www.ijecs.in
International Journal Of Engineering And Computer Science ISSN:2319-7242
Volume 3 Issue 3 March, 2014 Page No. 5196-5199

On Jordan Triple Derivations of prime Γ-Rings

A.K.Joardar[1] *and A.C.Paul*[2]
[1] Department of Mathematics, Islamic University, Kushtia-7003, Bangladesh

e-mail : atishjoardar@yahoo.com

[2] Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh

e-mail : acpaulrubd_math@yahoo.com

Abstract

In this article, we develop some important results relating to the concepts of triple derivation and Jordan define triple derivation and Jordan triple derivation of gamma rings. Through every triple derivation of a gamma ring M is obviously a Jordan triple derivation of M, but the converse statement is in general not true. Here we prove that every Jordan triple derivation of a 2-torsion free prime gamma ring is a derivation.

Mathematics Subject Classification : Primary 16Y30, 16W25; Secondary 16U80.

Keywords: Derivation and triple derivation, Jordan triple derivation, gamma rings and prime rings.

Introduction

Let M and Γ be additive abelian groups. M is said to be a Γ-ring if there exists a mapping $M\times\Gamma\times M\rightarrow M$ (sending (x,α,y) into xαy) such that

(a) $(x+y)\alpha z = x\alpha z+y\alpha z$
$x(\alpha+\beta)y=x\alpha y+x\beta y$
$x\alpha(y+z)=x\alpha y+x\alpha z$
(b) $(x\alpha y)\beta z=x\alpha(y\beta z)$

For all $x,y,z \in M$ and $\alpha, \beta \in \Gamma$.

A subset A of a Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A=\{m\alpha a: m\in M, \alpha\in\Gamma \text{ and } a\in A\}$, $(A\Gamma M)$ is contained in A. An ideal P of a Γ-ring M is prime if $P\neq M$ and for any ideal A and B of M, $A\Gamma M\subseteq P$, then $A\subseteq P$ or $B\subseteq P$. M is prime if $a\Gamma M\Gamma b=0$ with $a,b\in M$, implies either a=0 or b=0. M is 2-torsion free if 2m=0, for $m\in M$ implies m=0.

Let R be an associative ring. An additive mapping $d: R\rightarrow R$ is called a Triple derivation if

$$d(abc)=d(a)bc + ad(b)c + abd(c)$$

and Jordan Triple derivation if $d(aba)=d(a)ba + ad(b)a + abd(a)$.

It is clear that every triple derivation is a jordan triple derivation but the converse is not in general true.

Bell and Koppe [2] worked on triple derivations and developed some remarkable fruitful result on the classical rings. They also prove that every Jordan triple derivation is a derivation if R is a 2-torsion free prime ring.

M. Bresar [3] worked on Jordan triple derivations of semiprime rings and he proved that if R is a 2-torsion free semiprime ring, then every Jordan triple derivation is a derivation.

We Jing and Shijie [6] defined generalized Jordan triple derivation. They showed that every generalized Jordan triple derivation is a generalized derivation.

N. Nobusawa [5] was first introduced the notion of gamma ring. The gamma ring due to N. Nobusawa is now denoted by Γ_N-ring. Next Barnes [1] generalized it and gave the above definition. It is clear that every ring is a gamma ring.

M. Sapanci and A. Nakajima [4] worked on Jordan derivation on completely prime gamma rings. They prove that every Jordan derivation on a two torsion free completely prime gamma rings is a derivation.

In this paper, we define triple derivation and Jordan triple derivation of a gamma ring. We give an example of triple derivation and Jordan triple derivation for gamma rings. We also prove that every Jordan triple derivation is a derivation if it is a two torsion free prime Γ-ring.

2 Jordan triple derivation

Let M be a Γ ring, An additive mapping d: M→M is called a triple derivation if d(aαbβc)=d(a)αbβc + aαd(b)βc + aαbβd(c) for every a,b,c ∈ M and α, β ∈ Γ.

An additive mapping d: M→M is called a Jordan triple derivation if d(aαbβa)=d(a)αbβa + aαd(b)βa + aαbβd(a) for every a,b,c ∈ M and α, β ∈ Γ.

It is clear that every triple derivation is a jordan triple derivation. But every Jordan triple derivation is not in general a triple derivation.

Now we give the following examples:

2.1 Example

Let R be an associative ring with unity element 1. Let M = $M_{1,2}$(R) and

Γ=$\{\binom{n.1}{0}, n \in Z)\}$. Then M is a Γ-ring. Let d: R→R be a derivation. Now define D((x,y))=(d(x),d(y)). Then we show that D is a triple derivation associated to jordan derivation d. For this, let a=$(x_1\ ,y_1)$, b=$(x_2\ ,y_2)$, c=$(x_3\ ,y_3)$, α=$\binom{n_1.1}{0}$, β=$\binom{n_2.1}{0}$. We have to prove that D(aαbβc)=D(a)αbβc + aαD(b)βc + aαbβD(c). Now we have aαbβc=$(x_1n_1x_2n_2x_3\ ,x_1n_1x_2n_2y_3\ )$. So D(aαbβc)=(d($x_1n_1x_2n_2x_3$), d($x_1n_1x_2n_2y_3$)). Similarly, we get D(a)αbβc + aαD(b)βc + aαbβD(c)= (d($x_1n_1x_2n_2x_3$), d($x_1n_1x_2n_2y_3$)).

