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Gauss Legendre Quadrature Formulas for Polygons Using an All Quadrilateral Finite Element Mesh of Triangular Surfaces

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Abstract

This paper presents a numerical integration formula for the evaluation of $II_{\Omega}(f) = \iint_{\Omega} f(x,y)dxdy$, ***where*** $f \in C(\Omega)$ ***and*** Ω ***is any polygonal domain in*** $\Re^2$ ***That is a domain with boundary composed of piecewise straight lines.We then express*** $II_{P_N}(f) = \sum_{n=1}^{M} II_{T_n}(f) = \sum_{n=1}^{M}(\sum_{a=0}^{2}\sum_{b=1}^{4} II_{Q_{3n-a,b}}(f))$ ***in which*** P_N ***is a polygonal domain of N oriented edges*** l_{ik} $(k = i+1,\ i = 1,2,3,...,N)$, ***with end points*** $(x_i, y_i),\ (x_k, y_k)$ ***and*** $(x_1, y_1) = (x_{N+1}, y_{N+1})$. ***We have also assumed that*** P_N ***can be discretised into a set of M triangles,*** T_n ***and each triangle*** T_n ***is further discretised into three special quadrilaterals*** $Q_{3n-p}\ (p = 0,1,2)$ ***which are obtained by joining the centroid to the midpoint of its sides. We choose*** $T_n = T_{pqr}^{xy}$ ***an arbitrary triangle with vertices*** $((x_\alpha, y_\alpha),\ \alpha = p,q,r)$ ***in Cartesian space*** (x,y). ***We have shown that an efficient formula for this purpose is given by***

$$II_{T_n}(f) = (c_{pqr})\iint_S (4+\xi+\eta)\left(\sum_{e=1}^{3} f(x^{(e)}(u,v), y^{(e)}(u,v))\right)d\xi d\eta,$$

where, $u = u(\xi,\eta) = (1-\xi)(5+\eta)/24,\ v = v(\xi,\eta) = (1-\eta)(5+\xi)/24$

$$z^{(e)}(u,v) = z_1^{(e)} + (z_2^{(e)} - z_1^{(e)})u + (z_3^{(e)} - z_1^{(e)})v,\ \ z = (x,y)$$

$$\left((z_1^{(e)}, z_2^{(e)}, z_3^{(e)}), e = 1,2,3\right) = \left((z_p, z_q, z_r),(z_q, z_r, z_p),(z_r, z_p, z_q)\right)$$

$$c_{pqr} = (area\ of\ T_n)/48,$$

and $S = \{(\xi,\eta)/-1 \le \xi,\eta \le 1\}$ ***is the standard 2 square in*** (ξ,η) ***space. This 2 square S in*** (ξ,η) ***is discritised into four 1squares in (ξ,η) space.We then use four linear transformations*** $\xi = \xi^b(r,s), \eta = \eta^b(r,s), b = 1,2,3,4$ ***to map these 1 squares into 2 squares in*** (r,s) ***space. Further on using the Gauss Legendre Quadrature Rules of order 5(5)40, we obtain the weight coefficients and sampling points which can be used for any polygonal domain,*** $\Omega = P_N$ ***or*** T_n ***or*** $Q_{m,b}$ ***(m=3n-2,3n-1,3n;b=1,2,3,4)***

The present composite integration scheme integrates gives the same accuracy for half the number of triangles for each discretisation of a polygon used in our earlier work[16].

Keywords: Polygonal domain, Triangular and Convex Quadrilateral regions, Finite Element Mesh,Gauss Legendre Quadrature , Numerical Integration.

1 Introduction

The finite element method is a computational scheme to solve field problems in engineering and science. The technique has very wide application, and has been used on problems involving *stress analysis, fluid mechanics, heat transfer, diffusion, vibrations, electrical and magnetic fields,* etc. The fundamental concept involves dividing the body under study into a finite number of pieces (subdomains) called *elements* (see Figure). Particular assumptions are then made on the variation of the unknown dependent variable(s) across each element using so-called *interpolation or approximation functions*. This approximated variation is quantified in terms of solution values at special element locations called *nodes*. Through this discretization process, the method sets up an algebraic system of equations for unknown nodal values which approximate the continuous solution. Because element size, shape and approximating scheme can be varied to suit the problem, the method can accurately simulate solutions to problems of complex geometry and loading and thus this technique has become a very useful and practical tool

The finite element method is one of the most powerful computational technique for approximate solution of a variety of "real world" engineering and applied science problems for over half a century since its inception in the mid 1960. Today, finite element analysis (FEA) has become an integral and major component in the design or modelling of a physical phenomenon in various disciplines. The triangular and quadrilateral elements with either straight sides or curved sides are very widely used in a variety of applications [1-3]. The basic problem of integrating a function of two variables over the surface of the triangle is the subject of extensive research by many authors [4-5]. Derivation of high precision formulas is now possible over the triangular region by application of product formulas based only on the sampling points and weights of the well known Gauss Legendre quadrature rules [6-8]. There are reasons which support the development of composite integration for practical applications. In some recent investigations composite integration is illustrated with reference to the standard triangle [9-10]. Recently, in [11] Green's integral formula is used in the numerical evaluation of $II_{\Omega}(f)=\iint_{\Omega} f(x,y)dxdy$ by transforming a two dimensional problem into a one dimensional problem and by using univariate Gauss Legendre quadrature products. In [12], a cubature formula over polygons is proposed which is based on a 8-node spline finite elements. They use very dense meshes to prove the convergence of test function integrals for which error is shown to be in the range of 10^{-1} to 10^{-9}. In this paper we develop composite integration rules for polygonal domains which are fully discretised by special quadrilaterals and the test function integrals are shown to agree with the exact values up to 16 significant digits for smooth functions, this implies that the absolute error is of the order 10^{-16}. The composite integration rules of this paper as well as the cubature formulas of 8- node spline elements [12] converge to the exact values a little slowly for some nonsmooth functions**.** This again confirms the superiority of product formulas. In section 2 of this paper, we begin with a brief description of the special discretisation of arbitrary and the standard (right isosceles) triangular elements into a set of three special quadrilaterals which are obtained by joining the centroids to the midpoints of sides. In section 3 of this paper, we define some relevant linear transformations. In section 3.1, we prove lemma 1, which establishes the relation between the special quadrilaterals of an arbitrary triangle in (x,y) space and the special quadrilaterals of the standard triangle in (u,v) space by use of a single linear transformation between the global space (x,y) and the local parametric space (u,v). Then in section 3.2, we prove lemma 2 which establishes the relation between the three special quadrilaterals in (x,y) and a unique special quadrilateral interior to the standard triangle in (u,v) space by using three linear transformations. Section 4 of this paper is regarding the explicit form of the Jacobians. In section 4.1, we determine the explicit form of Jacobian when the arbitrary triangle in the global space (x,y) is mapped into a standard triangle in the local space (u,v) for the linear transformations used in lemma 1 and lemma 2. Section 4.2 of the paper begins with the derivation of explicit form of Jacobian for an arbitrary linear convex quadrilateral. In section 4.3, we determine the Jacobian for the special quadrilaterals $Q_e\,(e=1,2,3)$ in the global space (x,y) and the $\hat{Q}_e\,(e=1,2,3)$ in the local space (u,v), in either case we obtain the Jacobian as $c(4+\xi+\eta)$, where c is some appropriate constant. We prove this result in lemma 3 when the $\hat{Q}_e\,(e=1,2,3)$ are mapped into 2-squares $-1\le\xi,\eta\le+1$. In section 5, we establish two composite integration formulas which use lemmas 1, 2 and 3 proved in sections 3.1, 3.2 and 4.3. In section 5.1, we establish a composite integration formula which uses three bilinear transformations and a single linear transformation and in section 5.2, we also establish a composite integration formula which depends on three linear transformations and a single bilinear transformation. We see that composite integration formulas of section 5.1 are of the form $x(u^e(\xi,\eta),v^e(\xi,\eta)),\ y(u^e(\xi,\eta),v^e(\xi,\eta)),\ e=1,2,3$ and require the computation of three sets (u^e,v^e), $e=1,2,3$ whereas the composite formulas of section 5.2 are of the form $x^e(u^1(\xi,\eta),v^1(\xi,\eta))$, $y^e(u^1(\xi,\eta),v^1(\xi,\eta))$, and require the computation of one set $(u^1(\xi,\eta),v^1(\xi,\eta))$. Thus we prefer to use composite integration formula of section 5.2. The composite integration formulas of sections 5.1 or 5.2 is based on the discritisation of the original arbitrary triangle into 3- special quadrilaterals which are created by joining the centroid of the triangle to the midpoints of its sides. The new composite integration formula derived in section 5.3 takes this concept further and creates in all 12-new quadrilaterals by joining the centroids of the 3-special quadrilaterals to the midpoints of the sides.The derivation of this new composite integration formula first uses the centroid of the triangle and then the centroid of the 3-special quadrilaterals.These new 12-quadrilaterals have a unform angular variation in the four corners.This further allows uniformity in the spread of Gauss integration points and the convergence is faster than our earlier work[ ] based on 3-special quadrilaterals over a triangle for the composite integration formula.

