadrilateral_mesh4MOINEX_q9LG(n1,n2,n3,nmax,numtri,ndiv,mesh,xlength,ylengt
h)
clf
%(1)=generate 2-D quadrilateral mesh
%for a rectangular shape of domain
%quadrilateral_mesh_q4(xlength,ylength)
%xnode=number of nodes along x-axis
%ynode=number of nodes along y-axis
%xzero=x-coord of bottom left corner
%yzero=y-coord of bottom left corner
%xlength=size of domain alog x-axis
%ylength=size of domain alog y-axis
%quadrilateral_mesh4MOINEX_q4([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,1,2,1,1
,1)
%quadrilateral_mesh4MOINEX_q4([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,4,4,1,1
,1)
%quadrilateral_mesh4MOINEX_q4([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,1
,2,2,1,1)
%quadrilateral_mesh4MOINEX_q4([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,4
,4,2,1,1)
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates(n1,n2,n3,nmax,numt
ri,ndiv,mesh)
%quadrilateral_mesh4MOINEX_q8([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,1
,2,2,1,1)
%quadrilateral_mesh4MOINEX_q8([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,1,2,3,1
,1)
%[nel,nnel]=size(nodes);
```

```
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_orderLG([1;1;1
;1],[2;3;4;5],[3;4;5;2],5,1,2,3)
%quadrilateral_mesh4MOINEX_q9LG([1;1;1;1],[2;3;4;5],[3;4;5;2],5,1,2,3,1,1)
%quadrilateral_mesh4MOINEX_q9LG([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,1,2,1
,1,1)
%quadrilateral_mesh4MOINEX_q9LG([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9
,1,2,2,1,1)
%quadrilateral_mesh4MOINEX_q9LG([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9
,1,2,4,1,1)
[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_orderLG(n1,n2,n
3,nmax,numtri,ndiv,mesh)
[nel,nnel]=size(nodes);
disp([xlength,ylength,nnode,nel,nnel])
%gcoord(i,j),where i->node no. and j->x or y
%___________________________________________________________

%plot the mesh for the generated data
%x and y coordinates
xcoord(:,1)=gcoord(:,1);
ycoord(:,1)=gcoord(:,2);
%extract coordinates for each element
clf
for i=1:nel
for j=1:nnel
x(1,j)=xcoord(nodes(i,j),1);
y(1,j)=ycoord(nodes(i,j),1);
end;%j loop
xvec(1,1:5)=[x(1,1),x(1,2),x(1,3),x(1,4),x(1,1)];
yvec(1,1:5)=[y(1,1),y(1,2),y(1,3),y(1,4),y(1,1)];
axis tight
switch mesh
    case 1
axis([0 xlength 0 ylength])
case 2
axis([0 xlength 0 ylength])
    case 3
axis([0 xlength 0 ylength])
    case 4
axis([-xlength/2 xlength/2 -ylength/2 ylength/2])
end
figure(1)
plot(xvec,yvec);%plot element
hold on;
%place element number
midx=mean(xvec(1,1:4))
midy=mean(yvec(1,1:4))
if ndiv<=2
text(midx+.01,midy-.03,['[',num2str(i),']']);
end
end;%i loop
xlabel('x axis')
ylabel('y axis')
st1='Mesh of ';
st2=num2str(nel);
st3=' nine node ';
st4='quadrilateral';
st5=' elements&';
st6=num2str(nnode);
st7=',nodes'
title([st1,st2,st3,st4,st5,st6,st7])
%put node numbers
disp(nnode)
if ndiv<=2
for jj=1:nnode
text(gcoord(jj,1),gcoord(jj,2),['o',num2str(jj)]);
end
```

```
end
%axis off
if ndiv>=4
for jj=1:nnode
text(gcoord(jj,1),gcoord(jj,2),['o']);
end
end
hold on
figure(1),scatter(gcoord(:,1),gcoord(:,2),'MarkerFaceColor','g')
```

% (2)polygonal_domain_coordinates_2nd_orderLG.m

```
function[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_orderLG
(n1,n2,n3,nmax,numtri,n,mesh)
%n1=node number at(0,0)for  a choosen triangle
%n2=node number at(1,0)for  a choosen triangle
%n3=node number at(0,1)for  a choosen triangle
%eln=6-node triangles with centroid
%spqd=4-node special convex quadrilateral
%n must be even,i.e.n=2,4,6,.......i.e number of divisions
%nmax=one plus the number of segments of the polygon
%nmax=the number of segments of the polygon plus a node interior to the polygon
%numtri=number of T6 triangles in each segment i.e a triangle formed by
%joining the end poits of the segment to the interior point(e.g:the centroid) of the
polygon
%[eln,spqd]=nodaladdresses_special_convex_quadrilaterals_trial(n1=1,n2=2,n3=3,nmax=3,n=
2,4,6,...)
%[eln,spqd]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4
;5;2],5,1,2)
%[eln,spqd]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4
;5;2],5,4,4)
%[eln,spqd]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4
;5;2],5,9,6)
%[eln,spqd]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4
;5;2],5,16,8)
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_order([1;1;1;1
],[2;3;4;5],[3;4;5;2],5,1,2,1)
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_order([1;1;1;1
;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,1,2,2)
%PARVIZ MOIN EXAMPLE
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_order([1;1;1;1
;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,1,2,1)
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_order([1;1;1;1
;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,4,4,1)
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_orderLG([1;1;1
;1],[2;3;4;5],[3;4;5;2],5,1,2,3)
syms U V W xi yi
syms x y
switch mesh
case 1%domain with seven triangles(8-nodes)
x=sym([1/2;1/2;1;  1;1/2;0;  0;0])%for MOIN EXAMPLE
y=sym([1/2;  0;0;1/2;  1;1;1/2;0])%for MOIN EXAMPLE
case 2%square domain with eight triangles(9-nodes)
x=sym([1/2;1/2;1;  1;  1;1/2;0;  0;0])%FOR UNIT SQUARE
y=sym([1/2;  0;0;1/2;  1;  1;1;1/2;0])%FOR UNIT SQUARE

case 3%square domain with four triangles(5-nodes)
 x=sym([1/2;0;1;1;0])
 y=sym([1/2;0;0;1;1])
 case 4%square domain with eight triangles(9-nodes)
%      1    2    3   4   5    6    7   8    9
x=sym([0;   0; 1/2;1/2;1/2;  0;-1/2;-1/2;-1/2])%FOR UNIT SQUARE
y=sym([0;-1/2;-1/2;  0;1/2;1/2; 1/2;   0;-1/2])%FOR UNIT SQUARE

end
[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial_2nd_ord
erLG(n1,n2,n3,nmax,numtri,n);
```

```
[U,V,W]=generate_area_coordinate_over_the_standard_triangle(n);

ss1='number of 6-node triangles with centroid=';
[p1,q1]=size(eln);
disp([ss1 num2str(p1)])
%
eln
%
ss2='number of special convex quadrilaterals elements&nodes per element =';
[nel,nnel]=size(spqd);
disp([ss2 num2str(nel) ','  num2str(nnel)])
%
spqd
%
nnode=max(max(spqd));
ss3='number of nodes of the triangular domain& number of special quadrilaterals=';
disp([ss3 num2str(nnode) ',' num2str(nel)])

xi(1:nnode,1)=zeros(nnode,1);yi(1:nnode,1)=zeros(nnode,1);

nitri=nmax-1;
for itri=1:nitri
disp('vertex nodes of the itri triangle')
[n1(itri,1) n2(itri,1) n3(itri,1)]
x1=x(n1(itri,1),1)
x2=x(n2(itri,1),1)
x3=x(n3(itri,1),1)
%
y1=y(n1(itri,1),1)
y2=y(n2(itri,1),1)
y3=y(n3(itri,1),1)
rrr(:,:,itri)
U'
V'
W'
kk=0;
for ii=1:n+1
    for jj=1:(n+1)-(ii-1)
        kk=kk+1;
        mm=rrr(ii,jj,itri);
        uu=U(kk,1);vv=V(kk,1);ww=W(kk,1);
        xi(mm,1)=x1*ww+x2*uu+x3*vv;
        yi(mm,1)=y1*ww+y2*uu+y3*vv;
    end%for jj
end%for ii
[xi yi]
%add coordinates of centroid
 ne=(n/2)^2;
% stdnode=kk;
 for iii=1+(itri-1)*ne:ne*itri
     %kk=kk+1;
     node1=eln(iii,1)
     node2=eln(iii,2)
     node3=eln(iii,3)
     mm=eln(iii,7)
     xi(mm,1)=(xi(node1,1)+xi(node2,1)+xi(node3,1))/3;
     yi(mm,1)=(yi(node1,1)+yi(node2,1)+yi(node3,1))/3;

 end %for iii
 [xi yi]

end%for itri=1:nitri
for mmm=1:nel
    mmm1=nodes(mmm,1)
    mmm2=nodes(mmm,2)
```