2.2 Example

Let M be a Γ-ring defined as an example 2.1. Let N ={(x, x) : xϵM}. Then N is a Γ-ring contained in M. Let d be a derivation given in example 2.1. Define D: N→N by D((x,x))=(d(x),d(x)). Then we show that D is a Jordan triple derivation. Note that it is not a triple derivation.

To show this, let a=(x, x), b=(y, y), α=$\binom{n_1.1}{0}$, β=$\binom{n_2.1}{0}$. We have to prove D(aαbβa)=D(a)αbβa + aαD(b)βa + aαbβD(a). Now we have aαbβa=(xn_1yn_2x, xn_1yn_2y)

So D(aαbβa)=$(d(xn_1yn_2x), d(xn_1yn_2y)$). Similarly, we get D(a)αbβa + aαD(b)βa + aαbβD(a)= $(d(xn_1yn_2x), d(\ xn_1yn_2x)$).

Now we prove some lemma which are essential to prove our main theorem.

Lemma 2.1 : Let M be a Γ-ring and d be a Jordan triple derivation of a Γ-ring M. Then for all a,b,c ϵ M d(aαbβc + cαbβa) = d(a)αbβc + d(c)αbβa + aαd(b)βc + cαd(b)βa + aαbβd(c)+ cαbβd(a).

Proof: Computing d((a + c)αbβ(a+c)) and canceling the like terms from both sides, we prove the lemma.

Definition 1: Let M be a Γ-ring. Then for all a, b, c ϵM and α, β ϵ Γ we define

$$[a, b, c]_{\alpha, \beta}= a\alpha b\beta c-c\alpha b\beta a.$$

Lemma 2.2 : If M is a Γ-ring, then for all a, b, c ϵM and α, β ϵ Γ

(1) $[a, b, c]_{\alpha, \beta} +[c, b, a]_{\alpha, \beta} =0$

(2) $[a+c, b, d]_{\alpha, \beta} = [a, b, d]_{\alpha, \beta} + [c, b, d]_{\alpha, \beta}$
(3) $[a, b, c+d]_{\alpha, \beta} = [a, b, c]_{\alpha, \beta} + [a, b, d]_{\alpha, \beta}$
(4) $[a, b+d, c]_{\alpha, \beta} = [a, b, c]_{\alpha, \beta} + [a, d, c]_{\alpha, \beta}$
(5) $[a, b, c]_{\alpha+\beta, \gamma} = [a, b, c]_{\alpha, \gamma} + [a, b, c]_{\beta, \gamma}$
(6) $[a, b, c]_{\alpha, \beta+\gamma} = [a, b, d]_{\alpha, \beta} + [a, b, c]_{\alpha, \gamma}$

Proof : Obvious

Definition 2 : Let d be a Jordan triple derivation of a Γ-ring M. Then for all a, b, c ϵM and α, β ϵ Γ we define $G_{\alpha, \beta}(a\alpha b\beta c) = d(a\alpha b\beta c) - d(a)\alpha b\beta c - a\alpha d(b)\beta c - a\alpha b\beta d(c)$.

Lemma 2.3 : Let d be a Jordan triple derivation of a Γ-ring M. Then for all a, b, c ϵM and α, β ϵ Γ, we have

(1) $G_{\alpha, \beta}(a\alpha b\beta c) + G_{\alpha, \beta}(c\alpha b\beta a) = 0$
(2) $G_{\alpha, \beta}((a+c)\alpha b\beta d) = G_{\alpha, \beta}(a\alpha b\beta d) + G_{\alpha, \beta}(c\alpha b\beta d)$
(3) $G_{\alpha, \beta}(a\alpha b\beta(c+d)) = G_{\alpha, \beta}(a\alpha b\beta c) + G_{\alpha, \beta}(a\alpha b\beta d)$
(4) $G_{\alpha, \beta}(a\alpha(b+c)\beta d) = G_{\alpha, \beta}(a\alpha b\beta d) + G_{\alpha, \beta}(a\alpha c\beta d)$
(5) $G_{\alpha+\beta, \gamma}(a\alpha b\beta c) = G_{\alpha, \gamma}(a\alpha b\beta c) + G_{\beta, \gamma}(a\alpha b\beta c)$
(6) $G_{\alpha, \beta+\gamma}(a\alpha b\beta c) = G_{\alpha, \beta}(a\alpha b\beta c) + G_{\alpha, \gamma}(a\alpha b\beta c)$

Proof : Obvious

Lemma 2.4 : If M is a Γ-ring, then

$G_{\alpha, \beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha, \beta} + [a, b, c]_{\alpha, \beta}\gamma x\delta G_{\alpha, \beta}(a\alpha b\beta c) = 0$ for all x ϵM and γ, δ ϵ Γ