In section 6, we present the composite numerical integration formulas. We may note that the problem domain must be discretised into special quadrilaterals. The problem domain must contain at least one triangle for this purpose. The composite integration formulas are then obtained by application of Gauss Legendre quadrature rules [4] to the formula established in section 5.3. In In section 8, we consider the evaluation of some typical integrals. This demonstrates the efficiency of the derived formulas of last section. We have also appended the relevant and necessary computer codes.

2 A Special Discretization of Triangles

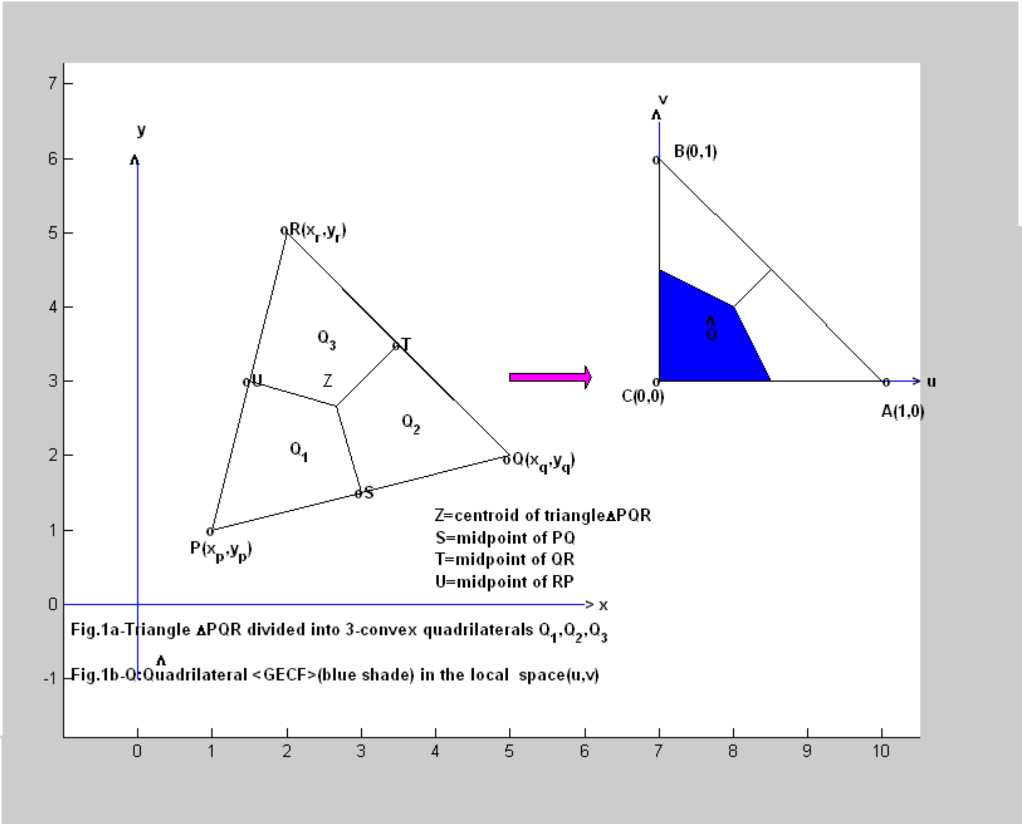
In this section, we describe a special discretisation scheme to generate quadrilaterals from triangles. In the proposed scheme, three unique quadrilaterals are obtained by joining the centroid of any triangle to the midpoints of its sides. We define such quadrilaterals as special quadrilaterals for our present investigations.

2.1 Special Quadrilaterals of an Arbitrary Triangle

We first consider an arbitrary triangle ΔPQR in the Cartesian space (x,y) with vertices $P(x_p,y_p)$, $Q(x_q,y_q)$ and $R(x_r,y_r)$. Let $Z((x_p+x_q+x_r)/3,\ (y_p+y_q+y_r)/3)$ be its centroid and also let *S*, *T*, *U* be the mid points of sides *PQ*, *QR* and *RP* respectively. Now by joining the centroid Z to the midpoints *S*, *T*, *U* by straight lines, we divide the triangle ΔPQR into three special quadrilaterals Q_1, Q_2 and Q_3 (say) which are spanned by vertices $\langle Z, U, P, S\rangle$, $\langle Z, S, Q, T\rangle$, and $\langle Z, T, R, U\rangle$ respectively.This is shown in Fig.1a.

2.2 Special Quadrilaterals of a Standard Triangle

We next consider the triangle ΔABC in the Cartesian space (u,v) with vertices, centroid and midpoints: $A(1,0),\ B(0,1),\ C(0,0),\ G(1/3,1/3),\ D(1/2,1/2), E(0,1/2)$ and $F(1/2,0)$. We now divide the triangle ΔABC into three special quadrilaterals $\hat{Q}_1,\ \hat{Q}_2$, and $\hat{Q}_3$ (say) which are spanned by vertices $\langle G, E, C, F\rangle$, $\langle G, F, A, D\rangle$, and $\langle G, D, B, E\rangle$ respectively. This is shown in Fig.1b.



3 Linear Transformations

We apply linear transformations to map an arbitrary triangle into a triangle of our choice. In this section, we use the well known linear transformation which maps an arbitrary triangle into a standard triangle (a right isosceles triangle). We also assume the special discretization scheme of the previous section for the following developments.

3.1 Lemma 1. There exists a unique linear transformation which map the special quadrilaterals Q_i into $\hat{Q}_i$ (i =1, 2, 3) satisfying the conditions

(i) $\sum_{i=1}^{3} Q_i = \Delta PQR$, the arbitrary triangle in the (x, y) space.

(ii) $\sum_{i=1}^{3} \hat{Q}_i = \Delta ABC$, the standard triangle (right isosceles) in the (u,v) space.

Proof: We shall now refer to Fig.1a, 1b and Fig.1c, 1d and consider the following linear transformation between (x, y) and (u,v) spaces.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} w + \begin{pmatrix} x_q \\ y_q \end{pmatrix} u + \begin{pmatrix} x_r \\ y_r \end{pmatrix} v, \quad w = 1 - u - v \tag{1}$$

We can verify that the linear transformation of eqn. (1) maps the arbitrary triangle ΔPQR into the standard triangle ΔABC. The points *P*, *Q*, *R*, *S*, *T*, *U* and *Z* are respectively mapped into the points *A*, *B*, *C*, *D*, *E*, *F* and *G* respectively. The quadrilaterals Q_i are mapped into quadrilaterals $\hat{Q}_i$. This proves the existence of the required transformation.