```
    mmm3=nodes(mmm,3)
    mmm4=nodes(mmm,4)
   mmm5=nodes(mmm,5)
   mmm6=nodes(mmm,6)
   mmm7=nodes(mmm,7)
   mmm8=nodes(mmm,8)
   mmm9=nodes(mmm,9)
   xi1=xi(mmm1,1)
   xi2=xi(mmm2,1)
   xi3=xi(mmm3,1)
   xi4=xi(mmm4,1)
   %(xi1+xi2)/2
   %
   yi1=yi(mmm1,1)
   yi2=yi(mmm2,1)
   yi3=yi(mmm3,1)
   yi4=yi(mmm4,1)
   %(yi1+yi2)/2
 xi(mmm5,1)=(xi1+xi2)/2;
 xi(mmm6,1)=(xi2+xi3)/2;
 xi(mmm7,1)=(xi3+xi4)/2;
 xi(mmm8,1)=(xi4+xi1)/2;
 xi(mmm9,1)=(xi1+xi2+xi3+xi4)/4;
 yi(mmm5,1)=(yi1+yi2)/2;
 yi(mmm6,1)=(yi2+yi3)/2;
 yi(mmm7,1)=(yi3+yi4)/2;
 yi(mmm8,1)=(yi4+yi1)/2;
 yi(mmm9,1)=(yi1+yi2+yi3+yi4)/4;
end%for nel
%[xi(18,1) yi(18,1)]

N=(1:nnode)'
[N xi yi]
%
coord(:,1)=(xi(:,1));
coord(:,2)=(yi(:,1));
gcoord(:,1)=double(xi(:,1));
gcoord(:,2)=double(yi(:,1));
%disp(gcoord)
```

(3) % **nodaladdresses_special_convex_quadrilaterals_trial_2nd_orderLG.m**

```
function[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial_2nd_orderLG(n1,n2,n3,nmax,numtri,n)
%n1=node number at(0,0)for  a choosen triangle
%n2=node number at(1,0)for  a choosen triangle
%n3=node number at(0,1)for  a choosen triangle
%eln=6-node triangles with centroid
%spqd=4-node special convex quadrilateral
%n must be even,i.e.n=2,4,6,.......i.e number of divisions
%nmax=one plus the number of segments of the polygon
%nmax=the number of segments of the polygon plus a node interior to the polygon
%numtri=number of T6 triangles in each segment i.e a triangle formed by
%joining the end poits of the segment to the interior point(e.g:the centroid) of the polygon
%[eln,spqd]=nodaladdresses_special_convex_quadrilaterals_trial(n1=1,n2=2,n3=3,nmax=3,n=2,4,6,...)
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4;5;2],5,1,2)
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4;5;2],5,4,4)
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4;5;2],5,9,6)
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1;1],[2;3;4;5],[3;4;5;2],5,16,8)
%PARVIZ MOIN EXAMPLE
```

```
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1
;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,1,2)
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial([1;1;1
;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,4,4)
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial_2nd_or
der([1;1;1;1],[2;3;4;5],[3;4;5;2],5,1,2)
%[eln,spqd,rrr,nodes,nodetel]=nodaladdresses_special_convex_quadrilaterals_trial_2nd_or
derLG([1;1;1;1],[2;3;4;5],[3;4;5;2],5,1,2)
%syms mst_tri x
ne=0;
nitri=nmax-1;
for itri=1:nitri
    elm(1:(n+1)*(n+2)/2,1)=zeros((n+1)*(n+2)/2,1)
elm(1,1)=n1(itri,1)
elm(n+1,1)=n2(itri,1)
elm((n+1)*(n+2)/2,1)=n3(itri,1)
disp('vertex nodes of the itri triangle')
[n1(itri,1) n2(itri,1) n3(itri,1)]
if itri==1
kk=nmax;
for k=2:n
    kk=kk+1
    elm(k,1)=kk
end
disp('base nodes=')
%elm(2:n)
edgen1n2(1:n+1,itri)=elm(1:n+1,1)
end%itri==1
if itri>1
    elm(1:n+1,1)=edgen1n3(1:n+1,itri-1);
end%if itri>1
if itri==1
    lmax=nmax+3*(n-1);
end%if itri==1
if (itri>1)&(itri<nitri)
    lmax=nmax+2*(n-1);
end% if (itri>1)&(itri<nitri)
mmax=nmax;
if itri==1
    mmax=max(max(edgen1n2(1:n+1,1)))
end%f itri==1
disp('right edge nodes')
nni=n+1;hh=1;qq(1,1)=n2(itri,1);
for i=0:(n-2)
    hh=hh+1;
    nni=nni+(n-i);
    elm(nni,1)=(mmax+1)+i;
    qq(hh,1)=(mmax+1)+i;

end
qq(n+1,1)=n3(itri,1);
edgen2n3(1:n+1,itri)=qq;

if itri<nitri
disp('left edge nodes')
nni=1;gg=1;pp(1,1)=n1(itri,1);
for i=0:(n-2)
    gg=gg+1;
    nni=nni+(n-i)+1;
    elm(nni,1)=lmax-i;
    pp(gg,1)=lmax-i;
end
pp(n+1,1)=n3(itri,1);
edgen1n3(1:n+1,itri)=pp
end%if itri<nitri
```

```
%if itri==n
% elm(1:n+1,1)=edgen1n2(1:n+1,1)
%end

if itri==nitri
disp('left edge nodes')
nni=1;gg=1;
for i=0:(n-2)
    gg=gg+1;
    nni=nni+(n-i)+1;
    elm(nni,1)=edgen1n2(gg,1);
end
%pp(n+1,1)=n3(itri,1);
%edgen1n3(1:n+1,itri)=pp
end%if itri==nitri
if itri==nitri
lmax=max(max(edgen2n3(1:n+1,itri)));
end%if itri==nitri




%elm
disp('interior nodes')
nni=1;jj=0;
for i=0:(n-3)
    nni=nni+(n-i)+1;
    for j=1:(n-2-i)
        jj=jj+1;
        nnj=nni+j;
        elm(nnj,1)=lmax+jj;
        [nnj lmax+jj];
    end
end
%disp(elm);
%disp(length(elm));

jj=0;kk=0;
for j=0:n-1
    jj=j+1;
for k=1:(n+1)-j
    kk=kk+1;
    row_nodes(jj,k)=elm(kk,1);
end
end
row_nodes(n+1,1)=n3(itri,1);
%for jj=(n+1):-1:1
%    (row_nodes(jj,:));
%end
%[row_nodes]
rr=row_nodes;
rr
rrr(:,:,itri)=rr;
disp('element computations')
if rem(n,2)==0
N=n+1;

for k=1:2:n-1
N=N-2;
i=k;
for j=1:2:N
    ne=ne+1
```

```
eln(ne,1)=rr(i,j);
eln(ne,2)=rr(i,j+2);
eln(ne,3)=rr(i+2,j);
eln(ne,4)=rr(i,j+1);
eln(ne,5)=rr(i+1,j+1);
eln(ne,6)=rr(i+1,j);
end%j
%me=ne
%N-2
if (N-2)>0
for jj=1:2:N-2
ne=ne+1
eln(ne,1)=rr(i+2,jj+2);
eln(ne,2)=rr(i+2,jj);
eln(ne,3)=rr(i,jj+2);
eln(ne,4)=rr(i+2,jj+1);;
eln(ne,5)=rr(i+1,jj+1);
eln(ne,6)=rr(i+1,jj+2);
end%jj
end%if(N-2)>0
end%k

end% if rem(n,2)==0
ne
%for kk=1:ne
%[eln(kk,1:6)]
%end
%add node numbers for element centroids

nnd=max(max(eln))
if (n>3)
for kkk=1+(itri-1)*numtri:ne
    nnd=nnd+1;
    eln(kkk,7)=nnd;
end
end
if n==2
 for kkk=itri:ne
    nnd=nnd+1;
    eln(kkk,7)=nnd;
 end
end
%for kk=1:ne
%[eln(kk,1:7)]
%end
%to generate special quadrilaterals
%mm=0;

%for iel=1:ne
 %   for jel=1:3
  %  mm=mm+1;
   %     switch jel
   %     case 1
   %       spqd(mm,1:4)=[eln(iel,7) eln(iel,6) eln(iel,1) eln(iel,4)];
    %      nodes(mm,1:4)=spqd(mm,1:4);
     %     nodetel(mm,1:3)=[eln(iel,2) eln(iel,3) eln(iel,1)];
     %    case 2
     %     spqd(mm,1:4)=[eln(iel,7) eln(iel,4) eln(iel,2) eln(iel,5)];
      %    nodes(mm,1:4)=spqd(mm,1:4);
       %   nodetel(mm,1:3)=[eln(iel,3) eln(iel,1) eln(iel,2)];
       %   case 3
      %    spqd(mm,1:4)=[eln(iel,7) eln(iel,5) eln(iel,3) eln(iel,6)];
      %    nodes(mm,1:4)=spqd(mm,1:4);
      %    nodetel(mm,1:3)=[eln(iel,1) eln(iel,2) eln(iel,3)];
      %  end%switch
```

