Shape Analysis of the Quintic Trigonometric Bèzier curve with two shape parameter

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Abstract: A quintic trigonometric Bèzier curve with two shape parameters, is presented in this work. The shape of the curve can be adjusted as desired, by simply altering the value of shape parameter, without changing the control polygon. The quintic trigonometric Bèzier curve can be made close to the quintic Bèzier curve or closer to the given control polygon than the quintic Bèzier curve.

Keywords: Trigonometric Bèzier Basis Function, Shape Parameter, Open curves, Close curves.

1. Introduction

Trigonometric splines are an important class of splines, which were first discovered by schoenberg [8]. In recent years, in geometrical modelling, for the development of the CAD/CAM software tools special attention were maid in the application of trigonometric splines. With the help of these observations, the problems of surface modelling can be handled more easily by trigonometric splines, specially problems relating to the data fitting on spherical objects. In recent years, trigonometric splines and polynomials plays a very important role in the Computer Aided Geometric Design (CAGD), specially in curve design: see [2], [3], [4], [5], [6], [10], [11].

The theory of the Bèzier curves is key feature in CAGD. These are considered as ideal geometric standard for the representation of piecewise polynomial curves. In recent years, trigonometric polynomial curves like Bèzier type are taken in discussion. A cubic trigonometric Bèzier curve with two shape parameters was discussed by Han et al [7]. It enjoyed all the geometric properties of the ordinary cubic Bèzier curve and was used for spur gear tooth design with S-shaped transition curve Abbas et al [1]. A study on class of TC- Bèzier curve with shape parameters was presented by Liu, et al [9].

The paper is organized as follows. In section 2, the basis functions of the quintic trigonometric Bèzier curve with two shape parameters are established and the properties of the basis function has been described. In section 3, quintic trigonometric Bèzier curves and their properties are discussed. In section 4, By using shape parameter, shape control of the curves is studied and open, closed quintic trigonometric Bèzier curves are presented.

2. Quintic Trigonometric Bèzier Basis Functions

In this section, definition and some properties of quintic trigonometric Bèzier basis functions with two shape parameters are given as follows:

Definition 2.1: For two arbitrarily real values of λ and μ, where λ, μ ∈ [−1,1] the following six functions of t(t ∈ [0,1]) are defined as quintic trigonometric Bèzier basis functions with two parameters λ and μ:

\[ b_0(t) = \left(1 - \sin \frac{\pi}{2} t\right)^3 \left(1 - \lambda \sin \frac{\pi}{2} t\right)^2 \]
\[ b_1(t) = \sin \frac{\pi}{2} t \left(1 - \sin \frac{\pi}{2} t\right)^2 \left(1 - \lambda \sin \frac{\pi}{2} t\right) \left(1 + \lambda \sin \frac{\pi}{2} t\right) \]
\[ b_2(t) = \sin \frac{\pi}{2} t \left(1 - \sin \frac{\pi}{2} t\right) \left(2 + \lambda - \lambda \sin \frac{\pi}{2} t\right) \]
\[ b_3(t) = \cos \frac{\pi}{2} t \left(1 - \cos \frac{\pi}{2} t\right)^2 \left(2 + \mu - \mu \cos \frac{\pi}{2} t\right) \]
\[ b_4(t) = \cos \frac{\pi}{2} t \left(1 - \cos \frac{\pi}{2} t\right)^2 \left(1 - \mu \cos \frac{\pi}{2} t\right) \left(1 + \mu \cos \frac{\pi}{2} t\right) \]
\[ b_5(t) = \left(1 - \cos \frac{\pi}{2} t\right)^3 \left(1 - \mu \cos \frac{\pi}{2} t\right)^2 \]

For \(\lambda = \mu = 0\), the basis functions are cubic trigonometric polynomials. For \(\lambda, \mu \neq 0\), the basis functions are quintic trigonometric polynomials.
Theorem 2.1: The basis functions (2.1) have the following properties:

(a) Nonnegativity: \( b_i(t) \geq 0, i = 0,1,2,3,4,5. \)
(b) Partition of Unity: \( \sum_{i=0}^{5} b_i(t) = 1. \)
(c) Monotonicity: For a given parameter \( t, b_0(t) \) and \( b_5(t) \) are monotonically decreasing \( \lambda \) and \( \mu \) respectively; \( b_1(t) \) and \( b_4(t) \) are monotonically increasing for the shape parameters \( \lambda \) and \( \mu \) respectively.
(d) Symmetry: \( b_i(t; \lambda, \mu) = b_{5-i}(1-t; \mu, \lambda), \) for \( i=0,1,2,3,4,5. \)

Proof: (a) For \( t \in [0,1] \) and \( \lambda, \mu \in [-1,1], \) then 
\[
1 - \sin \pi t \geq 0,1 - \lambda \sin \pi t \geq 0,1 - \sin \pi t \geq 0,1 - \mu \sin \pi t \geq 0,1 - \mu \sin \pi t \geq 0.
\]
It is obvious that \( b_i(t) \geq 0, i = 0,1,2,3,4,5. \)
(b) \( \sum_{i=0}^{5} b_i(t) = \left(1 - \sin \frac{\pi}{2}t\right)^3 \left(1 - \sin \frac{\pi}{2}t\right)^2 + \sin \frac{\pi}{2}t \left(1 - \sin \pi t - \lambda \sin \pi t + \mu \sin \pi t \right) \geq 1. \)

The remaining cases follow obviously.