3.2 Lemma 2. There exists three linear transformations which map the special quadrilaterals Q_i (i =1,2,3) in ΔPQR into a unique special quadrilateral $\hat{Q} = \hat{Q}_1$ (say) of the standard triangle ΔABC satisfying the conditions

(i) $\sum_{i=1}^{3} Q_i = \Delta PQR$, the arbitrary triangle in the (x, y) space.

(ii) $\sum_{i=1}^{3} \hat{Q}_i = \Delta ABC$, the standard triangle (right isosceles) in the (u, v) space.

Proof: We again refer to Fig.1a, 1b and consider the following linear transformations between (x, y) and (u, v) spaces.

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} w + \begin{pmatrix} x_q \\ y_q \end{pmatrix} u + \begin{pmatrix} x_r \\ y_r \end{pmatrix} v, \quad w = 1 - u - v \tag{2}$$

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \end{pmatrix} w + \begin{pmatrix} x_r \\ y_r \end{pmatrix} u + \begin{pmatrix} x_p \\ y_p \end{pmatrix} v, \quad w = 1 - u - v \tag{3}$$

$$\begin{pmatrix} x^{(3)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \end{pmatrix} w + \begin{pmatrix} x_p \\ y_p \end{pmatrix} u + \begin{pmatrix} x_q \\ y_q \end{pmatrix} v, \quad w = 1 - u - v \tag{4}$$

It is quite clear that each of the above transformations map the arbitrary triangle ΔPQR into the standard triangle ΔABC. We may further note the following.

(i) The transformation of eqn. (2) maps the vertices $P,\ Q,\ R$ in (x, y) space into vertices $C(0,0),\ A(1,0),\ B(0,1)$ in (u, v) space.

(ii) The transformation of eqn. (3) maps the vertices $Q,\ R,\ P$ in (x, y) space into vertices $C(0,0),\ A(1,0),\ B(0,1)$ in (u, v) space.

(iii) The transformation of eqn. (4) maps the vertices $R,\ P,\ Q$ in (x, y) space into vertices $C(0,0),\ A(1,0),\ B(0,1)$ in (u, v) space.

We can now verify that the linear transformation of eqn. (2) maps the quadrilateral Q_1 spanning the vertices $\langle Z, U, P, S \rangle$ in (x, y) space into the quadrilateral $\hat{Q} = Q_1$ spanning the vertices $\langle G, E, C, F \rangle$ in the (u, v) space. In a similar manner, we find that using the linear transformation of eqn. (3) the quadrilateral Q_2 spanned by vertices $\langle Z, S, Q, T \rangle$ in (x, y) space is mapped into the quadrilateral $\hat{Q} = Q_1$ spanning the vertices $\langle G, E, C, F \rangle$ in the (u, v) space. Finally on using the linear transformation of eqn. (4) the quadrilateral Q_3 spanned by vertices $\langle Z, T, R, U \rangle$ in (x, y) space is mapped into the quadrilateral $\hat{Q} = Q_1$ spanning the vertices $\langle G, E, C, F \rangle$ in the (u, v) space. This completes the proof of Lemma 2.

We may note here that the linear transformations $(x^{(1)}, y^{(1)})^T$ in eqn. (2) and $(x, y)^T$ in eqn. (1) are identical. We wish to say in advance that the application of the above lemmas will be of immense help in the development of this paper.

4 Explicit forms of the Jacobians

We have shown in the previous section that the quadrilaterals Q_e in Cartesian/global space (x, y) can be mapped into $\hat{Q}_e$ in the (u, v) space. Our ultimate aim is to find explicit integration formulas over the region Q_e. In this process, we first transform the integrals over Q_e into $\hat{Q}_e$, then the integrals over $\hat{Q}_e$ will be transformed to integrals over the 2-squares $(-1 \le \xi, \eta \le 1)$ in (ξ, η) space using the bilinear transformations from (u, v) space to (ξ, η) space. The main reason in adopting this process is that, the integration over the quadrilaterals is independent of the nodal coordinates of the global/Cartesian space $(x,\ y)$. This requires explicit form of the Jacobian which uses linear transformations to map Q_e into $\hat{Q}_e$ and the explicit form of the Jacobian which uses the bilinear transformation to map the $\hat{Q}_e$ into the 2-squares in (ξ, η) space.

4.1 Explicit form of the Jacobian using Linear Transformations

First, we consider the linear transformation $(x^{(e)}, y^{(e)})^T$ of eqns.(2-4) for lemma 2 which map the Q_e in (x, y) space into $\hat{Q}_e$ in (u, v) space.

Then it can be easily verified that

$$\frac{\partial(x^{(e)}, y^{(e)})}{\partial(u,v)} = \frac{\partial x^{(e)}}{\partial u}\frac{\partial y^{(e)}}{\partial v} - \frac{\partial x^{(e)}}{\partial v}\frac{\partial y^{(e)}}{\partial u}$$

$$= 2 \times \text{area of the triangle } \Delta PQR$$

$$= \begin{vmatrix} 1 & x_p & y_p \\ 1 & x_q & y_q \\ 1 & x_r & y_r \end{vmatrix} = 2 \times \Delta_{pqr} \text{ (say)} \qquad (5)$$

We also note that for lemma1 $(x, y)^T = (x^{(1)}, y^{(1)})^T$. Hence again, we obtain the same value for J.

4.2 Explicit form of the Jacobian using Bilinear Transformations

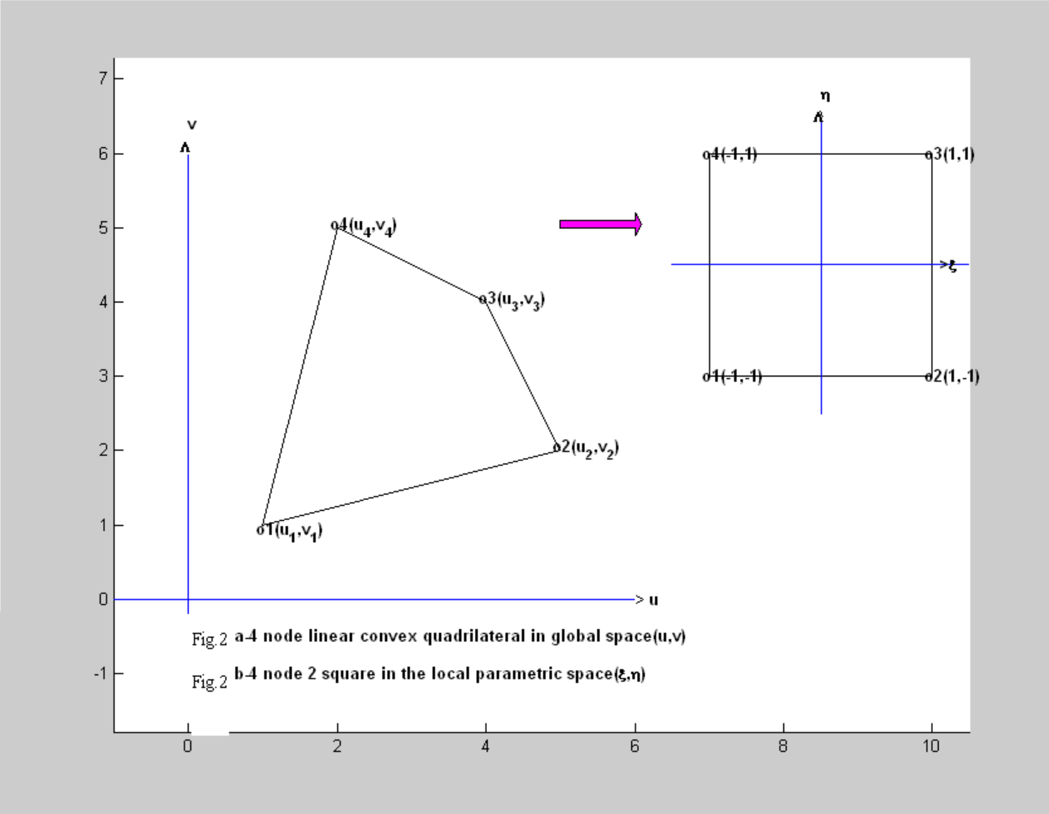


Fig.2 a-4 node linear convex quadrilateral in global space(u,v)