```
      %end
      %end
nmax=max(max(eln));
%nel=mm;
%
%ne
%spqd


end%itri

%to generate special quadrilaterals
mm=0;

for iel=1:ne
    for jel=1:3
    mm=mm+1;
        switch jel
         case 1
          spqd(mm,1:4)=[eln(iel,7) eln(iel,6) eln(iel,1) eln(iel,4)];
          nodes(mm,1:4)=spqd(mm,1:4);
          nodetel(mm,1:3)=[eln(iel,2) eln(iel,3) eln(iel,1)];
          case 2
          spqd(mm,1:4)=[eln(iel,7) eln(iel,4) eln(iel,2) eln(iel,5)];
          nodes(mm,1:4)=spqd(mm,1:4);
          nodetel(mm,1:3)=[eln(iel,3) eln(iel,1) eln(iel,2)];
          case 3
          spqd(mm,1:4)=[eln(iel,7) eln(iel,5) eln(iel,3) eln(iel,6)];
          nodes(mm,1:4)=spqd(mm,1:4);
          nodetel(mm,1:3)=[eln(iel,1) eln(iel,2) eln(iel,3)];
        end%switch
     end%for jel=1:3
   end


for inum=1:nnd
    for jnum=1:nnd
        mdpt(inum,jnum)=0;
    end
end
nd=nnd;
for mmm=1:mm
    mmm1=nodes(mmm,1);
    mmm2=nodes(mmm,2);
    mmm3=nodes(mmm,3);
    mmm4=nodes(mmm,4);
%midpoint side-1 of 4-node special quadrilateral
if((mdpt(mmm1,mmm2)==0)&(mdpt(mmm2,mmm1)==0))
    nd=nd+1;
   mdpt(mmm1,mmm2)=nd;
   mdpt(mmm2,mmm1)=nd;
end%if
%midpoint side-2 of 4-node special quadrilateral
if((mdpt(mmm2,mmm3)==0)&(mdpt(mmm3,mmm2)==0))
    nd=nd+1;
   mdpt(mmm2,mmm3)=nd;
   mdpt(mmm3,mmm2)=nd;
end%if
%midpoint side-3 of 4-node special quadrilateral
if((mdpt(mmm3,mmm4)==0)&(mdpt(mmm4,mmm3)==0))
    nd=nd+1;
   mdpt(mmm3,mmm4)=nd;
   mdpt(mmm4,mmm3)=nd;
end%if
%midpoint side-4 of 4-node special quadrilateral
```

```
if((mdpt(mmm4,mmm1)==0)&(mdpt(mmm1,mmm4)==0))
    nd=nd+1;
   mdpt(mmm4,mmm1)=nd;
   mdpt(mmm1,mmm4)=nd;
end
   nodes(mmm,5)=mdpt(mmm1,mmm2);
   nodes(mmm,6)=mdpt(mmm2,mmm3);
   nodes(mmm,7)=mdpt(mmm3,mmm4);
   nodes(mmm,8)=mdpt(mmm4,mmm1);
   nd=nd+1;
   nodes(mmm,9)=nd;
end
nnode=nd;
nel=mm;
spqd=nodes;
ss1='number of 6-node triangles with centroid=';
[p1,q1]=size(eln);
disp([ss1 num2str(p1)])
%
eln
%
ss2='number of special convex quadrilaterals elements&nodes per element =';
[nel,nnel]=size(spqd);
disp([ss2 num2str(nel) ','  num2str(nnel)])
%
nnode=max(max(spqd));
ss3='number of nodes of the triangular domain& number of special quadrilaterals=';
disp([ss3 num2str(nnode) ',' num2str(nel)])
```

%(4)generate_area_coordinate_over_the_standard_triangle.m

```
function[U,V,W]=generate_area_coordinate_over_the_standard_triangle(n)
syms ui vi wi U V W
kk=0;
for j=1:n+1
    for i=1:(n+1)-(j-1)
        kk=kk+1;
        ui(i,j)=(i-1)/n;
        vi(i,j)=(j-1)/n;
        wi(i,j)=1-ui(i,j)-vi(i,j);
        U(kk,1)=ui(i,j);
        V(kk,1)=vi(i,j);
        W(kk,1)=wi(i,j);
    end
end
%  ui
 % vi
  %wi
 % U'
  %V'
  %W'
 %kk
```

%(5)**glsampleptsweights.m**

```
function [s,wwww]=glsampleptsweights(n)
switch n
    case 1
        s(1)=0;wwww(1)=2;
    case 2
        table=[  .57735026918962576450914878050195,  1.000000000000000000000000000000
                -.57735026918962576450914878050195,  1.000000000000000000000000000000];
s=table(:,1);wwww=table(:,2);
case 3
            table=[  .77459666924148337703585307995650,  .55555555555555555555555555555571
                    -.77459666924148337703585307995650,  .55555555555555555555555555555571
                                   0,                                               8/9];
s=table(:,1);wwww=table(:,2);
case 4
    table=[   .86113631159405257522394648889280,  .34785484513745385737306394922182
             -.86113631159405257522394648889280,  .34785484513745385737306394922182
              .33998104358485626480266575910326,  .65214515486254614262693605077797
             -.33998104358485626480266575910326,  .65214515486254614262693605077797];
```

```
         s=table(:,1);wwww=table(:,2);
     case 5
     table=[                                 0,  .5688888888888888888888888888889
              .9061798459386639927976268782994,  .2369268850561890875142640407202
             -.9061798459386639927976268782994,  .2369268850561890875142640407202
              .5384693101056830910363144207002,  .4786286704993664680412915148356
             -.5384693101056830910363144207002,  .4786286704993664680412915148356];
         s=table(:,1);wwww=table(:,2);
     case 6
     table=[  .9324695142031520278123015544940,  .1713244923791703450402961421731
             -.9324695142031520278123015544940,  .1713244923791703450402961421731
              .2386191860831969086305017216807,  .4679139345726910473898703439895
             -.2386191860831969086305017216807,  .4679139345726910473898703439895
              .6612093864662645136613995950198,  .3607615730481386075698335138373
             -.6612093864662645136613995950198,  .3607615730481386075698335138373];
             s=table(:,1);wwww=table(:,2);
     case 7
      table=[                                 0,  .4179591836734693877551020408163
               .9491079123427585245261896840478,  .1294849661688696932706114326786
              -.9491079123427585245261896840478,  .1294849661688696932706114326786
               .4058451513773971669066064120769,  .3818300505051189449503697754890
              -.4058451513773971669066064120769,  .3818300505051189449503697754890
               .7415311855993944398638647732807,  .2797053914892766679014677714243
              -.7415311855993944398638647732807,  .2797053914892766679014677714243];
                s=table(:,1);wwww=table(:,2);
     case 8
               table=[  .9602898564975362316835608685694,  .1012285362903762591525313543105
                       -.9602898564975362316835608685694,  .1012285362903762591525313543105
                        .7966664774136267395915539364758,  .2223810344533744705443559944278
                       -.7966664774136267395915539364758,  .2223810344533744705443559944278
                        .5255324099163289858177390491892,  .3137066458778872873379622019867
                       -.5255324099163289858177390491892,  .3137066458778872873379622019867
                        .1834346424956498049394761423601,  .3626837833783619829651504492771
                       -.1834346424956498049394761423601,  .3626837833783619829651504492771];

                         s=table(:,1);wwww=table(:,2);
     case 9
         table=[                                   0,    .3302393550012597631645250692869
                    .9681602395076260898355762029036, .8127438836157441197189215811499e-1
                   -.9681602395076260898355762029036, .8127438836157441197189215811499e-1
                    .8360311073266357942994297880697,    .1806481606948574040584720312462
                   -.8360311073266357942994297880697,    .1806481606948574040584720312462
                    .6133714327005903973087020393414,    .2606106964029354623187428694187
                   -.6133714327005903973087020393414,    .2606106964029354623187428694187
                    .3242534234038089290385380146433,    .3123470770400028400686304065846
                   -.3242534234038089290385380146433,    .3123470770400028400686304065846];

                             s=table(:,1);wwww=table(:,2);
   case 10
       table=[   -.1488743389816312108848260011297,    .2955242247147528701738929946513
                  .1488743389816312108848260011297,    .2955242247147528701738929946513
                 -.4333953941292471907992659431657,    .2692667193099963550912269215693
                  .4333953941292471907992659431657,    .2692667193099963550912269215693
                 -.6794095682990244062343273651148,    .2190863625159820439955349342295
                  .6794095682990244062343273651148,    .2190863625159820439955349342295
                 -.8650633666889845107320966884235,    .1494513491505805931457763396548
                  .8650633666889845107320966884235,    .1494513491505805931457763396548
                 -.9739065285171717200779640120844, .6667134430868813759356880989621e-1
                  .9739065285171717200779640120844, .6667134430868813759356880989621e-1];

   s=table(:,1);wwww=table(:,2);
end
```

%(6)D2PoissonEquationQ9MoinEx_MeshgridContour.m