Fig. 1. shows the curves of the quintic trigonometric basis function for \( \lambda = \mu = 1 \) (red solid) and \( \lambda = \mu = 1 \) (blue dashed).

3. Qunitic trigonometric Bézier curve

We constant the quintic trigonometric Bézier curve with two shape parameters as follows:

**Definition 3.1:** Given the control points \( P_i(i = 0,1,2,3,4,5) \) in \( R^2 \) or \( R^3, \) then

\[
C(t) = \sum_{i=0}^{5} P_i b_i(t), \quad t \in [0,1], \lambda, \mu \in [-1,1].
\]

The curve defined by (3.1) possesses some properties which can be obtained easily from the properties of the basis function.

**Theorem 3.1:** The Quintic trigonometric Bézier curve (3.1) have the following properties.

(a) End point properties:
\[
C(0) = P_0, \quad C(1) = P_5.
\]
\[
C'(0) = \pi \left[(3 + 2 \lambda)P_0 - (1 + \lambda)P_1 - (2 + \lambda)P_2 \right],
\]
\[
C'(1) = \pi \left[(3 + 2 \mu)P_5 - (1 + \mu)P_4 - (2 + \mu)P_3 \right].
\]

(b) Symmetry: The control points \( P_i \) and \( P_{5-i} \) define the same curve in different parametrizations, that is \( C(t; \lambda, \mu, P_i) = C(1-t; \mu, \lambda, P_{5-i}), \) \( t \in [0,1], \lambda, \mu \in [-1,1]. \)

(c) Geometric invariance: The shape of the curve (3.1) is independent of the choice of coordinates, i.e., for \( (t = 0,1,2,3,4,5, \lambda, \mu), \) it satisfies the following two equation:
\[
C(t; \lambda, \mu, P_i) + q = C(t; \lambda, \mu, P_i) + q,
\]
where \( q \) is an arbitrary vector in \( R^2 \) or \( R^3, T \) is an arbitrary \( d \times d \) matrix, \( d = 2 \) or \( 3. \)

(d) Convex hull property: From the non-negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by its control points \( P_i. \)

![Figure 2: The quintic trigonometric Bézier curves with two shape parameters](image)

4. Shape control of the quintic Trigonometric Bézier curve

The parameters \( \lambda \) and \( \mu \) controls the shape of the curve (3.1). In figures 2, the Quintic trigonometric Bézier curve \( C(t) \) gets closer to the control polygon as the values of the parameters \( \lambda \) and \( \mu \) increases. In figures 2, the curves are generated by setting the values of \( \lambda, \mu \) as \( \lambda = \mu = -1 \) (black dash-dotted lines), \( \lambda = \mu = 0 \) (blue dashed lines), \( \lambda = \mu = 1 \) (red solid lines).

In figures 3(a), the curves are generated by changing \( \lambda \) to \( \lambda = -1 \) (black dash-dotted lines), \( \lambda = 0 \) (blue dashed lines), \( \lambda = 1 \) (red solid lines), and setting \( \mu = 1. \) In figure 3(b), the curves are generated by changing \( \mu \) to \( \mu = -1 \) (black dash-dotted lines), \( \mu = 0 \) (blue dashed lines), \( \mu = 0 \) (red solid lines), and setting \( \lambda = 1. \)

In figures 4, the curves are generated by changing \( \lambda, \mu \) to \( \lambda = \mu = -1 \) (black dash-dotted lines), \( \lambda = \mu = 0 \) (blue dashed lines), \( \lambda = \mu = 0 \) (red solid lines).
In order to construct a closed quintic trigonometric Bézier curves, we can set $P_n = P_0$. In figure 5(a), The closed quintic trigonometric Bézier curves of altering the values of the shape parameters $\lambda$ and $\mu$ at the same time. The quintic trigonometric Bézier curves are generated by setting $\lambda = -1, \mu = -1$ (black solid lines), $\lambda = 0, \mu = 0$ (blue dashed lines) and $\lambda = 1, \mu = 1$ (red solid lines).

In order to construct a open quintic trigonometric Bézier curves, we can set $P_n \neq P_0$. In figure 5(b), The open quintic trigonometric Bézier curves of altering the values of the shape parameters $\lambda$ and $\mu$ at the same time. The quintic trigonometric Bézier curves are generated by setting $\lambda = -1, \mu = -1$(black solid lines), $\lambda = 0, \mu = 0$ (blue dashed lines) and $\lambda = 1, \mu = 1$ (red solid lines).

In figure 6, Now we show some relation of the quintic trigonometric Bézier curves and quintic Bézier curves corresponding to their control polygons. From figure 6(a) and 6(b), we can see that the quintic trigonometric Bézier curve is closer to the given control polygon than the quintic Bézier curve for some $\lambda$ and $\mu$.

5. Conclusion
In this paper, we have presented the quintic trigonometric Bézier curve with two shape parameters. Each section of the curve only refers to the six control points. We can design different shape curves by changing parameters. The proposed curve can be used to generate open and close curves. In future, it can be extended to tensor surfaces.

References