Fig.2 b-4 node 2 square in the local parametric space(ξ,η)

Let us consider an arbitrary four noded linear convex quadrilateral element in the global Cartesian space (u,v) as shown in Fig.2a which is mapped into a 2-square in the local parametric space (ξ,η) as shown in Fig.2b. The necessary bilinear transformation is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k=1}^{4} \begin{pmatrix} u_k \\ vk \end{pmatrix} M_k(\xi,\eta) \qquad (6)$$

where (u_k, v_k), $(k = 1,2,3,4)$ are the vertices of the quadrilateral element $Q^*{}_e$ in the (u,v) plane and $M_k(\xi,\eta)$ denotes the shape function of node k and they are expressed in the standard texts[1-3]:

$$M_k(\xi,\eta)=\frac{1}{4}(1+\xi\xi_k)(1+\eta\eta_k) \quad (7a)$$

$$\{(\xi_k,\eta_k),k=1,2,3,4\}=\{(-1,-1),\ (1,-1),\ (1,1),\ (-1,1)\} \quad (7b)$$

From eqns. (6) and (7), we have

$$\frac{\partial u}{\partial \xi}=\sum_{k=1}^{4}u_k\frac{\partial M_k}{\partial \xi}=\frac{1}{4}[(-u_1+u_2+u_3-u_4)+(u_1-u_2+u_3-u_4)\eta] \quad (8a)$$

$$\frac{\partial u}{\partial \eta}=\sum_{k=1}^{4}u_k\frac{\partial M_k}{\partial \eta}=\frac{1}{4}[(-u_1-u_2+u_3+u_4)+(u_1-u_2+u_3-u_4)\xi] \quad (8b)$$

Similarly,

$$\frac{\partial v}{\partial \xi}=\sum_{k=1}^{4}v_k\frac{\partial M_k}{\partial \xi}=\frac{1}{4}[(-v_1+v_2+v_3-v_4)+(v_1-v_2+v_3-v_4)\eta] \quad (8c)$$

$$\frac{\partial v}{\partial \eta}=\sum_{k=1}^{4}v_k\frac{\partial M_k}{\partial \eta}=\frac{1}{4}[(-v_1-v_2+v_3+v_4)+(v_1-v_2+v_3-v_4)\xi] \quad (8d)$$

Hence, from eqns.(8), the Jacobian can be expressed as

$$J^*=\frac{\partial(u,v)}{\partial(\xi,\eta)}=\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta}-\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi}=\alpha+\beta\xi+\gamma\eta \quad (9a)$$

where, $\alpha=[(u_4-u_2)(v_1-v_3)+(u_3-u_1)(v_4-v_2)]/8,$

$\beta=[(u_4-u_3)(v_2-v_1)+(u_1-u_2)(v_4-v_3)]/8,$

$\gamma=[(u_4-u_1)(v_2-v_3)+(u_3-u_2)(v_4-v_1)]/8$ (9b)

4.3 Explicit form of Jacobian for Special Quadrilaterals

Lemma 3. Let ΔABC be an arbitrary triangle with vertices $A(1,0)$, $B(0,1)$, $C(0,0)$ and let $D(1/2,1/2), E(0,1/2)$ and $F(1/2,0)$ be midpoints of sides AB, BC and CA respectively and also let $G(1/3,1/3)$ be its centroid. Then the Jacobian of the three special quadrilaterals $\hat{Q}_e(e$ = 1, 2, 3) viz $\langle G, E, C, F\rangle$, $\langle G, F, A, D\rangle$ and $\langle G, D, B, E\rangle$ have the same expression given by:

$$\hat{J}=\frac{\partial(u,v)}{\partial(\xi,\eta)}=\hat{J}^e=\frac{1}{96}(4+\xi+\eta),\ (e=1,2,3) \quad (10a)$$

Proof: We can immediately verify that eqn.(10a) is true by substituting the nodal values of $\hat{Q}_e$ in eqn. (9a-b).

The general result for special quadrilaterals $Q_e(e$ =1,2,3) follows by direct substitution of geometric coordinates of the vertices in eqns. (9a-9b) or by chain rule of partial differentiation and use of eqn.(1):

$$J=J^e=\frac{\partial(x,y)}{\partial(\xi,\eta)}=\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(\xi,\eta)}=\left(2\Delta_{pqr}\right)\left(\frac{4+\xi+\eta}{96}\right)=\frac{\Delta_{pqr}}{48}\left(4+\xi+\eta\right) \quad (10b)$$

5 Problem Statement

In some physical applications, we are required to compute integrals of some functions which are expressed in explicit form. In finite element and boundary element method, evaluation of two dimensional integrals with explicit functions as integrands is of great importance. This is the subject matter of several investigations [4-15]. We now consider the evaluation of the integral

$$II_\Omega(f)=\iint_\Omega f(x,y)dxdy,\ \Omega:\text{polygonal domain} \quad (11)$$

$II_\Omega(f)$ can be computed as finite sum of linear integrals and this can be expressed as

$$II_\Omega(f)=\sum_i\iint_{\Delta_i}f(x,y)dxdy \quad (12)$$

where it is assumed that $\Omega=\bigcup_i\Delta_i$, Δ_i = an arbitrary triangle of the domain Ω.

5.1 Composite integration over an arbitrary triangle

Integration over a triangular domain is computed by use of linear transformation between Cartesian and area coordinates. We use the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \end{pmatrix} \begin{pmatrix} 1-u-v \\ u \\ v \end{pmatrix} \qquad (13)$$

to map the arbitrary triangle ΔPQR with vertices $((x_p, y_p), (x_q, y_q), (x_r, y_r))$ in (x, y) space into a standard triangle with vertices $(0,0), (1,0), (0, 1)$ in (u,v) space. The original triangle ΔPQR in (x, y) space and the transformed triangle in (u,v) space are shown in Fig 1*a*,*b* and hence from eqn.(5) and above eqn.(13)

$$\frac{\partial(x,y)}{\partial(u,v)} = 2\times \text{area of triangle } \Delta PQR = 2\Delta_{pqr}$$

$$= (x_q - x_p)(y_r - y_p) - (x_r - x_p)(y_q - y_p) \qquad (14)$$

We now define

$$II_{\Delta PQR}(f) = \iint_{\Delta PQR} f(x, y)dxdy$$

$$= 2\Delta_{pqr} \int_0^1 \int_0^{1-u} f(x(u,v), y(u,v))dudv \qquad (15)$$

We now divide the triangle ΔPQR into three special quadrilaterals Q_e as discussed in the previous section. By use of Lemma1 we know that the special quadrilaterals Q_e in (x, y) space are transformed into special quadrilaterals $\hat{Q}_e$ in (u,v) space. We use Lemma2 to transform each of these $\hat{Q}_e$ into a 2-square in (ξ,η) space by means of the following linear transformations between (x, y) and (u,v) spaces.

