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Transient Analysis of Unreliable Server $M^X/G/1$ Queue with Bernoulli Vacation Schedule and Second Optional Repair under Controlled Admissibility Policy

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Abstract

In this paper, we propose to study such a model which deals with the aspects concerning the control of the arrival process with second optional repair and Bernoulli vacation schedule. The paper deals with $M^x/G/1$ queueing system where after completion of a service the server either goes for a vacation of random length with probability $\theta (0 \le \theta \le 1)$ or may continue to serve the next customer with probability $(1-\theta)$, if any. Both service time and vacation time follow general distribution. Server is subject to random breakdowns according to Poisson process, followed by instantaneous repair. If the server could not be repaired with the first essential repair, subsequent optional repair is needed for the restoration of the server. Both essential and optional repair times follow exponential distribution. Unlike the usual batch arrivals queueing model, there is restriction over the admissibility of batch arrivals in which not all the arriving batches are allowed to join the queue at all times. The restricted admissibility policy differs during a busy period and a vacation period. We obtain the time dependent probability generating functions in terms of their Laplace transforms and corresponding steady state results explicitly. In addition, some performance measures such as expected queue size and expected waiting time of a customer are also obtained. The numerical results for various performance measures are displayed via graphs.
Keywords: $M^X/G/1$ queue, First essential repair, Second optional repair, Controlled admissibility policy, Bernoulli vacation schedule. Transient analysis.

Introduction

Control of queues is one of the most significant and interesting area of queueing theory. One of the queue control is, control on admission of arriving customers. In these problems, either the arrival rate can be modified or the customer can be refused admission. In some of queueing models, queue length is controlled by the rejection of some of incoming arriving customers, in some models, the customers themselves control the decision to enter in to the system. Such a control policy is called restricted admissibility policy and which has been studied by many authors. Ghosh and Weerasinghe (2010) addressed a rate control problem associated with a single server Markovian queueing system with customer abandonment in heavy traffic. Transient solution of an $M^X/G/1$ queue queueing model under restricted admissibility policy has been analyzed by Ayyappan and Shyamala (2013).

In many real cases, the server may experience breakdowns, so that a more realistic queueing model is that which incorporates the assumption of unreliable server. When the server could not be repaired or restored by the first essential repair, subsequent repairs are needed to restore the server. The most realistic aspect in modeling of an unreliable server is multi optional repair which has been discussed by Jain et al. (2011). Ayyappan and Shyamala (2013) have explicated an $M^X/G/1$ queue with second optional repair.

Presently, most of the studies are devoted to batch arrival vacation models under different vacation policies because of its interdisciplinary characteristic. Steady state behavior of an $M^X/G/1$ queue with general retrial time and Bernoulli vacation schedule for an unreliable server with delayed repair has been studied by Choudhury and Ke (2012) . Gao and Liu (2013) obtained stationary performance measures for M/G/1 queue with vacation interruption under Bernoulli vacation schedule.

In the present paper, we consider unreliable M^x/G/1 queue under controlled admissibility policy with the concept of Bernoulli vacation schedule. Repair is provided in two phases. If the server could not be repaired with the first essential repair, subsequent optional repair is provided to the single server for the refurbishment of the server. The paper is organized as follows. In section 2, we define the underlying assumptions and notations of the system under study. The analysis based on supplementary variables and generating function approach, is given in section 3. The average queue size and mean waiting time of a customer are calculated in section 4. Last section is devoted to concluding remarks.

1. Model Description

The following assumptions are made to describe the model mathematically:

- Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda\, a_i \Delta t(i=1,2,3,.......)$ be first order probability that a batch of 'i' customers arrives at the system during a short interval of the time $(t, t+\Delta t)$, where $0 \le a_i \le 0,\ \sum_{i=1}^{\infty} a_i = 1$ and $\lambda > 0$ is the mean arrival rate of the batches.
- Each customer undergoes service provided by a single server on FCFS basis. The service time follows general probability distribution with distribution function (DF) $B(x)$, density function $b(x)$ Laplace Stieltjes transform (LST) $B^*(s)$ and finite moments $E\left(B^k\right)(k \ge 1)$.
- Let $\mu(x)\,dx$ be the conditional probability of completion of the service during the interval $(x, x+dx]$ given that elapsed time is x , so that $\mu(x) = \dfrac{b(x)}{1-B(x)}$ and therefore , $b(v) = \mu(v)\, e^{-\int_0^v \mu(x)\,dx}$
- Further it is assumed that not all batches are allowed to join the system at all times. Let ξ_1 $(0 \le \xi_1 \le 1)$ be the probability that an arriving batch will be allowed to join the system when server is not on vacation and ξ_2 $(0 \le \xi_2 \le 1)$ be the probability that an arriving batch will be allowed to join the system during the vacation period.
- As soon as the service of a customer is completed the server may go for a vacation of random length V with probability $\theta\,(0 \le \theta \le 1)$ or may continue to serve the next customer, if any, with probability $(1-\theta)$. Assuming that the vacation random variable $V(s)$ follows a general probability distribution with Distribution function $v(s)$, LST $V^*(s)$ and finite moment $E\left(V^k\right)(k \ge 1)$ and is independent of service time random variable and the arrival process.
- Let $\vartheta(x)\,dx$ be the conditional probability of completion of a vacation during the interval $(x, x+dx]$ given that elapsed time is x , so that $\vartheta(x) = \dfrac{v(x)}{1-V(x)}$ and therefore , $v(s) = \vartheta(s)\, e^{-\int_0^s \vartheta(x)\,dx}$
- Server is subject to random breakdowns according to Poisson process with mean breakdown rate $\alpha > 0$. As soon as the server is broken down it is instantaneously sent for repair. If the server could not be repaired with the first essential repair, subsequent optional repair is needed for the rectification of the server, i.e., after the completion of first essential repair, the server may opt for second optional repair with probability ψ or may join the system with complementary probability

$(1-\psi)$ to provide the service to the customers. Both essential and optional repair times follow exponential distribution with mean $\frac{1}{\beta_1}$ and $\frac{1}{\beta_2}$ respectively.

Chapman Kolmogorov equations governing the model are constructed as follows:

$$\frac{\partial}{\partial t} P_n(x,t) + \frac{\partial}{\partial x} P_n(x,t) + (\lambda + \mu(x) + \alpha) P_n(x,t) = \lambda(1-\xi_1) P_n(x,t) + \lambda \xi_1 \sum_{i=1}^{n-1} a_i\, P_{n-i}(x,t);\ n \geq 1 \quad (1)$$

$$\frac{\partial}{\partial t} P_0(x,t) + \frac{\partial}{\partial x} P_0(x,t) + (\lambda + \mu(x) + \alpha) P_0(x,t) = \lambda(1-\xi_1) P_0(x,t) \quad (2)$$

$$\frac{\partial}{\partial t} V_n(x,t) + \frac{\partial}{\partial x} V_n(x,t) + (\lambda + \vartheta(x)) V_n(x,t) = \lambda(1-\xi_2) V_n(x,t) + \lambda \xi_2 \sum_{i=1}^{n-1} a_i\, V_{n-i}(x,t);\ n \geq 1 \quad (3)$$

$$\frac{\partial}{\partial t} V_0(x,t) + \frac{\partial}{\partial x} V_0(x,t) + (\lambda + \vartheta(x)) V_0(x,t) = \lambda (1-\xi_2) V_0(x,t) \quad (4)$$

$$\frac{d}{dt} R_n^{(1)}(t) + (\lambda + \beta_1) R_n^{(1)}(t) = \lambda(1-\xi_1) R_n^{(1)}(t) + \lambda \xi_1 \sum_{i=1}^{n} a_i\, R_{n-i}^{(1)}(t) + \alpha \int_0^\infty P_{n-1}(x,t)\, dx;\ n \geq 1 \quad (5)$$

$$\frac{d}{dt} R_0^{(1)}(t) + (\lambda + \beta_1) R_0^{(1)}(t) = \lambda(1-\xi_1) R_0^{(1)}(t) \quad (6)$$

$$\frac{d}{dt} R_n^{(2)}(t) + (\lambda + \beta_2) R_n^{(2)}(t) = \lambda(1-\xi_1) R_n^{(2)}(t) + \lambda \xi_1 \sum_{i=1}^{n} a_i\, R_{n-i}^{(2)}(t) + \psi\, \beta_1\, R_n^{(1)}(t);\ n \geq 1 \quad (7)$$

$$\frac{d}{dt} R_0^{(2)}(t) + (\lambda + \beta_2) R_0^{(2)}(t) = \lambda(1-\xi_1) R_0^{(2)}(t) + \psi\, \beta_1\, R_0^{(1)}(t) \quad (8)$$

$$\frac{d}{dt} Q(t) + \lambda \xi_1 Q(t) = \lambda\xi_1 (1-\xi_2) Q(t) + \beta_1 \xi_2 (1-\psi) R_0^{(1)}(t) + \beta_2 \xi_2 R_0^{(2)}(t) + \xi_2(1-\theta) \int_0^\infty P_0(x,t)\, \mu(x)\, dx + \xi_1 \int_0^\infty V_0(x,t)\, \vartheta(x)\, dx \quad (9)$$