```
function[]=D2PoissonEquationQ9MoinEx_MeshgridContour(n1,n2,n3,nmax,numtri,ndiv,mesh,fcn
,ng)

%note that input vlues of X and Y must be symbolic constants
%for the example triangle input for X is sym([-1/2 1/2 0])
%for the example triangle input for Y is sym([0 0 sqrt(3/4)])
%LaplaceEquationQ4twoD(3,sym([-1/2 1/2 0]),sym([0 0 sqrt(3/4)]))
%ndiv=2,4,6,8,......
%polygonal_domain_coordinates([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,1,2)
%polygonal_domain_coordinates([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8,4,4)
%D2LaplaceEquationQ4MoinExautomeshgen(n1,n2,n3,nmax,numtri,ndiv)
```

```
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8
,1,2,1)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8;2],8
,4,4,1)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;
9;2],9,1,2,2)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;
9;2],9,4,4,2)
%quadrilateral_mesh4MOINEX_q4(n1,n2,n3,nmax,numtri,ndiv,mesh,xlength,ylength)([1;1;1;1;
1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,1,2,2,1,1)
%D2POISSONEQUATION_NODALINTERPOLATION_VALUES(n1,n2,n3,nmax,numtri,ndiv,mesh)([1;1;1;1;1
;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;9;2],9,1,2,2)
%D2LaplaceEquationQ4MoinExautomeshgen([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6;7;8;
9;2],9,100,20,2)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,4,4,1)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,9,6,1)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,16,8,1)
%D2PoissonEquationQ4MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,25,10,1)
%D2PoissonEquationQ8MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2)
%D2PoissonEquationQ8MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;8
;2],8,1,2,1)
%%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1],[2;3;4;5;6;7;8],[3;4;5;6;7;
8;2],8,1,2,1,1,10)
%D2PoissonEquationQ9MoinEx_MeshgridContour([1;1;1;1;1;1;1;1],[2;3;4;5;6;7;8;9],[3;4;5;6
;7;8;9;2],9,1,2,2,1,10)

syms coord
%[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates(n1,n2,n3,nmax,numt
ri,ndiv,mesh)
%nnel=4;
[coord,gcoord,nodes,nodetel,nnode,nel]=polygonal_domain_coordinates_2nd_orderLG(n1,n2,n
3,nmax,numtri,ndiv,mesh)
nnel=9;
ndof=1;
%nc=(ndiv/2)^2;
%nnode=(ndiv+1)*(ndiv+2)/2+nc;
%nel=3*nc;
sdof=nnode*ndof;
ff=(zeros(sdof,1));ss=(zeros(sdof,sdof));
clf
clc
%nnode=17,nel=12,nnel=4,ndof=1
%>>LaplaceEquationQuad4twodimension(12,17,4,1)
%
%Ex1:nnode=41,nel=36,,nnel=4,nodf=1
%>>LaplaceEquationQuad4twodimensionEx1(36,41,4,1)
%>>improvedLaplaceEquationQuad4twodimensionEx1_explicit(36,41,4,1)
%Ex2:nnode=83,nel=69,,nnel=4,nodf=1
%>>improvedLaplaceEquationQuad4twodimensionEx2_explicit(69,83,4,1)#
%>>improvedLaplaceEquationQuad4twodimensionEx2_explicitfnmesh(69,83,4,1)#
%improvedLaplaceEquationQuad4twodimensionEx2_explicitvfnmesh(72,87,4,1)#new
```

```
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=3,nnode=7,nnel=4,ndof=1)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel,nnode,nnel=4,ndof=1,quadt
ype=0/3,mesh=1,2,3...)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=12,nnode=19,nnel=4,ndof=1
,quadtype=0/3,mesh=3)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=27,nnode=37,nnel=4,ndof=1
,quadtype=0/3,mesh=4)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=48,nnode=61,nnel=4,ndof=1
,quadtype=0/3,mesh=5)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(nel=75,nnode=91,nnel=4,ndof=1
,quadtype=0/3,mesh=6)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(108,127,4,1,3,7)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(147,169,4,1,3,8)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(192,217,4,1,3,9)
%improvedLaplaceEquationQuad4twodimensionEx3_explicitmesh(243,271,4,1,3,10)
disp([nel nnode nnel ndof])
format long g
for i=1:nel
N(i,1)=i;
end
for i=1:nel
NN(i,1)=i;
end
    %[coord,gcoord]=coordinate_rtisoscelestriangle00_h0_hh(ndiv);
    %[nodetel,nodes]=nodaladdresses4special_convex_quadrilaterals(ndiv)
    %
    %bcdof=[2;5;3]
   %boundary conditions-1
   switch mesh
       case 1
    nnn=0;
    for nn=1:nnode
        xnn=gcoord(nn,1);ynn=gcoord(nn,2);
        if (xnn==0)&((ynn>=0)&(ynn<=1))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
    %boundary conditions-2
     for nn=1:nnode
        xnn=gcoord(nn,1);ynn=gcoord(nn,2);
        if (ynn==0)&((xnn>=0)&(xnn<=1))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
    %boundary conditions-3
    for nn=1:nnode
        xnn=gcoord(nn,1);ynn=gcoord(nn,2);
        if (ynn==1)&((xnn>=0)&(xnn<=1/2))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
   %boundary conditions-4
   for nn=1:nnode
      xnn=gcoord(nn,1);ynn=gcoord(nn,2);
        if (xnn==1)&((ynn>=0)&(ynn<=1/2))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
```

```
  %boundary conditions-5
  for nn=1:nnode
      xnn=coord(nn,1);ynn=coord(nn,2);
        if ((xnn+ynn)==3/2)
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=double((sin(pi*xnn))*(sin(pi*ynn)))
        end
  end
    bcdof
    mm=length(bcdof);

format long g
%analytical solution

 xi=(zeros(nnode,1));
for m=1:nnode
    xm=(gcoord(m,1));ym=(gcoord(m,2));
    xi(m,1)=sin(pi*xm)*sin(pi*ym);
end



case 2
      nnn=0;
    for nn=1:nnode
        xnn=coord(nn,1);ynn=gcoord(nn,2);
        if (xnn==0)&((ynn>=0)&(ynn<=1))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
    %boundary conditions-2
     for nn=1:nnode
        xnn=gcoord(nn,1);ynn=coord(nn,2);
        if (ynn==0)&((xnn>=0)&(xnn<=1))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
    %boundary conditions-3
    for nn=1:nnode
        xnn=gcoord(nn,1);ynn=coord(nn,2);
        if (ynn==1)&((xnn>=0)&(xnn<=1))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end
   %boundary conditions-4
   for nn=1:nnode
      xnn=coord(nn,1);ynn=gcoord(nn,2);
        if (xnn==1)&((ynn>=0)&(ynn<=1))
            nnn=nnn+1;
            bcdof(nnn,1)=nn;
            bcval(nnn,1)=0;
        end
    end


%end
    bcdof
    mm=length(bcdof);
```

```
format long g
%analytical solution

 xi=(zeros(nnode,1));
for m=1:nnode
    xm=(gcoord(m,1));ym=(gcoord(m,2));
    xi(m,1)=sin(pi*xm)*sin(pi*ym);
end


   end %switch
 for L=1:nel
   for M=1:3
      LM=nodetel(L,M);
      xx(L,M)=gcoord(LM,1);
      yy(L,M)=gcoord(LM,2);
    end
  end
%_________________________________________________________________
%ng=10
 [sp,wt]=glsampleptsweights(ng)
table2=[N xx yy];
%disp([xx yy])
%______________________________________________________________________
```