$$x(u^e, v^e) = x^{(e)}(\xi,\eta) = \left(1-u^{(e)} - v^{(e)}\right)x_p + u^{(e)} x_q + x_r v^{(e)}, (e = 1,2,3)$$

$$y(u^e, v^e) = y^{(e)}(\xi,\eta) = \left(1-u^{(e)} - v^{(e)}\right)y_p + u^{(e)} y_q + y_r v^{(e)}, (e = 1,2,3) \qquad (16)$$

and the bilinear transformation between (u,v) and (ξ,η) spaces

$$u^{(e)} = u^{(e)}(\xi,\eta) = u_1^{(e)} M_1 + u_2^{(e)} M_2 + u_3^{(e)} M_3 + u_4^{(e)} M_4$$

$$v^{(e)} = v^{(e)}(\xi,\eta) = v_1^{(e)} M_1 + v_2^{(e)} M_2 + v_3^{(e)} M_3 + v_4^{(e)} M_4 \qquad (17)$$

where

$$\left((u_k^{(1)}, v_k^{(1)}), k = 1,2,3,4\right) = \left(\left(\frac{1}{3}, \frac{1}{3}\right), \left(0, \frac{1}{2}\right), (0,0), \left(\frac{1}{2}, 0\right)\right),$$

$$\left((u_k^{(2)}, v_k^{(2)}), k = 1,2,3,4\right) = \left(\left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{2}, 0\right), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right)\right),$$

$$\left((u_k^{(3)}, v_k^{(3)}), k = 1,2,3,4\right) = \left(\left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0,1), \left(0, \frac{1}{2}\right)\right),$$

$$M_k = M_k(\xi,\eta) = \frac{1}{4}(\xi + \xi\xi_k)(1 + \eta\eta_k), k = 1,2,3,4$$

$$(\xi_k, \eta_k), k = 1,2,3,4 = ((-1,-1), (1,-1), (1,1), (-1,1)) \qquad (18)$$

We now verify that

$$u^{(1)} = u^{(1)}(\xi,\eta) = \frac{(1-\xi)(5+\eta)}{24},$$

$$v^{(1)} = v^{(1)}(\xi,\eta) = \frac{(1-\eta)(5+\xi)}{24},$$ We can

$$1 - u^{(1)} - v^{(1)} = \frac{(7 + 2\xi + 2\eta + \xi\eta)}{12}. \qquad (19)$$

also further verify that

$$u^{(2)} = 1 - u^{(1)} - v^{(1)}, \quad v^{(2)} = u^{(1)}, \quad 1 - u^{(2)} - v^{(2)} = v^{(1)},$$
$$u^{(3)} = v^{(1)}, \quad v^{(3)} = 1 - u^{(1)} - v^{(1)}, \quad 1 - u^{(3)} - v^{(3)} = u^{(1)}. \tag{20}$$

Thus, we find the following three unique transformations which map the special quadrilaterals $Q_i, (i = 1, 2, 3)$ in (x, y) space into a 2-square in (ξ, η) space:

$$x^{(1)}(\xi, \eta) = \left(1 - u^{(1)} - v^{(1)}\right)x_p + u^{(1)} x_q + v^{(1)} x_r$$
$$y^{(1)}(\xi, \eta) = \left(1 - u^{(1)} - v^{(1)}\right)y_p + u^{(1)} y_q + v^{(1)} y_r \tag{21}$$

$$x^{(2)}(\xi, \eta) = \left(1 - u^{(1)} - v^{(1)}\right)x_q + u^{(1)} x_r + v^{(1)} x_p$$
$$y^{(2)}(\xi, \eta) = \left(1 - u^{(1)} - v^{(1)}\right)y_q + u^{(1)} y_r + v^{(1)} y_p \tag{22}$$

$$x^{(3)}(\xi, \eta) = \left(1 - u^{(1)} - v^{(1)}\right)x_r + u^{(1)} x_p + v^{(1)} x_q$$
$$y^{(3)}(\xi, \eta) = \left(1 - u^{(1)} - v^{(1)}\right)y_r + u^{(1)} y_p + v^{(1)} y_q \tag{23}$$

The transformations of eqns.(21)- (23) are of the form stated in eqn.(2), (3), (4) and from eqn.(19), we now define

$$u^{(1)} = \lambda = \lambda(\xi, \eta), v^{(1)} = \mu = \mu(\xi, \eta), 1 - u^{(1)} - v^{(1)} = (1 - \lambda - \mu) \tag{24}$$

The above findings again prove the hypothesis of Lemma 2.

5.2 Composite Integration Formula for the Arbitrary Triangle

We again consider the integral defined earlier in eqn.(15) and use Lemma1, 2 and our findings of section 5.1.

$$\begin{aligned}
II_{\Delta PQR}(f) &= \iint_{\Delta PQR} f(x, y)\,dxdy \\
&= 2\Delta_{pqr} \int_0^1 \int_0^{1-u} f(x(u, v), y(u, v))\,dudv \\
&= \sum_{e=1}^{3} \iint_{Q_e} f(x, y)\,dxdy \\
&= 2\Delta_{pqr} \sum_{e=1}^{3} \iint_{\hat{Q}_e} f(x(u^{(e)}, v^{(e)}), y(u^{(e)}, v^{(e)}))\,du^{(e)} dv^{(e)} \\
&= 2\Delta_{pqr} \sum_{e=1}^{3} \iint_{\hat{Q}_e} f(x^{(e)}(u^{(1)}, v^{(1)}), y^{(e)}(u^{(1)}, v^{(1)}))\,du^{(1)} dv^{(1)} \\
&= 2\Delta_{pqr} \sum_{e=1}^{3} \iint_{\hat{Q}_e} f(x^{(e)}(\lambda, \mu), y^{(e)}(\lambda, \mu))\,d\lambda d\mu \\
&= 2\Delta_{pqr} \int_{-1}^{1} \int_{-1}^{1} \sum_{e=1}^{3} f(x^{(e)}(\lambda, \mu), y^{(e)}(\lambda, \mu)) \frac{\partial(x^{(e)}, y^{(e)})}{\partial(\xi, \eta)}\,d\xi d\eta \\
&= 2\Delta_{pqr} \int_{-1}^{1} \int_{-1}^{1} \frac{(4 + \xi + \eta)}{96} \left(\sum_{e=1}^{3} f(x^{(e)}(\lambda, \mu), y^{(e)}(\lambda, \mu)) \right) d\xi d\eta
\end{aligned} \tag{25}$$

where we have from eqns.(21) – (24)

$$x^{(1)}(\xi, \eta) = \left(1 - \lambda - \mu\right)x_p + \lambda x_q + \mu x_r$$
$$y^{(1)}(\xi, \eta) = \left(1 - \lambda - \mu\right)y_p + \lambda y_q + \mu y_r \tag{26}$$

$$x^{(2)}(\xi, \eta) = \left(1 - \lambda - \mu\right)x_q + \lambda x_r + \mu x_p$$
$$y^{(2)}(\xi, \eta) = \left(1 - \lambda - \mu\right)y_q + \lambda y_r + \mu y_p \tag{27}$$

$$x^{(3)}(\xi, \eta) = \left(1 - \lambda - \mu\right)x_r + \lambda x_p + \mu x_q$$
$$y^{(3)}(\xi, \eta) = \left(1 - \lambda - \mu\right)y_r + \lambda y_p + \mu y_q \tag{28}$$

with $x^{(e)}(\xi,\eta) = x^{(e)}(\lambda,\mu), y^{(e)}(\xi,\eta) = y^{(e)}(\lambda,\mu)$, and $\lambda = \lambda(\xi,\eta)$, $\mu = \mu(\xi,\eta)$ as given in eqns. (19) and (24).

The above composite integration formula is based on the discritisation of the original arbitrary quadrilateral into 3- special quadrilaterals which are created by joining the centroid of the triangle to the midpoints of its sides. The new composite integration takes this concept further and creates in all 12-new quadrilaterals by joining the centroids of the 3-special quadrilaterals to the midpoints of the sides as shown Fig.3a and Fig.3b. The derivation of this new composite integration formula is explained in the next section.