The above equations are solved to be solved with the following boundary conditions at $x=0$

$$P_0(0,t) = a_1 \lambda \xi_1 \xi_2 Q(t) + (1-\psi) \xi_2 \beta_1 R_1^{(1)}(t) + \xi_2 \beta_2 R_1^{(2)}(t) + \xi_2(1-\theta) \int_0^\infty P_1(x,t)\, \mu(x)\, dx + \xi_1 \int_0^\infty V_1(x,t)\, v(x)\, dx \quad (10)$$

$$P_n(0,t) = a_{n+1} \lambda \xi_1 \xi_2 Q(t) + (1-\psi) \xi_2 \beta_1 R_{n+1}^{(1)}(t) + \xi_2 \beta_2 R_{n+1}^{(2)}(t) + \xi_2(1-\theta) \int_0^\infty P_{n+1}(x,t)\, \mu(x)\, dx + \xi_1 \int_0^\infty V_{n+1}(x,t)\, \vartheta(x)\, dx;\ n \geq 0 \quad (11)$$

$$\xi_1 V_n(0,t) = \xi_2\, \theta \int_0^\infty P_n(x,t)\, \mu(x)\, dx;\ n \geq 0 \quad (12)$$

Next we define the following probability Generating Functions:

$$P(x,z,t)=\sum_{n=0}^{\infty} z^n P_n(x,t);\; P(z,t)=\sum_{n=0}^{\infty} z^n P_n(t);\; V(x,z,t)=\sum_{n=0}^{\infty} z^n V_n(x,t);\; V(z,t)=\sum_{n=0}^{\infty} z^n V_n(t)$$

$$R^{(1)}(z,t)=\sum_{n=0}^{\infty} z^n R_n^{(1)}(t);\; R^{(2)}(z,t)=\sum_{n=0}^{\infty} z^n R_n^{(2)}(t);\; a(z)=\sum_{n=1}^{\infty} a_n z^n$$

2. The Analysis

Theorem 1: The marginal generating functions when the server is busy, on vacations, under repair with first essential repair and optional second repair, respectively, are

$$\bar{P}(z,s)=\bar{P}(0,z,s)\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{s+\lambda\xi_1(1-a(z))+\alpha}\right\}$$

$$\bar{V}(z,s)=\left(\frac{\xi_2}{\xi_1}\right)\theta\, P(0,z,s)\, B\left(s+\lambda\xi_2(1-a(z))+\alpha\right)\left\{\frac{1-\bar{V}\left(s+\lambda\xi_2(1-a(z))\right)}{s+\lambda\xi_2(1-a(z))}\right\}$$

$$\bar{R}^{(1)}(z,s)=\alpha\, z\bar{P}(0,z,s)\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)}\right\}$$

$$\bar{R}^{(2)}(z,s)=\psi\,\beta_1\,\alpha\, z\bar{P}(0,z,s)\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)\left(s+\lambda\xi_1(1-a(z))+\beta_2\right)}\right\}$$

Where

$$\bar{P}(0,z,s)=\frac{\lambda\,\xi_1\xi_2\left(a(z)-1\right)\bar{Q}(s)+\left(1-s\;\bar{Q}(s)\right)}{\eta}$$

Proof:

Taking Laplace transforms of equations (1)-(12)

$$\frac{\partial}{\partial x}\bar{P}_n(x,s)+\left(s+\lambda+\mu(x)+\alpha\right)\bar{P}_n(x,s)=\lambda\left(1-\xi_1\right)\bar{P}_n(x,s)+\lambda\,\xi_1\sum_{i=1}^{n-1}a_i\,\bar{P}_{n-i}(x,s);\; n\ge 1 \tag{13}$$

$$\frac{\partial}{\partial x}\bar{P}_0(x,s)+\left(s+\lambda+\mu(x)+\alpha\right)\bar{P}_0(x,s)=\lambda\left(1-\xi_1\right)\bar{P}_0(x,s) \tag{14}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s)+\left(s+\lambda+\vartheta(x)\right)\bar{V}_n(x,s)=\lambda\left(1-\xi_2\right)\bar{V}_n(x,s)+\lambda\,\xi_2\sum_{i=1}^{n-1}a_i\,\bar{V}_{n-i}(x,s);\; n\ge 1 \tag{15}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s)+\left(s+\lambda+\vartheta(x)\right)\bar{V}_0(x,s)=\lambda\left(1-\xi_2\right)\bar{V}_0(x,s) \tag{16}$$