```
intJdn1dn1uvrs =[vpa(sym('  .68800369699992917040751928889')) vpa(sym(' .63026863732902919654693031748'));...
vpa(sym(' .63026863732902919654693031748')) vpa(sym('  .68800369699992917040751928889'))];

intJdn1dn2uvrs =[vpa(sym('  .49469766679465431322423128849e-1')) vpa(sym(' -.92492499790138905674372588873e-1'));...
vpa(sym('  .74174166876527760992294077793e-1')) vpa(sym(' -.58041502241256898516961218830e-1'))];

intJdn1dn3uvrs =[vpa(sym(' -.27639187071864196478244612770e-1')) vpa(sym(' -.36406227705135716084660539928e-1'));...
vpa(sym(' -.36406227705135716084660539928e-1')) vpa(sym(' -.27639187071864196478244612770e-1'))];

intJdn1dn4uvrs =[vpa(sym(' -.58041502241256898516961218830e-1')) vpa(sym('  .74174166876527760992294077793e-1'));...
vpa(sym(' -.92492499790138905674372588873e-1')) vpa(sym('  .49469766679465431322423128849e-1'))];

intJdn1dn5uvrs =[vpa(sym(' -.42034020155248898256147959993')) vpa(sym('  .23835534506556861836117847003'));...
vpa(sym(' -.42831132160109804830548819664')) vpa(sym('  .10203781293970562330870803320'))];

intJdn1dn6uvrs =[vpa(sym(' .12626870005164588340746772853')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .15224655673422927054938212414'))];

intJdn1dn7uvrs =[vpa(sym(' .15224655673422927054938212414')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .12626870005164588340746772853'))];

intJdn1dn8uvrs =[vpa(sym('  .10203781293970562330870803320')) vpa(sym(' -.42831132160109804830548819664'));...
vpa(sym('  .23835534506556861836117847003')) vpa(sym(' -.42034020155248898256147959993'))];

intJdn1dn9uvrs =[vpa(sym(' -.61200564253936530143881487716')) vpa(sym(' -.72289148243880098858257938847'));...
vpa(sym(' -.72289148243880098858257938847')) vpa(sym(' -.61200564253936530143881487716'))];

intJdn2dn1uvrs =[vpa(sym('  .49469766679465431322423128849e-1')) vpa(sym('  .74174166876527760992294077793e-1'));...
vpa(sym(' -.92492499790138905674372588873e-1')) vpa(sym(' -.58041502241256898516961218830e-1'))];

intJdn2dn2uvrs =[vpa(sym('  .24822242723570359859415910165')) vpa(sym(' -.13943577781062161645243966287'));...
vpa(sym(' -.13943577781062161645243966287')) vpa(sym('  .22525137401606776447455758469'))];

intJdn2dn3uvrs =[vpa(sym('  -.93683290648218768462051538669e-1')) vpa(sym(' -.82852283681797324065968279846e-1'));...
vpa(sym('  .83814382984869342600698386820e-1')) vpa(sym('   .37904800898126436475032582264e-1'))];

intJdn2dn4uvrs =[vpa(sym('   .62925700025302140451751960e-3')) vpa(sym(' .26100052047122979851449223466e-1'));...
vpa(sym(' .26100052047122979851449223466e-1')) vpa(sym('   .62925700025302140451751960e-3'))];

intJdn2dn5uvrs =[vpa(sym('   -.46868216752790132335135919662')) vpa(sym('  -.34646194452109534302782806358'));...
vpa(sym('   .32020472214557132363883860309')) vpa(sym(' .85836729209541381287229242888e-1'))];

intJdn2dn6uvrs =[vpa(sym('   .20486711030201898764584585532')) vpa(sym('  .25718774442435573105221634722'));...
vpa(sym(' -.40947892224231093561445031945')) vpa(sym(' -.28073862677651621463393803502'))];

intJdn2dn7uvrs =[vpa(sym(' .50885917641966498462888866644e-1')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' -.17609016019421764322513358 0e-2'))];
```

```
intJdn2dn8uvrs =[vpa(sym(' -.41736856619169902011341267580e-1')) vpa(sym(' -.94765689307712237483661393070e-1'));...
vpa(sym(' -.94765689307712237483661393070e-1')) vpa(sym('  .33881015844374607003400451610e-1'))];

intJdn2dn9uvrs =[vpa(sym('   .50027835935882456394917533200e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym(' -.42962146348647921061586792260e-1'))];

intJdn3dn1uvrs =[vpa(sym(' -.27639187071864196478244612700e-1')) vpa(sym(' -.36406227705135716084660539280e-1'));...
vpa(sym(' -.36406227705135716084660539280e-1')) vpa(sym(' -.27639187071864196478244612700e-1'))];

intJdn3dn2uvrs =[vpa(sym('  -.93683290648218768462051538690e-1')) vpa(sym('  .83814382984869342600698386820e-1'));...
vpa(sym(' -.82852283681797324065968279846e-1')) vpa(sym('   .37904800898126436475032582640e-1'))];

intJdn3dn3uvrs =[vpa(sym('  .33203856905137126709551224250')) vpa(sym(' .29812794600013651335576030277 1'));...
vpa(sym(' .29812794600013651335576030277 1')) vpa(sym('  .33203856905137126709551224250'))];

intJdn3dn4uvrs =[vpa(sym('   .37904800898126436475032582640e-1')) vpa(sym(' -.82852283681797324065968279846e-1'));...
vpa(sym('  .83814382984869342600698386820e-1')) vpa(sym('  -.93683290648218768462051538690e-1'))];

intJdn3dn5uvrs =[vpa(sym('   .15710844821544667058002407565')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym(' .43059644229262196507731226970e-1'))];

intJdn3dn6uvrs =[vpa(sym('  .14159217838663875388665351332')) vpa(sym(' -.35360191630747700837476451283'));...
vpa(sym('  .31306475035918965829190215383')) vpa(sym(' -.34086853405320051524855521401'))];

intJdn3dn7uvrs =[vpa(sym(' -.34086853405320051524855521401')) vpa(sym('  .31306475035918965829190215383'));...
vpa(sym(' -.35360191630747700837476451283')) vpa(sym('  .14159217838663875388665351332'))];

intJdn3dn8uvrs =[vpa(sym(' .43059644229262196507731226970e-1')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym('   .15710844821544667058002407565'))];

intJdn3dn9uvrs =[vpa(sym('  -.24951262900756184435610227570')) vpa(sym(' -.48105764260800696388913166479'));...
vpa(sym(' -.48105764260800696388913166479')) vpa(sym('  -.24951262900756184435610227570'))];

intJdn4dn1uvrs =[vpa(sym(' -.58041502241256898516961218300e-1')) vpa(sym(' -.92492499790138905674372588730e-1'));...
vpa(sym('  .74174166876527760992294077930e-1')) vpa(sym('  .49469766679465431322423128490e-1'))];

intJdn4dn2uvrs =[vpa(sym('   .62925700025302140451751960e-3')) vpa(sym(' .26100052047122979851449223460e-1'));...
vpa(sym(' .26100052047122979851449223460e-1')) vpa(sym('   .62925700025302140451751960e-3'))];

intJdn4dn3uvrs =[vpa(sym('   .37904800898126436475032582640e-1')) vpa(sym('  .83814382984869342600698386820e-1'));...
vpa(sym(' -.82852283681797324065968279846e-1')) vpa(sym('  -.93683290648218768462051538690e-1'))];

intJdn4dn4uvrs =[vpa(sym('  .22525137401606776447455758469')) vpa(sym(' -.13943577781062161645243966287'));...
vpa(sym(' -.13943577781062161645243966287')) vpa(sym('  .24822242723570359859415910165'))];

intJdn4dn5uvrs =[vpa(sym('  .33881015844374607003400451610e-1')) vpa(sym(' -.94765689307712237483661393070e-1'));...
vpa(sym(' -.94765689307712237483661393070e-1')) vpa(sym(' -.41736856619169902011341267580e-1'))];

intJdn4dn6uvrs =[vpa(sym(' -.17609016019421764322513358000e-2')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' .50885917641966498462888866440e-1'))];

intJdn4dn7uvrs =[vpa(sym(' -.28073862677651621463393803502')) vpa(sym(' -.40947892224231093561445031945'));...
vpa(sym('  .25718774442435573105221634722')) vpa(sym('   .20486711030201898764584585320'))];

intJdn4dn8uvrs =[vpa(sym(' .85836729209541381287229242880e-1')) vpa(sym('   .32020472214557132363883860309'));...
vpa(sym('  -.34646194452109534302782806358')) vpa(sym('   -.46868216752790132335135919620'))];

intJdn4dn9uvrs =[vpa(sym(' -.42962146348647921061586792260e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym('   .50027835935882456394917533200e-1'))];

intJdn5dn1uvrs =[vpa(sym(' -.42034020155248898256147959930')) vpa(sym(' -.42831132160109804830548819640'));...
vpa(sym('  .23835534506556861836117847030')) vpa(sym('  .10203781293970562330870803200'))];

intJdn5dn2uvrs =[vpa(sym('   -.46868216752790132335135919620')) vpa(sym('   .32020472214557132363883860309'));...
vpa(sym('  -.34646194452109534302782806358')) vpa(sym(' .85836729209541381287229242880e-1'))];

intJdn5dn3uvrs =[vpa(sym('   .15710844821544667058002407565')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym(' .43059644229262196507731226970e-1'))];

intJdn5dn4uvrs =[vpa(sym('  .33881015844374607003400451610e-1')) vpa(sym(' -.94765689307712237483661393070e-1'));...
vpa(sym(' -.94765689307712237483661393070e-1')) vpa(sym(' -.41736856619169902011341267580e-1'))];