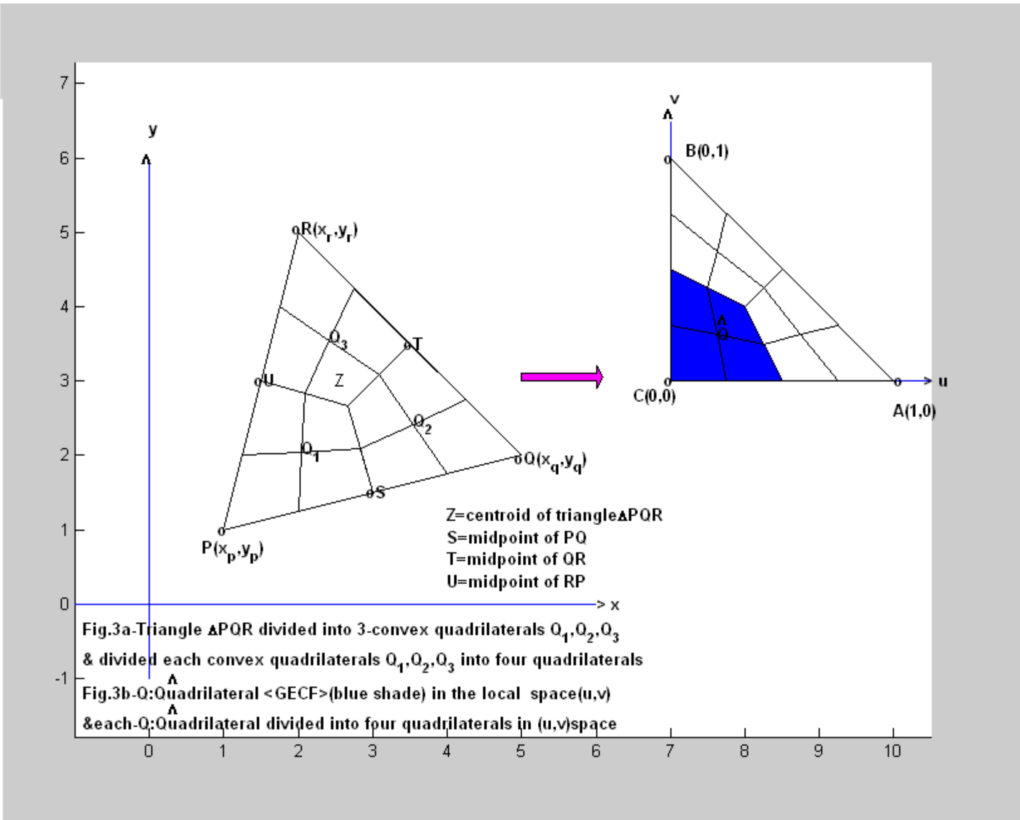


Fig.3a-Triangle ΔPQR divided into 3-convex quadrilaterals Q_1, Q_2, Q_3 & divided each convex quadrilaterals Q_1, Q_2, Q_3 into four quadrilaterals

Fig.3b-$\hat{Q}$:Quadrilateral <GECF>(blue shade) in the local space(u,v) &each-$\hat{Q}$:Quadrilateral divided into four quadrilaterals in (u,v)space

5.3 A New Composite Integration Formula for the Arbitrary Triangle

We have from eqn(25) of the previous section 5.2:

$$II_{\Delta PQR}(f) = \iint_{\Delta PQR} f(x,y)dxdy$$

$$= 2\Delta_{pqr}\int_{-1}^{1}\int_{-1}^{1}\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f(x^{(e)}(\lambda,\mu), y^{(e)}(\lambda,\mu))\right)d\xi d\eta$$

$$=2\Delta_{pqr}\left(\int_{-1}^{0}\int_{-1}^{0} + \int_{0}^{1}\int_{-1}^{0} + \int_{0}^{1}\int_{0}^{1} + \int_{-1}^{0}\int_{0}^{1}\right)\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f\left(x^{(e)}(\lambda,\mu), y^{(e)}(\lambda,\mu)\right)\right)d\xi d\eta \quad \text{.....(29)}$$

$$=\sum_{e=1}^{3}\sum_{j=1}^{4}\left(\iint_{Q_{e,j}} f(x,y)\,dxdy\right) \quad \text{.....(30)}$$

Where, λ= λ(ξ,η)=(1-ξ)(5+η)/24,μ=μ(ξ,η)=(1-η)(5+ξ)/24 ,and $\boldsymbol{Q_{e,j}}(\boldsymbol{j = 1,2,3,4})$ are the four quadrilaterals obtained from the quadrilateral $\boldsymbol{Q_e}$(e=1,2,3) by joining its centroid to the midpoint of four sides and $\boldsymbol{Q_e} = \sum_{j=1}^{4}\boldsymbol{Q_{e,j}}$.We also have n$\sum_{e=1}^{4}\boldsymbol{Q_e} = \boldsymbol{\Delta_{PQR}}$

Thus,from the above eqns(25),(29) and(30),we obtain

$$2\Delta_{pqr}\left(\int_{-1}^{0}\int_{-1}^{0}\right)\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f\left(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu)\right)\right)d\xi d\eta=\sum_{e=1}^{3}\iint_{Q_{e,1}} f(x,y)dxdy \quad \text{.........(31)}$$

$$2\Delta_{pqr}\left(\int_{0}^{1}\int_{-1}^{0}\right)\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f\left(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu)\right)\right)d\xi d\eta=\sum_{e=1}^{3}\iint_{Q_{e,2}} f(x,y)dxdy \quad \text{.........(32)}$$

$$2\Delta_{pqr}\left(\int_{0}^{1}\int_{0}^{1}\right)\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f\left(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu)\right)\right)d\xi d\eta=\sum_{e=1}^{3}\iint_{Q_{e,3}} f(x,y)dxdy \quad \text{.........(33)}$$

$$2\Delta_{pqr}\left(\int_{-1}^{0}\int_{0}^{1}\right)\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f\left(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu)\right)\right)d\xi d\eta=\sum_{e=1}^{3}\iint_{Q_{e,4}} f(x,y)dxdy \quad \text{.........(34)}$$

We now apply bilinear transformations to map each $Q_{e,j}$,(j=1,2,3,4) into a 2-square in a natural space (r,s) and write:

$$\iint_{Q_{e,j}} f(x,y)dxdy=2\Delta_{pqr}\int_{-1}^{1}\int_{-1}^{1} f\left(x^{(e)}\left(\lambda^j,\mu^j\right),y^{(e)}\left(\lambda^j,\mu^j\right)\right)J^j \text{drds} \quad \text{..........(35)}$$

Where,

$\lambda^j=\lambda(\xi^j,\eta^j);\mu^j=\mu(\xi^j,\eta^j);$

$J^j=\frac{(4+\xi^j+\eta^j)}{96}\frac{\partial(\xi^j,\eta^j)}{\partial(r,s)};$

$\xi^1(r,s)=\frac{-1}{2}(1+r);\xi^2(r,s)=\frac{1}{2}(1+s);\xi^3(r,s)=\frac{1}{2}(1+r);\xi^4(r,s)=\frac{-1}{2}(1+s);$

$\eta^1(r,s)=\frac{-1}{2}(1+s);\eta^2(r,s)=\frac{-1}{2}(1+r);\eta^3(r,s)=\frac{1}{2}(1+s);\eta^4(r,s)=\frac{1}{2}(1+r);$