$$\left(s+\lambda+\beta_1\right)\bar{R}_n^{(1)}(s)=\lambda\left(1-\xi_1\right)\bar{R}_n^{(1)}(s)+\lambda\,\xi_1\sum_{i=1}^{n}a_i\,R_{n-i}^{(1)}(s)+\alpha\int_0^{\infty}\bar{P}_{n-1}(x,s)\,dx;\; n\ge 1 \tag{17}$$

$$\left(s+\lambda+\beta_1\right)\overline{R}_0^{(1)}(s)=\lambda\left(1-\xi_1\right)\overline{R}_0^{(1)}(s) \tag{18}$$

$$\left(s+\lambda+\beta_2\right)\overline{R}_n^{(2)}(s)=\lambda\left(1-\xi_1\right)\overline{R}_n^{(2)}(s)+\lambda\,\xi_1\sum_{i=1}^{n}a_i\,\overline{R}_{n-i}^{(2)}(s)+\psi\,\beta_1\,\overline{R}_n^{(1)}(s);\ n\geq 1 \tag{19}$$

$$\frac{d}{dt}R_0^{(2)}(t)+\left(\lambda+\beta_2\right)R_0^{(2)}(t)=\lambda\left(1-\xi_1\right)R_0^{(2)}(t)+\psi\,\beta_1\,R_0^{(1)}(t) \tag{20}$$

$$\begin{aligned}\left(s+\lambda\right)\xi_1\,\overline{Q}(s)=\lambda\xi_1\,(1-\xi_2)\,\overline{Q}(s)+\beta_1\,\xi_2\left(1-\psi\right)\overline{R}_0^{(1)}(s)+\beta_2\,\xi_2\,\overline{R}_0^{(2)}(s)+\xi_2(1-\theta)\int_0^\infty\overline{P}_0(x,s)\,\mu(x)\,dx\\+\xi_1\int_0^\infty\overline{V}_0(x,s)\,\vartheta(x)\,dx\end{aligned} \tag{21}$$

Boundary conditions:

$$\xi_2\overline{P}_0(0,s)=a_1\,\lambda\,\xi_1\xi_2\,\overline{Q}(s)+\left(1-\psi\right)\xi_2\,\beta_1\,\overline{R}_1^{(1)}(s)+\xi_2\,\beta_2\,\overline{R}_1^{(2)}(s)+\xi_2\left(1-\theta\right)\int_0^\infty\overline{P}_1(x,s)\,\mu(x)\,dx+\xi_1\int_0^\infty\overline{V}_1(x,s)\,\vartheta(x)\,dx \tag{22}$$

$$\xi_2\overline{P}_n(0,s)=a_{n+1}\,\lambda\,\xi_1\xi_2\,\overline{Q}(s)+\left(1-\psi\right)\xi_2\,\beta_1\,\overline{R}_{n+1}^{(1)}(s)+\xi_2\,\beta_2\,\overline{R}_{n+1}^{(2)}(s)+\xi_2\left(1-\theta\right)\int_0^\infty\overline{P}_{n+1}(x,s)\,\mu(x)\,dx+\xi_1\int_0^\infty\overline{V}_{n+1}(x,s)\,\vartheta(x)\,dx;\ n\geq 0$$

$$\xi_1\,\overline{V}_n(0,s)=\xi_2\,\theta\int_0^\infty\overline{P}_n(x,s)\,\mu(x)\,dx;\ n\geq 0$$

(23)

(24)

Now multiply equation (13) and (14) by z^n respectively and summing over all possible $'n'$ and using probability generating functions, we obtain

$$\frac{\partial}{\partial x}\overline{P}(x,z,s)+\left(s+\lambda\xi_1\left(1-a(z)\right)+\mu(x)+\alpha\right)\overline{P}(x,z,s)=0 \tag{25}$$

Performing similar operation on equations (15) to (20)

$$\frac{\partial}{\partial x}\overline{V}(x,z,s)+\left(s+\lambda\xi_2\left(1-a(z)\right)+\vartheta(x)\right)\overline{V}(x,z,s)=0 \tag{26}$$

$$\left(s+\lambda\xi_1\left(1-a(z)\right)+\beta_1\right)\overline{R}^{(1)}(z,s)=\alpha\,z\int_0^\infty\overline{P}(x,z,s)\,dx;\ n\geq 1 \tag{27}$$

$$\left(s+\lambda\xi_1\left(1-a(z)\right)+\beta_2\right)\overline{R}^{(2)}(z,s)=\psi\,\beta_1\,\overline{R}^{(1)}(z,s);\ n\geq 1 \tag{28}$$