intJdn5dn5uvrs =[vpa(sym(' 1.0663384833344564856750376651')) vpa(sym('  .56556859863119073241392390190'));...
vpa(sym('  .56556859863119073241392390190')) vpa(sym(' 1.1211333295329756651950424278'))];

intJdn5dn6uvrs =[vpa(sym('  -.58306188548401601151324941690')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym(' -.22579554943131879593427777753'))];

intJdn5dn7uvrs =[vpa(sym('   -.16767681050348731002396257420')) vpa(sym(' -.17096478113284630633060979200e-2'));...
vpa(sym(' -.17096478113284630633060979200e-2')) vpa(sym('   .17132457681637017904834081102'))];

intJdn5dn8uvrs =[vpa(sym(' .90271486053820367920675971200e-1')) vpa(sym('   .41024643803333829631989611840'));...
vpa(sym('   .41024643803333829631989611840')) vpa(sym(' .90271486053820367920675971200e-1'))];
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intJdn5dn9uvrs =[vpa(sym('   .29216163161979549627091262 30')) vpa(sym('  -.358160408697768775645395583 6'));...
vpa(sym('  -.35816040869776877564539558 36')) vpa(sym(' -1.3461311727311867153221086666'))];

intJdn6dn1uvrs =[vpa(sym(' .12626870005164588340746772853')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .15224655673422927054938212414'))];

intJdn6dn2uvrs =[vpa(sym('   .20486711030201898764584585 32')) vpa(sym(' -.40947892224231093561445031945'));...
vpa(sym('  .25718774442435573105221634722')) vpa(sym(' -.28073862677651621463393803502'))];

intJdn6dn3uvrs =[vpa(sym('  .14159217838663875388665351332')) vpa(sym('  .31306475035918965829190215383'));...
vpa(sym(' -.35360191630747700837476451283')) vpa(sym(' -.34086853405320051524855521401'))];

intJdn6dn4uvrs =[vpa(sym(' -.17609016019421764322513358 0e-2')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' .50885917641966498462888866 44e-1'))];

intJdn6dn5uvrs =[vpa(sym('  -.58306188548401601151324941 69')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym(' -.22579554943131879593427777753'))];

intJdn6dn6uvrs =[vpa(sym(' 1.5998510858095308546375928228')) vpa(sym(' .24469327225090109904911663174'));...
vpa(sym(' .24469327225090109904911663174')) vpa(sym('  .64675514174853057100283264 58'))];

intJdn6dn7uvrs =[vpa(sym(' -.74612686016266341101136360 5e-1')) vpa(sym('   .42496611860422613701397774221'));...
vpa(sym('   .42496611860422613701397774221')) vpa(sym(' -.74612686016266341101136360 5e-1'))];

intJdn6dn8uvrs =[vpa(sym('   .17132457681637017904834081102')) vpa(sym(' -.17096478113284630633060979 2e-2'));...
vpa(sym(' -.17096478113284630633060979 2e-2')) vpa(sym('   -.16767681050348731002396257 42'))];

intJdn6dn9uvrs =[vpa(sym('   -1.5844681782639801295792636156')) vpa(sym(' -.94301127139103615215193728 96e-1'));...
vpa(sym(' -.94301127139103615215193728 96e-1')) vpa(sym('    .23980459065606283692676632 48'))];

intJdn7dn1uvrs =[vpa(sym(' .15224655673422927054938212414')) vpa(sym(' .16865169113202404137334892156'));...
vpa(sym(' .16865169113202404137334892156')) vpa(sym(' .12626870005164588340746772853'))];

intJdn7dn2uvrs =[vpa(sym(' .50885917641966498462888866 44e-1')) vpa(sym('  -.10335794828229434487750587371'));...
vpa(sym('  -.10335794828229434487750587371')) vpa(sym(' -.17609016019421764322513358 0e-2'))];

intJdn7dn3uvrs =[vpa(sym(' -.34086853405320051524855521401')) vpa(sym(' -.35360191630747700837476451283'));...
vpa(sym('  .31306475035918965829190215383')) vpa(sym('  .14159217838663875388665351332'))];

intJdn7dn4uvrs =[vpa(sym(' -.28073862677651621463393803502')) vpa(sym('  .25718774442435573105221634722'));...
vpa(sym(' -.40947892224231093561445031945')) vpa(sym('   .20486711030201898764584585 32'))];

intJdn7dn5uvrs =[vpa(sym('   -.16767681050348731002396257 42')) vpa(sym(' -.17096478113284630633060979 2e-2'));...
vpa(sym(' -.17096478113284630633060979 2e-2')) vpa(sym('   .17132457681637017904834081102'))];

intJdn7dn6uvrs =[vpa(sym(' -.74612686016266341101136360 5e-1')) vpa(sym('   .42496611860422613701397774221'));...
vpa(sym('   .42496611860422613701397774221')) vpa(sym(' -.74612686016266341101136360 5e-1'))];

intJdn7dn7uvrs =[vpa(sym('  .64675514174853057100283264 58')) vpa(sym(' .24469327225090109904911663174'));...
vpa(sym(' .24469327225090109904911663174')) vpa(sym(' 1.5998510858095308546375928228'))];

intJdn7dn8uvrs =[vpa(sym(' -.22579554943131879593427777753')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym('  -.58306188548401601151324941 69'))];

intJdn7dn9uvrs =[vpa(sym('    .23980459065606283692676632 48')) vpa(sym(' -.94301127139103615215193728 96e-1'));...
vpa(sym(' -.94301127139103615215193728 96e-1')) vpa(sym('   -1.5844681782639801295792636156'))];

intJdn8dn1uvrs =[vpa(sym('  .10203781293970562330870803 20')) vpa(sym('  .23835534506556861836117847 03'));...
vpa(sym(' -.42831132160109804830548819 64')) vpa(sym(' -.42034020155248898256147959 93'))];

intJdn8dn2uvrs =[vpa(sym(' -.41736856619169902011341267 58e-1')) vpa(sym(' -.94765689307712237483661393 07e-1'));...
vpa(sym(' -.94765689307712237483661393 07e-1')) vpa(sym('  .33881015844374607003400451 61e-1'))];

intJdn8dn3uvrs =[vpa(sym(' .43059644229262196507731226 97e-1')) vpa(sym('   .12945549547911074908308207668'));...
vpa(sym('   .12945549547911074908308207668')) vpa(sym('   .15710844821544667058002407565'))];

intJdn8dn4uvrs =[vpa(sym(' .85836729209541381287229242 88e-1')) vpa(sym('  -.34646194452109534302782806358'));...
vpa(sym('   .32020472214557132363883860309')) vpa(sym('   -.46868216752790132335135919 62'))];

intJdn8dn5uvrs =[vpa(sym(' .90271486053820367920675971 2e-1')) vpa(sym('   .41024643803333829631989611 84'));...
vpa(sym('   .41024643803333829631989611 84')) vpa(sym(' .90271486053820367920675971 2e-1'))];

intJdn8dn6uvrs =[vpa(sym('   .17132457681637017904834081102')) vpa(sym(' -.17096478113284630633060979 2e-2'));...
vpa(sym(' -.17096478113284630633060979 2e-2')) vpa(sym('   -.16767681050348731002396257 42'))];

intJdn8dn7uvrs =[vpa(sym(' -.22579554943131879593427777753')) vpa(sym(' -.54252818687130357695788942932'));...
vpa(sym(' -.54252818687130357695788942932')) vpa(sym('  -.58306188548401601151324941 69'))];

intJdn8dn8uvrs =[vpa(sym(' 1.1211333295329756651950424278')) vpa(sym('  .56556859863119073241392390 19'));...
vpa(sym('  .56556859863119073241392390 19')) vpa(sym(' 1.0663384833344564856750376651'))];

intJdn8dn9uvrs =[vpa(sym(' -1.3461311727311867153221086666')) vpa(sym('  -.35816040869776877564539558 36'));...
vpa(sym('  -.35816040869776877564539558 36')) vpa(sym('   .29216163161979549627091262 30'))];
```