Proof : First we compute d(aα(bβcγxδcαb)βa + cα(bβaγxδaαb)βc) by using the definition of Jordan triple derivation we get d(a)αbβcγxδcαbβa + aαd(b)βcγxδcαbβa + aαbβd(c)γxδcαbβa + aαbβcγd(x)δcαbβa + aαbβcγxδd(c)αbβa + aαbβcγxδcαd(b)βa + aα(bβcγxδcαbβd(a) + d(c)αbβaγxδaαbβc + cαd(b)βaγxδaαbβc + cαbβd(a)γxδaαbβc + cαbβaγd(x)δaαbβc + cαbβaγxδd(a)αbβc + cαbβaγxδaαd(b)βc + cαbβaγxδaαbβd(c). On the other hand, we have d((aαbβc)γxδ(cαbβa) +(cαbβa)γxδ(aαbβc)) and using lemma 2.1, we get d(aαbβc)γxδcαbβa + d(cαbβa)γxδaαbβc + aαbβcγd(x)δcαbβa + cαbβaγd(x)δaαbβc + aαbβcγxδd(cαbβa) + cαbβaγxδd(aαbβc). Since these two are equal, cancelling the like terms from both sides of this equality and rearranging them,we get

$G_{\alpha, \beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha, \beta} + [a, b, c]_{\alpha, \beta}\gamma x\delta G_{\alpha, \beta}(a\alpha b\beta c) = 0$

Lemma 2.5 : If M is a prime Γ-rings, then the following is true

(1) If I and J are non zero left (or right) ideals of M, then I ΓJ≠0.
(2) If I is a non zero left (or right) ideal of M, then Ann_l(I)=0 (respectively, Ann_r(I)=0).

Proof: For all 0≠x∈I and 0≠y∈J, then we have 0≠xΓMΓyΓIΓMΓJ⊂IΓJ (since IΓM⊂I). Since IΓM is a non zero ideal of M and 0=Ann_l(I) Γ(IΓM)= Ann_l(I) ΓIΓM ⊂Ann_l(I) ΓMΓM. Since M is prime, we have Ann_l(I)=0.

Lemma 2.6 : let M be a 2-torsion free semiprime Γ-ring and suppose that a, b ϵM. If aΓmΓb + bΓmΓa = 0 for all m ϵ M, then either a = 0 or b=0.

Proof : Since aαmβb+bαmβa=0 for all α,β∈Γ. Putting pγaδq by m we get 2aαpγbδqβa=0. Since M is a 2-torsion free, then aαpγbδqβa=0 for all p,qϵM if a≠0, then MΓa is a non zero left ideal. Hence by the above lemma we get aΓMΓb=0, for which it yields b=0.

Lemma 2.7 : Let M is a 2-torsion free prime Γ-ring. Then for all a, b, c, xϵM and α, β, γ, δ ϵ Γ. Then $G_{\alpha, \beta}(a\alpha b\beta c)=0$ or $[a, b, c]_{\alpha, \beta =0}$.

Proof : From lemma 2.4 we get $G_{\alpha, \beta}(a\alpha b\beta c)\gamma x\delta[a, b, c]_{\alpha, \beta} + [a, b, c]_{\alpha, \beta}\gamma x\delta G_{\alpha, \beta}(a\alpha b\beta c) = 0$. Now by above lemma, we get $G_{\alpha, \beta}(a\alpha b\beta c)=0$ or $[a, b, c]_{\alpha, \beta}=0$.

Now we have the position to prove our main theorem

Theorem 2.1 : Let M is a 2-torsion free prime Γ-ring, then every Jordan triple derivation is a triple derivation.

Proof : By lemma 2.7, we have $G_{\alpha, \beta}(a\alpha b\beta c)=0$ or $[a, b, c]_{\alpha, \beta}=0$.

Case 1: Suppose $[a, b, c]_{\alpha, \beta}=0$, then aαbβc=cαbβa. Therefore from lemma 2.1, d(aαbβc)=d(a)αbβc + aαd(b)βc + aαbβd(c) i.e. Jordan triple derivation is a triple derivation.

Case 2: Suppose $G_{\alpha, \beta}$(aαbβc)=0 then d(aαbβc)=d(a)αbβc + aαd(b)βc + aαbβd(c). i.e. Jordan triple derivation is a triple derivation.

Theorem 2.2 : Any Jordan triple derivation of a 2-torsion free prime Γ-ring is a derivation.

Proof : Consider w=d(aα(bγxδa)αb)

= d(a)αbγxδaαb + aαd(bγxδa)αb + aαbγxδaαd(b)

= d(a)αbγxδaαb + aαd(b)γxδaαb + aαbγd(x)δaαb + aαbγxδd(a)αb + aαbγxδaαd(b)

Again w= d((aαb)γxδ(aαb))= d(aαb)γxδaαb + aαbγd(x)δaαb + aαbγxδd(aαb)

Comparing the two expression we obtain (d(aαb) –d(a)αb-aαd(b)) γxδaαb+aαb γxδ(d(aαb) –d(a)αb-aαd(b))=0. Again by primeness of M , d(aαb) – d(a)αb-aαd(b)=0, i.e. d is derivation.

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