For boundary conditions, multiply equation (22) by z and (23) by z^{n+1} respectively and summing over all possible $'n'$ and using probability generating functions, we obtain

$$z\xi_2\overline{P}(0,z,s)=\lambda\,\xi_1\xi_2\,(a(z)-1)\,\overline{Q}(s)+\left(1-s\,\overline{Q}(s)\right)+(1-\psi)\xi_2\,\beta_1\,\overline{R}^{(1)}(z,s)+\xi_2\,\beta_2\,\overline{R}^{(2)}(z,s)$$

$$+\xi_2(1-\theta)\int_0^\infty \overline{P}\,(x,z,s)\,\mu(x)\,dx+\xi_1\int_0^\infty \overline{V}\,(x,z,s)\,\vartheta(x)\,dx;\ n\geq 0 \tag{29}$$

Performing the similar operation on (24)

$$\xi_1\,\overline{V}(0,z,s)=\xi_2\,\theta\int_0^\infty \overline{P}(x,z,s)\,\mu(x)\,dx;\ n\geq 0 \tag{30}$$

Integrating eq. (25), we get

$$\overline{P}(x,z,s)=\overline{P}\,(0,z,s)\exp\left\{-\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)x-\int_0^x \mu(t)\,dt\right\} \tag{31}$$

Where $\overline{P}(0,z,s)$ is given by eq. (29)

Integrate (31) by parts w.r.t. x,

$$\overline{P}(z,s)=\overline{P}\,(0,z,s)\,\left\{\frac{1-\overline{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{s+\lambda\xi_1\,(1-a(z))+\alpha}\right\} \tag{32}$$

Where,

$$\overline{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)=\int_0^\infty e^{-\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)x}\,dB(x) \tag{33}$$

is the Laplace- Stieltjes transform of the service time $B(x)$

Multiply (31) by $\mu(x)$ and integrating over x, we get

$$\int_0^\infty \overline{P}(x,z,s)\,\mu(x)\,dx=\overline{P}\,(0,z,s)\,\overline{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right) \tag{34}$$

From eqs. (30) & (34)

$$\xi_1\,\overline{V}(0,z,s)=\xi_2\,\theta\,\overline{P}\,(0,z,s)\,\overline{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right);\ n\geq 0 \tag{35}$$

Integrating (26) , we get

$$\overline{V}(x,z,s)=\overline{V}\,(0,z,s)\exp\left\{-\left(s+\lambda\xi_2\,(1-a(z))\right)x-\int_0^x \vartheta(t)\,dt\right\} \tag{36}$$

From eq. (35) and (36)

$$\overline{V}(x,z,s)=\left(\frac{\xi_2}{\xi_1}\right)\theta\,\overline{P}\,(0,z,s)\,\overline{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)\exp\left\{-\left(s+\lambda\xi_2\,(1-a(z))\right)x-\int_0^x \vartheta(t)\,dt\right\} \tag{37}$$

Again integrating eq. (37) by parts w.r.t x :

From (35)

$$\bar{V}(z,s)=\left(\frac{\xi_2}{\xi_1}\right)\theta\, P(0,z,s)\, B\left(s+\lambda\xi_2\,(1-a(z))+\alpha\right)\left\{\frac{1-\bar{V}\left(s+\lambda\xi_2\,(1-a(z))\right)}{s+\lambda\xi_2\,(1-a(z))}\right\} \tag{38}$$

Where,

$$\bar{V}\left(s+\lambda\xi_1\,(1-a(z))\right)=\int_0^\infty e^{-\left(s+\lambda\xi_2\,(1-a(z))+\alpha\right)x}\,dV(x) \tag{39}$$

is the Laplace- Stieltjes transform of the vacation time $V(x)$. Now multiply (37) by $\vartheta(x)$ and integrating over x , we get

$$\int_0^\infty \bar{V}(x,z,s)\,\vartheta(x)=\left(\frac{\xi_2}{\xi_1}\right)\theta\,\bar{P}(0,z,s)\,\bar{B}\left(s+\lambda\xi_2\,(1-a(z))+\alpha\right)\bar{V}\left(s+\lambda\xi_2\,(1-a(z))\right) \tag{40}$$

using eq. (34), eq. (27) becomes

$$\bar{R}^{(1)}(z,s)=\alpha\, z\bar{P}\,(0,z,s)\,\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)}\right\} \tag{41}$$

Using eq. (41), eq. (28) becomes

$$\bar{R}^{(2)}(z,s)=\psi\,\beta_1\,\alpha\, z\bar{P}\,(0,z,s)\,\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)\left(s+\lambda\xi_1\,(1-a(z))+\beta_2\right)}\right\} \tag{42}$$