```
intJdn9dn1uvrs =[vpa(sym(' -.6120056425393653014388148716')) vpa(sym(' -.7228914824388009885825793847'));...
vpa(sym(' -.7228914824388009885825793847')) vpa(sym(' -.6120056425393653014388148716'))];

intJdn9dn2uvrs =[vpa(sym('   .500278359358824563949175332e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym(' -.4296214634864792106158679226e-1'))];

intJdn9dn3uvrs =[vpa(sym('  -.2495126290075618443561022757')) vpa(sym(' -.4810576426080069638891316479'));...
vpa(sym(' -.4810576426080069638891316479')) vpa(sym('  -.2495126290075618443561022757'))];

intJdn9dn4uvrs =[vpa(sym(' -.4296214634864792106158679226e-1')) vpa(sym('    .40941168025551439401144362445'));...
vpa(sym('    .40941168025551439401144362445')) vpa(sym('   .500278359358824563949175332e-1'))];

intJdn9dn5uvrs =[vpa(sym('   .2921616316197954962709126230')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym(' -1.3461311727311867153221086666'))];

intJdn9dn6uvrs =[vpa(sym('   -1.5844681782639801295792636156')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('     .2398045906560628369267663248'))];

intJdn9dn7uvrs =[vpa(sym('     .2398045906560628369267663248')) vpa(sym(' -.9430112713910361521519372896e-1'));...
vpa(sym(' -.9430112713910361521519372896e-1')) vpa(sym('   -1.5844681782639801295792636156'))];

intJdn9dn8uvrs =[vpa(sym(' -1.3461311727311867153221086666')) vpa(sym('  -.3581604086977687756453955836'));...
vpa(sym('  -.3581604086977687756453955836')) vpa(sym('   .2921616316197954962709126230'))];

intJdn9dn9uvrs =[vpa(sym(' 3.2530857106790011221652797406')) vpa(sym(' 1.2900488362095239461700024257'));...
vpa(sym(' 1.2900488362095239461700024257')) vpa(sym(' 3.2530857106790011221652797406'))];


%integrals of products of glbal derivatives
intJdndn=double([intJdn1dn1uvrs intJdn1dn2uvrs intJdn1dn3uvrs intJdn1dn4uvrs intJdn1dn5uvrs intJdn1dn6uvrs intJdn1dn7uvrs intJdn1dn8uvrs intJdn1dn9uvrs ;...
          intJdn2dn1uvrs intJdn2dn2uvrs intJdn2dn3uvrs intJdn2dn4uvrs intJdn2dn5uvrs intJdn2dn6uvrs intJdn2dn7uvrs intJdn2dn8uvrs intJdn2dn9uvrs;...
          intJdn3dn1uvrs intJdn3dn2uvrs intJdn3dn3uvrs intJdn3dn4uvrs intJdn3dn5uvrs intJdn3dn6uvrs intJdn3dn7uvrs intJdn3dn8uvrs intJdn3dn9uvrs;...
          intJdn4dn1uvrs intJdn4dn2uvrs intJdn4dn3uvrs intJdn4dn4uvrs intJdn4dn5uvrs intJdn4dn6uvrs intJdn4dn7uvrs intJdn4dn8uvrs intJdn4dn9uvrs;...
          intJdn5dn1uvrs intJdn5dn2uvrs intJdn5dn3uvrs intJdn5dn4uvrs intJdn5dn5uvrs intJdn5dn6uvrs intJdn5dn7uvrs intJdn5dn8uvrs intJdn5dn9uvrs;...
          intJdn6dn1uvrs intJdn6dn2uvrs intJdn6dn3uvrs intJdn6dn4uvrs intJdn6dn5uvrs intJdn6dn6uvrs intJdn6dn7uvrs intJdn6dn8uvrs intJdn6dn9uvrs;...
          intJdn7dn1uvrs intJdn7dn2uvrs intJdn7dn3uvrs intJdn7dn4uvrs intJdn7dn5uvrs intJdn7dn6uvrs intJdn7dn7uvrs intJdn7dn8uvrs intJdn7dn9uvrs;...
          intJdn8dn1uvrs intJdn8dn2uvrs intJdn8dn3uvrs intJdn8dn4uvrs intJdn8dn5uvrs intJdn8dn6uvrs intJdn8dn7uvrs intJdn8dn8uvrs intJdn8dn9uvrs;...
          intJdn9dn1uvrs intJdn9dn2uvrs intJdn9dn3uvrs intJdn9dn4uvrs intJdn9dn5uvrs intJdn9dn6uvrs intJdn9dn7uvrs intJdn9dn8uvrs intJdn9dn9uvrs]);


%_________________________________________________________________________


%
for iel=1:nel
index=zeros(nnel*ndof,1);

X=xx(iel,1:3);
Y=yy(iel,1:3);
%disp([X Y])
xa=X(1,1);
xb=X(1,2);
xc=X(1,3);
ya=Y(1,1);
yb=Y(1,2);
yc=Y(1,3);
bta=yb-yc;btb=yc-ya;
gma=xc-xb;gmb=xa-xc;
delabc=gmb*bta-gma*btb;
G=[bta btb;gma gmb]/delabc;
GT=[bta gma;btb gmb]/delabc;
Q=GT*G;
sk(1:9,1:9)=(zeros(9,9));
for i=1:9
 for j=i:9
 sk(i,j)=(delabc*sum(sum(Q.*(intJdndn(2*i-1:2*i,2*j-1:2*j)))));
 sk(j,i)=sk(i,j);
 end
end
%f =[5/144;1/24;7/144;1/24]*(2*delabc);

 xe(1,1)=(xa+xb+xc)/3;
 xe(2,1)=(xa+xc)/2;
 xe(3,1)=xa;
 xe(4,1)=(xa+xb)/2;
 %
 ye(1,1)=(ya+yb+yc)/3;
```

```
 ye(2,1)=(ya+yc)/2;
 ye(3,1)=ya;
 ye(4,1)=(ya+yb)/2;
 %
 [sp,wt]=glsampleptsweights(ng);
 %for j=1:4
 %    qe(j,1)=(2*pi^2)*sin(pi*xe(j,1))*sin(pi*ye(j,1));
 %end
 %II =([  1/72, 7/864, 1/216, 7/864;...
 %     7/864,  1/54,  1/96, 1/216;...
  %    1/216,  1/96, 5/216,  1/96;...
   %   7/864, 1/216,  1/96,  1/54]);
 %f=(2*delabc)*(II*qe);
 %+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
  xe1=xe(1,1);xe2=xe(2,1);xe3=xe(3,1);xe4=xe(4,1);
 ye1=ye(1,1);ye2=ye(2,1);ye3=ye(3,1);ye4=ye(4,1);
 f(1:9,1)=zeros(9,1)
 for i=1:ng
     si=sp(i,1);wi=wt(i,1);
     for j=1:ng
         sj=sp(j,1);wj=wt(j,1);
         n1ij=(si^2-si)*(sj^2-sj)/4;
         n2ij=(si^2+si)*(sj^2-sj)/4;
         n3ij=(si^2+si)*(sj^2+sj)/4;
         n4ij=(si^2-si)*(sj^2+sj)/4;
         n5ij=(1-si^2)*(sj^2-sj)/2;
         n6ij=(si^2+si)*(1-sj^2)/2;
         n7ij=(1-si^2)*(sj^2+sj)/2;
         n8ij=(si^2-si)*(1-sj^2)/2;
         n9ij=(1-si^2)*(1-sj^2);

        %___________________________________
         N1ij=(((1-si)*(1-sj)))/4;
         N2ij=(((1+si)*(1-sj)))/4;
         N3ij=(((1+si)*(1+sj)))/4;
         N4ij=(((1-si)*(1+sj)))/4;
         xeij=xe1*N1ij+xe2*N2ij+xe3*N3ij+xe4*N4ij;
         yeij=ye1*N1ij+ye2*N2ij+ye3*N3ij+ye4*N4ij;
         %
         %
        fcnxyij=fcnxy(fcn,xeij,yeij);
        f1i=n1ij*fcnxyij*(4+si+sj)/96;
        f2i=n2ij*fcnxyij*(4+si+sj)/96;
        f3i=n3ij*fcnxyij*(4+si+sj)/96;
        f4i=n4ij*fcnxyij*(4+si+sj)/96;
        f5i=n5ij*fcnxyij*(4+si+sj)/96;
        f6i=n6ij*fcnxyij*(4+si+sj)/96;
        f7i=n7ij*fcnxyij*(4+si+sj)/96;
        f8i=n8ij*fcnxyij*(4+si+sj)/96;
        f9i=n9ij*fcnxyij*(4+si+sj)/96;
       % f1i=n1ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f2i=n2ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f3i=n3ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f4i=n4ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f5i=n5ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f6i=n6ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f7i=n7ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f8i=n8ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
        %f9i=n9ij*(2*pi^2)*sin(pi*xeij)*sin(pi*yeij)*(4+si+sj)/96;
%---------------------------------------------------------------
          f(1,1)=f(1,1)+f1i*wi*wj;
          f(2,1)=f(2,1)+f2i*wi*wj;
          f(3,1)=f(3,1)+f3i*wi*wj;
          f(4,1)=f(4,1)+f4i*wi*wj;
          f(5,1)=f(5,1)+f5i*wi*wj;
          f(6,1)=f(6,1)+f6i*wi*wj;
```

```
          f(7,1)=f(7,1)+f7i*wi*wj;
          f(8,1)=f(8,1)+f8i*wi*wj;
          f(9,1)=f(9,1)+f9i*wi*wj;
      end
  end
 f=(delabc)*f;