$J^1=\frac{1}{128}-\frac{1}{768}r-\frac{1}{768}s;$

$\lambda^1=\left(\frac{-3}{16}+\frac{1}{48}\right)(\frac{-3}{2}-\frac{1}{2}\ r);\ \mu^1=\left(\frac{1}{16}+\frac{1}{48}r\ \right)(\frac{11}{2}+\frac{1}{2}\ s);$

$J^2=\frac{1}{96}+\frac{1}{768}s-\frac{1}{768}r;$

$\lambda^2=\left(\frac{-3}{16}+\frac{1}{48}\ r\right)(\frac{-1}{2}+\frac{1}{2}\ s);\ \mu^2=\left(\frac{1}{16}+\frac{1}{48}\ r\right)(\frac{11}{2}+\frac{1}{2}\ s);$

$J^3=\frac{5}{384}+\frac{1}{768}s+\frac{1}{768}r;$

$\lambda^3=\left(\frac{-11}{48}-\frac{1}{48}s\ \right)(\frac{-1}{2}+\frac{1}{2}r\ );\ \mu^3=\left(\frac{1}{48}-\frac{1}{48}s\ \right)(\frac{11}{2}+\frac{1}{2}r\ );$

$J^4=\frac{1}{96}-\frac{1}{768}s+\frac{1}{768}r;$

$\lambda^4=\left(\frac{-11}{48}-\frac{1}{48}\ r\right)(\frac{-3}{2}-\frac{1}{2}s\ );\ \mu^4=\left(\frac{1}{48}-\frac{1}{48}r\ \right)(\frac{9}{2}-\frac{1}{2}\ s);$

……………………………………….(36)

Now on substituting from eqn(35) into eqn(30),we finally obtain the following new composite integration formula:

$$II_{\Delta PQR}(f)=\iint_{\Delta PQR} f(x,y)dxdy$$

$$=2\Delta_{pqr}\int_{-1}^{1}\int_{-1}^{1}\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu))\right)d\xi d\eta$$

$$=\sum_{e=1}^{3}\sum_{j=1}^{4}\left(\iint_{Q_{e,j}} f(x,y)\,dxdy\right))$$

$$=2\Delta_{pqr}\ \left[\sum_{e=1}^{3}\sum_{j=1}^{4}(\int_{-1}^{1}\int_{-1}^{1} f\left(x^{(e)}\left(\lambda^{j},\mu^{j}\right),y^{(e)}\left(\lambda^{j},\mu^{j}\right)\right)J^{j}\mathrm{drds})\right]$$

……………………………………………….(37)

In the above equation $J^{j}=J^{j}(r,s)$, $x^{(e)}\left(\lambda^{j},\mu^{j}\right)=x^{(e)}(r,s)$,$y^{(e)}\left(\lambda^{j},\mu^{j}\right)=y^{(e)}(r,s)$,because $\lambda^{j}=\lambda^{j}(r,s) and \mu^{j}=\mu^{j}(r,s)$

We have shown that the original triangle in Cartesian space (x,y) is mapped into a standard triangle in natural space (ξ,η).We divide the original triangle and the standard triangle by the following procedure:

step(i):We first join the centroids of triangles to their respective mid points of sides.This creates three special quadrilateral in both the triangles.

step(ii):We next divide each of these special quadrilaterals into four smaller quadrilaterals.This creates 12-smaller quadrilaterals in both the triangles.

The procedure of step(i) is shown in Fig.1a-1b and Fig.2a-2b.The procedure of step(ii) is depicted in Figs.3a-3b

In Fig.3a,we have shown discretisation of an arbitrary triangle(an equilateral triangle is selected for convenience),as stated above we first join the centroid of triangle to the midpoints of its three sides.This creates three special quadrilaterals.Then in each special quadrilateral,we locate their centroid and join it by straight lines to the respective midpoints of their four sidesWe repeat this procedure on a standard triangle(right isosceles triangle) as shown in Fig.3b.

6 Numerical Integration Formulas

In the following sections 6.1-6.2 explain the discuss about the formulas derived in our earlier paper[16] which is necessary to understand the present paper. In sections 6.3 and 6.4 we discuss the implementation of the new composite integration formulas derived in the present paper

6.1 Numerical integration over an Arbitrary Triangle ΔPQR

We could use either of the formulas in eqn.(16) or eqns.(26-28). We prefer to use eqn.(25), since it requires the computation of just one set of $(u,v)=(u^{(1)},v^{(1)})$ for all the three quadrilaterals. The transformation formulas of eqns. (26-28) are easy to implement as a computer code, since the coordinates of ΔPQR are to be used in cyclic permutation in $(x^{(e)},y^{(e)}),\ e=1,2,3)$. Note that in $(x^{(1)},\ y^{(1)})^{T}$ the coefficients of w, u, v are $(x_p,y_p)^T$, $(x_q,y_q)^T$, $(x_r,y_r)^T$ respectively. In $(x^{(2)},\ y^{(2)})^{T}$ the coefficients of w, u, v are $(x_q,y_q)^T$, $(x_r,y_r)^T$, $(x_p,y_p)^T$ respectively and in $(x^{(3)},\ y^{(3)})^{T}$ the coefficients of w, u, v are $(x_r,y_r)^T$, $(x_p,y_p)^T$, $(x_q,y_q)^T$ respectively. We can use Gauss Legendre quadrature rule to evaluate eqn.(25). The resulting numerical integration formula can be written as

$$II_{\Delta PQR}(f)\approx 2\Delta_{pqr}\sum_{k=1}^{N\times N}W_k^{(N)}\sum_{e=1}^{3}\left(f(x^{(e)}(U_k^{(N)},\,V_k^{(N)}),y^{(e)}(U_k^{(N)},\,V_k^{(N)}))\right) \qquad (38)$$

The weights and sampling points in the above formula satisfy the relation

$$\left((W_k^{(N)},\,U_k^{(N)},\,V_k^{(N)})\right)=\left((4+s_i^{(N)}+s_j^{(N)})w_i^{(N)}w_j^{(N)}\big/96,\,u(s_i^{(N)},\,s_j^{(N)}),\,v(s_i^{(N)},\,s_j^{(N)})\,\right)$$

$$k=1,2,3,\ldots..N\times N\,,\ \ i,j=1,2,3,\ldots..N \qquad (39)$$

and

$$u(s_i^{(N)},\,s_j^{(N)})=(1-s_i^{(N)})(1-s_j^{(N)})\big/12+(1-s_i^{(N)})(1+s_j^{(N)})\big/8,$$

$$v(s_i^{(N)},\,s_j^{(N)})=(1-s_i^{(N)})(1-s_j^{(N)})\big/12+(1+s_i^{(N)})(1-s_j^{(N)})\big/8 \qquad (40)$$

for a N- point Gauss Legendre rule of order N with $\left((w_n^{(N)},\,s_n^{(N)}),\ n=1,2,3,\ldots\ldots N\right)$ as the weights and sampling points respectively.

We can compute the arrays $\left((W_k^{(N)}, U_k^{(N)}, V_k^{(N)}), k = 1,2,3,...... N^2\right)$ for any available Gauss Legendre quadrature rule of order *N*. We have listed a code to compute the arrays $(W_k^{(N)}, U_k^{(N)}, V_k^{(N)}, k = 1,2,3,...... N^2)$ for *N* = 5, 10, 15, 20, 25, 30, 35, 40. This is necessary since explicit list of $(W_k^{(N)}, U_k^{(N)}, V_k^{(N)}, N = 5, 10, 15, 20, 25, 30, 35, 40)$ will generate a large amount of values, viz: 25, 100, 225, 400, 625, 900, 1225, and 1600 for each of $W_k^{(N)}, U_k^{(N)}, V_k^{(N)}$. The computer code will be simple with few statements and it requires the input values of $\left((w_n^{(N)}, s_n^{(N)}, n = 1,2,..., N), N = 5,10,15,20,25,30,35,40\right)$.