Using eqs. (34),(40,)(41), (42) , eq. (29) becomes

$$\begin{aligned}
z\xi_2\bar{P}(0,z,\ s) &= \lambda\,\xi_1\xi_2\left(a(z)-1\right)\bar{Q}(s)+\left(1-s\ \bar{Q}(s)\right)+\ (1-\psi)\xi_2\,\beta_1\,\alpha\, z\bar{P}\,(0,z,s)\,\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)}\right\}\\
&+\xi_2\,\beta_2\,\psi\,\beta_1\,\alpha\, z\bar{P}\,(0,z,s)\,\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)\left(s+\lambda\xi_1\,(1-a(z))+\beta_2\right)}\right\}\\
&+\xi_2(1-\theta)\bar{P}\,(0,z,s)\,\bar{B}\,\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)+\\
&+\xi_2\,\theta\,\bar{P}(0,z,s)\,\bar{B}\left(s+\lambda\xi_2\,(1-a(z))+\alpha\right)\bar{V}\left(s+\lambda\xi_2\,(1-a(z))\right)
\end{aligned}$$

$$\Rightarrow \bar{P}\,(0,z,s)=\frac{\lambda\,\xi_1\xi_2\left(a(z)-1\right)\bar{Q}(s)+\left(1-s\ \bar{Q}(s)\right)}{\eta} \tag{43}$$

Where

$$\begin{aligned}
\eta = \Big[z\xi_2 &-(1-\psi)\xi_2\,\beta_1\,\alpha\, z\,\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)}\right\}\\
&-\xi_2\,\beta_2\,\psi\,\beta_1\,\alpha\, z\,\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)\left(s+\lambda\xi_1\,(1-a(z))+\beta_2\right)}\right\}\\
&-\xi_2(1-\theta)\bar{B}\,\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)-\xi_2\,\theta\ \bar{B}\left(s+\lambda\xi_2\,(1-a(z))+\alpha\right)\bar{V}\left(s+\lambda\xi_2\,(1-a(z))\right)\Big]
\end{aligned} \tag{44}$$

Theorem 2: The steady state probabilities when the server is busy while rendering service, on vacations, under repair with first essential repair and optional second repair, respectively, are given as follow

$$P(z)=\left\{\frac{1-\bar{B}\left(s+\lambda\xi_1\,(1-a(z))+\alpha\right)}{s+\lambda\xi_1\,(1-a(z))+\alpha}\right\}\left\{\frac{\lambda\,\xi_1\xi_2\left(a(z)-1\right)\bar{Q}}{\eta}\right\} \tag{45}$$

$$V(z)=\left(\frac{\xi_2}{\xi_1}\right)\theta\ \ B\left(s+\lambda\xi_2\,(1-a(z))+\alpha\right)\left\{\frac{1-\bar{V}\left(s+\lambda\xi_2\,(1-a(z))\right)}{s+\lambda\xi_2\,(1-a(z))}\right\}\left\{\frac{\lambda\,\xi_1\xi_2\left(a(z)-1\right)\bar{Q}}{\eta}\right\} \tag{46}$$

(47)

$$R^{(2)}(z)=\psi\ \beta_1\ \alpha\ z\ \left\{\frac{1-\overline{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)\left(s+\lambda\xi_1(1-a(z))+\beta_2\right)}\right\}\left\{\frac{\lambda\ \xi_1\xi_2\left(a(z)-1\right)\ \overline{Q}}{\eta}\right\} \quad (48)$$

Proof: By Using Tauberian property $Lt_{s\to 0}s\bar{f}(s)=Lt_{t\to\infty}f(t)$ in eqs. (32) (38), (41), (42) and (43), we get steady state probabilities when the server is busy, on vacations, under repair with first essential repair and optional second repair, respectively.

Theorem 3: The probability generating function of the number of the customers in the queue is given by

$$\begin{aligned}W(z)=&\left[\left(\frac{1-\overline{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{s+\lambda\xi_1(1-a(z))+\alpha}\right)+\left(\frac{\xi_2}{\xi_1}\right)\theta\ B\left(s+\lambda\xi_2(1-a(z))+\alpha\right)\left(\frac{1-\overline{V}\left(s+\lambda\xi_2(1-a(z))\right)}{s+\lambda\xi_2(1-a(z))}\right)\right.\\&+\alpha\ z\left(\frac{1-\overline{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)}\right)+\\&\left.+\psi\ \beta_1\ \alpha\ z\left(\frac{1-\overline{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)\left(s+\lambda\xi_1(1-a(z))+\beta_2\right)}\right)\right]\left\{\frac{\lambda\ \xi_1\xi_2\left(a(z)-1\right)\ \overline{Q}}{\eta}\right\}\end{aligned} \quad (49)$$