 %+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
%______________________________________
 edof=nnel*ndof;
 k=0;
 for i=1:nnel
     nd(i,1)=nodes(iel,i);
     start=(nd(i,1)-1)*ndof;
     for j=1:ndof
         k=k+1;
         index(k,1)=start+j;
     end
 end
 %-------------------------------------------------------------------------
for i=1:edof
    ii=index(i,1);
    ff(ii,1)=ff(ii,1)+f(i,1);
 for j=1:edof
     jj=index(j,1);
     ss(ii,jj)=ss(ii,jj)+sk(i,j);
 end
end
end%for iel
 %-------------------------------------------------------------------------
%bcdof=[13;37;35;33;31;29;27;25;23;21;19;17;15];
%apply boundary conditions

%
mm=length(bcdof);
sdof=size(ss);
%
for i=1:mm
c=bcdof(i,1);
for j=1:sdof
ss(c,j)=0;
end
%
ss(c,c)=1;
ff(c,1)=bcval(i,1);
end
%solve the equations

phi=ss\ff;
for I=1:nnode
NN(I,1)=I;
end

disp('__________________________________________________________________')
disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
disp('_________________________________________________________________________________
____')
disp('                         fem-computed values             anlytical(theoretical)-
values           ')

disp([NN phi xi])
disp('_________________________________________________________________________________
____')
```

```
disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
nodes
gcoord


[x,y]=meshgrid(0:0.1:1,0:0.1:1);

for i=1:11
    for j=1:11
      for iel=1:nel
 %node numbers of quadrilateral
       nd1=nodes(iel,1);nd2=nodes(iel,2);nd3=nodes(iel,3);nd4=nodes(iel,4);
       nd5=nodes(iel,5);nd6=nodes(iel,6);nd7=nodes(iel,7);nd8=nodes(iel,8);
       nd9=nodes(iel,9);
 %coordinates of quadrilateral(u,v)

u(1,1)=gcoord(nd1,1);u(2,1)=gcoord(nd2,1);u(3,1)=gcoord(nd3,1);u(4,1)=gcoord(nd4,1);

v(1,1)=gcoord(nd1,2);v(2,1)=gcoord(nd2,2);v(3,1)=gcoord(nd3,2);v(4,1)=gcoord(nd4,2);
 %coordinates of the grid(x,y)

          in=inpolygon(x(i,j),y(i,j),u,v);
          if (in==1)
           X=x(i,j);Y=y(i,j);
           [t]=convexquadrilateral_coordinates(u,v,X,Y);
           r=t(1,1);
           s=t(2,1);
           shn1=(r^2-r)*(s^2-s)/4;
           shn2=(r^2+r)*(s^2-s)/4;
           shn3=(r^2+r)*(s^2+s)/4;
           shn4=(r^2-r)*(s^2+s)/4;
           shn5=(1-r^2)*(s^2-s)/2;
           shn6=(r^2+r)*(1-s^2)/2;
           shn7=(1-r^2)*(s^2+s)/2;
           shn8=(r^2-r)*(1-s^2)/2;
           shn9=(1-r^2)*(1-s^2);

PHI(i,j)=shn1*phi(nd1,1)+shn2*phi(nd2,1)+shn3*phi(nd3,1)+shn4*phi(nd4,1)+shn5*phi(nd5,1
)+shn6*phi(nd6,1)+shn7*phi(nd7,1)+shn8*phi(nd8,1)+shn9*phi(nd9,1);
              break
           end%if (in==1)
        end%for iel
   %THE PROGRAM EXECUTION JUMPS TO HERE if (in==1)
    end%for j
end%for i
  z=sin(pi*x).*sin(pi*y);

for i=1:11
    for j=1:11
        if (abs(PHI(i,j))<=1e-5)
            PHI(i,j)=0;
        end
        if (abs(z(i,j))<=1e-5)
            z(i,j)=0;
        end

    end
end
switch mesh
case 1
    hold off
 clf
figure(1)
```

```
 x=[0.0 1.0 1.0 0.5 0.0];
 y=[0.0 0.0 0.5 1.0 1.0];
 patch(x,y,'w')
hold on
[x,y]=meshgrid(0:.1:1,0:0.1:1)
y((y>1/2)&(y<=1)&(x>1/2)&(x<=1)&(x+y>3/2))=NaN;
[c,h]=contour(x,y,PHI)
xlabel('X-axis');
ylabel('Y-axis');
clabel(c,h);
axis square
st1='Contour level curves for ';
st2='FEM solution of ';
st3='Nine Noded  ';
st4='Special Quadrilateral';
st5=' Elements'
title([st1,st2,st3,st4,st5])
sst1='(MESH HAS '
sst2=num2str(nnode)
sst3=' NODES'
sst4=' AND '
sst5=num2str(nel)
sst6=' ELEMENTS)'
text(0.25,-.08,[sst1 sst2 sst3 sst4 sst5 sst6])
figure(2)
 x=[0.0 1.0 1.0 0.5 0.0];
 y=[0.0 0.0 0.5 1.0 1.0];
 patch(x,y,'w')
hold on
[x,y]=meshgrid(0:.1:1,0:0.1:1)
y((y>1/2)&(y<=1)&(x>1/2)&(x<=1)&(x+y>3/2))=NaN;
[c,h]=contour(x,y,z)
xlabel('X-axis');
ylabel('Y-axis');
clabel(c,h);
axis square
title('contour level curves for exact solution: sin(pi*x)*sin(pi*y)')
 mm=0;
 for i=1:11
  for j=1:11
      mm=mm+1;
      femsoln(mm,1)=PHI(i,j);
      exactsoln(mm,1)=z(i,j);
  end
end


case 2
    hold off
 clf
figure(1)
[x,y]=meshgrid(0:.1:1,0:0.1:1)
[c,h]=contour(x,y,PHI)
xlabel('X-axis');
ylabel('Y-axis');
clabel(c,h);
axis square
st1='Contour level curves for ';
st2='FEM solution of ';
st3='Nine Noded  ';
st4='Special Quadrilateral';
st5=' Elements'
title([st1,st2,st3,st4,st5])
sst1='(MESH HAS '
sst2=num2str(nnode)
```

```
sst3=' NODES'
sst4=' AND '
sst5=num2str(nel)
sst6=' ELEMENTS)'
text(0.25,-.08,[sst1 sst2 sst3 sst4 sst5 sst6])

figure(2)
[x,y]=meshgrid(0:.1:1,0:0.1:1)
[c,h]=contour(x,y,z)
xlabel('X-axis');
ylabel('Y-axis');
clabel(c,h);
axis square
title('contour level curves for exact solution: sin(pi*x)*sin(pi*y)')
 mm=0;
 for i=1:11
  for j=1:11
      mm=mm+1;
      femsoln(mm,1)=PHI(i,j);
      exactsoln(mm,1)=z(i,j);
  end
end

end%switch mesh
[femsoln exactsoln]

 disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
[1 phi(1,1) xi(1,1)]
disp('____________________________________________________________________')
disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
disp('___________________________________________________________________________________
____')
disp('                          fem-computed values             anlytical(theoretical)-
values           ')

disp([NN phi xi])
disp('___________________________________________________________________________________
____')

disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
for ncpt=1:nel
    centrpt=nodes(ncpt,9)
    elcentr(ncpt,1)=centrpt;
    phicpt(ncpt,1)=phi(centrpt,1);
    xicpt(ncpt,1)=xi(centrpt,1);
end
for I=1:nel
elnumm(I,1)=I;
end
disp('___________________________________________________________________________________
____')
disp(' serial no        center point                fem-computed values
anlytical(theoretical)-values           ')

disp([elnumm  elcentr phicpt xicpt])
disp('___________________________________________________________________________________
____')

format compact
%=========================================================================================
===========
```

```
disp('__________________________________________________________________')
disp('number of nodes,elements & nodes per element')
[nnode nel nnel ndof]
disp('_________________________________________________________________________________
_____________________________________________________')
disp('        NODE FEM SOLUTION EXACT SOLUTION       NODE FEM SOLUTION EXACT SOLUTION
NODE FEM SOLUTION EXACT SOLUTION          ')

disp('---------------------------------------------------------------------------------
------------------------------------------------------')
for I=1:3:nel
   % A=[elcentr(I) phicpt(I) xicpt(I)];B=[elcentr(I+1) phicpt(I+1)
xicpt(I+1)];C=[elcentr(I+2) phicpt(I+2) xicpt(I+2)];
    %disp([elcentr(I) phicpt(I) xicpt(I) elcentr(I+1) phicpt(I+1) xicpt(I+1)])
    %disp([elcentr(I+2) phicpt(I+2) xicpt(I+2)])
    %disp([A B C])
    fprintf('\n%5d %18.14f %18.14f %5d %18.14f %18.14f %5d %18.14f %18.14f',elcentr(I),
phicpt(I), xicpt(I), elcentr(I+1), phicpt(I+1), xicpt(I+1), elcentr(I+2), phicpt(I+2),
xicpt(I+2));
end
 fprintf('\n')
disp('---------------------------------------------------------------------------------
------------------------------------------------------')
```