Proof: Probability generating function of the number of the customers in the queue $W(z)$ irrespective to the state of system is given by adding eqs. (45), (46), (47) and (48)

$$W(z)=P(z)+V(z)+R^{(1)}(z)+R^{(2)}(z)$$

$$\begin{aligned}W(z)=&\left[\left(\frac{1-\overline{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{s+\lambda\xi_1(1-a(z))+\alpha}\right)+\left(\frac{\xi_2}{\xi_1}\right)\theta\ B\left(s+\lambda\xi_2(1-a(z))+\alpha\right)\left(\frac{1-\overline{V}\left(s+\lambda\xi_2(1-a(z))\right)}{s+\lambda\xi_2(1-a(z))}\right)\right.\\&+\alpha\ z\left(\frac{1-\overline{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)}\right)+\\&\left.+\psi\ \beta_1\ \alpha\ z\left(\frac{1-\overline{B}\left(s+\lambda\xi_1(1-a(z))+\alpha\right)}{\left(s+\lambda\xi_1(1-a(z))+\alpha\right)\left(s+\lambda\xi_1(1-a(z))+\beta_1\right)\left(s+\lambda\xi_1(1-a(z))+\beta_2\right)}\right)\right]\left\{\frac{\lambda\ \xi_1\xi_2\left(a(z)-1\right)\ \overline{Q}}{\eta}\right\}\end{aligned}$$

Corollary 1: If the system is in steady state, then

$$P[1]=\Pr[The\ server\ is\ busy\ in\ providing\ servcie]=\lim_{z\to 1}P(z)=\frac{\lambda\beta_1\beta_2\xi_1\xi_2Q[1-\overline{B}(\alpha)]E[I]}{\zeta} \quad (50)$$

$$V[1]=\Pr[The\ server\ is\ on\ vacation]=\lim_{z\to 1}V(z)=\frac{\lambda\theta\alpha\beta_1\beta_2\xi_1\xi_2Q\ \overline{B}(\alpha)\ E[I]\ E[V]}{\zeta} \quad (51)$$

$$R^{(1)}[1]=\Pr[The\ server\ is\ getting\ first\ essential\ repair]=\lim_{z\to 1}R^{(1)}(z)=\frac{\lambda\alpha\beta_2\xi_2Q[1-\overline{B}(\alpha)]E[I]}{\zeta} \quad (52)$$

$$R^{(2)}[1]=\Pr[The\ server\ is\ getting\ \sec ond\ optional\ repair]=\lim_{z\to 1}R^{(2)}(z)=\frac{\lambda\psi\alpha\beta_1\xi_1Q[1-\overline{B}(\alpha)]E[I]}{\zeta} \quad (53)$$

Where

$$\zeta = \alpha\beta_1\beta_2\xi_1\xi_2\overline{B}(\alpha)\{1-\theta\, E[I]E[V]\} - \lambda E[I][1-\overline{B}(\alpha)]\{\psi\alpha\beta_1\xi_1 + \beta_1\beta_2\xi_1\xi_2 + \alpha\beta_2\xi_2\} \tag{54}$$

Proof: The corresponding steady state results can be obtained by applying L'Hospital's Rule in eqs. (45), (46), (47) and (48) by letting $z=1$

Corollary 2: The probability that there are no customers in the system when the server is idle but available in the system is given by

$$Q = \zeta\left\{\zeta + \lambda\,\beta_1\beta_2\xi_1\xi_2[1-\overline{B}(\alpha)]E[I] + \lambda\theta\alpha\beta_1\beta_2\xi_1\xi_2\,\overline{B}(\alpha)\,E[I]\,E[V] + \lambda\alpha\beta_2\xi_2[1-\overline{B}(\alpha)]E[I] + \lambda\psi\alpha\beta_1\xi_1\,[1-\overline{B}(\alpha)]E[I]\right\}^{-1}$$

Proof: In order to obtain Q, we use normalizing condition

$$P[1]+V[1]+R^{(1)}[1]+R^{(2)}[2]+Q=1 \tag{55}$$

By using eqs (50)-(53) and normalizing condition, we get the required expression.

3. Average Queue Length

In this section we derive the expected number of customers in this $M^X G/1$ queueing system with non reliable server. Let L_q be the expected number of customers at a random epoch, then

$$L_q = \lim_{z\to 1}\frac{d}{dz}W(z)$$

(56)

$$\Rightarrow L_q = \frac{D'(1)\,N''(1) - N'(1)\,D''(1)}{2\left(D'(1)\right)^2}$$

Where $N'(1) = \lambda E[I]\,Q\,\left\{\left(\psi\alpha\beta_1\xi_1 + \beta_1\beta_2\xi_1\xi_2 + \xi_2\beta_2\alpha\right) + \overline{B}(\alpha)\left[\left(\theta\alpha\beta_1\beta_2\xi_1\xi_2 E[V]\right) - \left(\psi\alpha\xi_1\beta_1 + \beta_1\beta_2\xi_1\xi_2 + \xi_2\beta_2\alpha\right)\right]\right\}$

(57)

$$N''(1) = 2Q[\lambda E[I]]^2\left\{\left(\frac{\alpha}{\lambda E[I]}-1\right) + \overline{B}(\alpha)\left(1-\frac{\alpha}{\lambda\, E[I]} - \theta\psi\alpha\xi_1\beta_1 E[V] - \theta\beta_1\beta_2\xi_1\xi_2 E[V] - \theta\alpha\beta_2\xi_2 E[V] + \frac{1}{2}\,\theta\alpha\beta_1\beta_2\xi_1\xi_2 E[V^2]\right)\right.$$
$$\left. + \overline{B}(\alpha)\left\{\left(\psi\alpha\beta_1\xi_1 + \beta_1\beta_2\xi_1\xi_2 + \xi_2\beta_2\alpha\right) - \theta\alpha\beta_1\beta_2\xi_1\xi_2 E[V]\right\} + \lambda Q E[I(I-1)]\left(\psi\alpha\beta_1\xi_1 + \beta_1\beta_2\xi_1\xi_2 + \xi_2\beta_2\alpha\right)\right\}$$

(58)

$$D'(1) = \overline{B}(\alpha)\left\{\alpha\beta_1\beta_2\xi_1\xi_2 + \lambda E[I]\left(\psi\alpha\beta_1\xi_1 + \beta_1\beta_2\xi_1\xi_2 + \xi_2\beta_2\alpha\right)\right\} - \lambda E[I]\left(\psi\alpha\beta_1\xi_1 + \beta_1\beta_2\xi_1\xi_2 + \xi_2\beta_2\alpha\right) - \theta\alpha\beta_1\beta_2\xi_1\xi_2 E[I]E[V]$$

(59)

$$D''(1)=2(\lambda E[I])^2\left\{1-\frac{\psi\alpha\beta_1\xi_1+\beta_1\beta_2\xi_1\xi_2+\beta_2\xi_2\alpha}{\lambda E[I]}-\overline{B}(\alpha)\left\{1+\theta E[V]\left(\psi\alpha\beta_1\xi_1+\beta_1\beta_2\xi_1\xi_2+\beta_2\xi_2\alpha\right)+\frac{1}{2}\alpha\beta_1\beta_2\xi_1\xi_2E[V^2]\right\}\right.$$
$$+\overline{B}(\alpha)\left\{\frac{\alpha\beta_1\beta_2\xi_1\xi_2}{\lambda E[I]}-\left(\psi\alpha\beta_1\xi_1+\beta_1\beta_2\xi_1\xi_2+\beta_2\xi_2\alpha\right)+\alpha\theta\beta_1\beta_2\xi_1\xi_2E[V]\right\}-\lambda Q\left(\psi\alpha\beta_1\xi_1+\beta_1\beta_2\xi_1\xi_2+\beta_2\xi_2\alpha\right)E[I(I-1)]$$
$$\left.+\left(\psi\alpha\beta_1\xi_1+\beta_1\beta_2\xi_1\xi_2+\beta_2\xi_2\alpha\right)-\alpha\theta\beta_1\beta_2\xi_1\xi_2E[V]\right. \quad (60)$$

Mean waiting time of a customer can be obtained by Little's formula $W_q=\frac{L_q}{\lambda}$ (61)

4. Concluding Remarks

The research on queueing theory has been extensively developed due to a lot of significance in the problems relating with decision making process and it has made a tremendous impact in industry and logistic sector from its significant applications in many other areas like population studies, health sectors, manufacturing and production sections etc..In the present paper, we have considered $M^x/G/1$ queue with batch arrival and modified Bernoulli schedule server vacations. The concepts of server breakdown and server repair in two phases were incorporated into the model to make the study more realistic. In many congestion situations just before a service starts, the server has the option to control the queue by controlling the admissions of arriving batches. Such a model may find applications in many day to day real life queueing situations. Further our model assumes that the server vacations are based on Bernoulli schedule which means that just after completing a service selected by the customer, the server may take vacation of random length or may continue staying in the system. The concepts of Bernoulli schedule vacation, batch arrival and unreliable server have been incorporated together in our queueing model which has potential applicability in manufacturing, computer and communication systems, etc..

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