6.2 Composite Integration over a Polygonal Domain P_N

We now consider the evaluation of $II_\Omega(f) = \iint_\Omega f(x,y)dxdy$, where $f \in C(\Omega)$ and Ω is any polygonal domain in $\Re^2$. That is a domain with boundary composed of piecewise straight lines. We then write $\mathrm{II}_{P_N}(f) = \sum_{n=1}^{M} \mathrm{II}_{T_n}(f) = \sum_{n=1}^{M}\left(\sum_{p=0}^{2} \mathrm{II}_{Q_{3n-p}}(f)\right)$ ……….(41) in which, we define P_N as a polygonal domain of *N* oriented edges l_{ik} $(k = i+1,\ i = 1,2,3,..., N)$, with end points (x_i, y_i), (x_k, y_k) and $(x_1, y_1) = (x_{N+1}, y_{N+1})$. We have assumed in the above eqn. (41) that P_N can be discretised into a set of M triangles, $T_n\ (n = 1,2,3,...., M)$. In the numerical integration formula of section 6.1, we have $T_n = T_{pqr}^{xy} = \Delta PQR$, an arbitrary triangle with vertices $\left((x_\alpha, y_\alpha),\ \alpha = p,q,r\right)$ in Cartesian space (x,y). The numerical integration formula for $\mathrm{II}_{T_n}(f) = \mathrm{II}_{T_{pqr}^{xy}} = \mathrm{II}_{\Delta PQR_n}$ is already explained in section 6.1. We can get higher accuracy for the integral $\mathrm{II}_{P_N}(f)$ by using the refined triangular mesh of the polygonal domain. We have written a computer code in MATLAB for this purpose. We first decompose the given polygonal domain into a coarse mesh of *M* triangles T_n (say), as expressed in eqn.(41). We then refine the mesh containing *M* triangles into a new mesh with $n^2 \times M$ triangles, satisfying the relation $T_n = \sum_{p=1}^{n^2} t_p$. This division can be carried by using the linear transformations connecting the Cartesian space (x,y) and the local space (u,v).We divide the standard triangle (right isosceles) in the (u,v) space into n^2 right isosceles triangles and then use the linear transformation to obtain the corresponding Cartesian nodal coordinates using the linear transformation. The nodal connectivity data in (u,v) space is also determined for the subdivisions. This is prepared for each subdivision and incorporated into the computer code. The computer code can obtain integral values $II_\Omega(f_i(x,y)),\ \Omega = P_N$ by dividing the P_N into meshes with refinements. The first mesh is the coarse mesh with *M* triangles, the subsequent meshes will have $n^2 \times M$ ($n^2 = 2^2, 3^2, 4^2, 5^2, 6^2, 7^2,......$) triangles. We know that the numerical algorithm of section 6.1 is for an arbitrary triangle which is further divided into three special quadrilaterals. Thus the polygonal domain is divided into $3 \times n^2 \times M$ special quadrilaterals. This computer code written in MATLAB is appended for reference in our earlier paper [16]

6.3 New Formula for Numerical integration over an Arbitrary Triangle ΔPQR

We now use the formulas in eqn.(37). The transformation formulas of eqns. (35-37) are easy to implement as a computer code, since the coordinates of ΔPQR are to be used in cyclic permutation in $(x^{(e)}, y^{(e)}),\ e = 1,2,3)$. Note that in $(x^{(1)}, y^{(1)})^T$ the coefficients of (1-λ-μ),λ,μ are $(x_p, y_p)^T, (x_q, y_q)^T, (x_r, y_r)^T$ respectively. In $(x^{(2)}, y^{(2)})^T$ the coefficients of (1-λ-μ),λ,μ are $(x_q, y_q)^T, (x_r, y_r)^T, (x_p, y_p)^T$ respectively and in $(x^{(3)}, y^{(3)})^T$ the coefficients of (1-λ-μ),λ,μ are $(x_r, y_r)^T, (x_p, y_p)^T, (x_q, y_q)^T$ respectively. We can use Gauss Legendre quadrature rule to evaluate eqn.(37).We now explain the programming implementation of the new fomula in eqn(37).Using Gauss Legendre Quadrature Rules,we can write the following approximation of eqn(37)

The resulting numerical integration formula can be written as

$$II_{\Delta_{PQR}} \sim (2\Delta_{pqr}) \sum_{e=1}^{3} ((\sum_{j=1}^{4} (\sum_{k=1}^{N^2} f \left(x^{(e)} \left(U_{k,j}{}^{(N)}, V_{k,j}{}^{(N)} \right), y^{(e)} (U_{k,j}{}^{(N)}, V_{,k,j}{}^{(N)} \right)))) \text{ ....(42)}$$

Where, the triplet $(((W_{k,j}{}^{(N)}, U_{k,j}{}^{(N)}, V_{k,j}{}^{(N)}), k = 1: N^2), j = 1: 4)$,refer to the weight coefficients and sampling points of order N and they can be computed by using the N-th order sampling points and weight coefficients of one dimensional Gauss Legendre Quadrature rules.

Therefore the weight coefficients and the sampling points in eqn(42) ,satisfy the following relations:

$$(((W_{k,j}{}^{(N)}), k = 1: N^2), j = 1: 4) = (((J^j \ (s_a{}^{(N)}, s_b{}^{(N)}), a = 1: N), b = 1: N), j = 1: 4) \text{ ......(43)}$$

$$(((U_{k,j}{}^{(N)}), k = 1: N^2), j = 1: 4) = (((\lambda^j \ (s_a{}^{(N)}, s_b{}^{(N)}), a = 1: N), b = 1: N), j = 1: 4) \text{ ......(44)}$$

$$(((V_{k,j}{}^{(N)}), k = 1: N^2), j = 1: 4) = (((\mu^j \ (s_a{}^{(N)}, s_b{}^{(N)}), a = 1: N), b = 1: N), j = 1: 4) \text{ ....(45)}$$

and the mathematical expressions for $((J^j, \lambda^j, \mu^j), j = 1: 4)$ are as defined in eqn(36) which use

a *N*- point Gauss Legendre Quadrature rule of order *N* with $((w_n^{(N)}, s_n^{(N)}), n = 1, 2, 3, ... N)$ as the weights and sampling points respectively.

We can compute the arrays $(((W_{k,j}{}^{(N)}, U_{k,j}{}^{(N)}, V_{k,j}{}^{(N)}), k=1:N^2), j=1:4)$ for any available Gauss Legendre quadrature rule of order *N*. We have listed a code to compute this arrays for *N* = 5:5:40. This is necessary since explicit list of this array will generate a large amount of values, viz: 4x25, 4x100, 4x225, 4x400,4x 625, 4x900, 4x1225, and 4x1600 for each component of this triplet. The computer code will be simple with few statements and it requires the input values of $((w_n^{(N)}, s_n^{(N)}, n = 1, 2, ..., N), N = 5, 10, 15, 20, 25, 30, 35, 40)$.

6.4 New Formula for Composite Integration over a Polygonal Domain P_N

We now consider the evaluation of $II_\Omega(f) = \iint_\Omega f(x, y) dxdy$, where $f \in C(\Omega)$ and Ω is any polygonal domain in $\Re^2$. That is a domain with boundary composed of piecewise straight lines. We then write

$$II_{P_N}(f) = \sum_{n=1}^{M} II_{T_n}(f) = \sum_{n=1}^{M} (\sum_{a=0}^{2} \sum_{b=1}^{4} II_{Q_{3n-a,b}}(f)) \quad \text{..................................(46)}$$

in which, we define P_N as a polygonal domain of *N* oriented edges l_{ik} $(k = i+1, i = 1, 2, 3, ..., N),$ with end points $(x_i, y_i), (x_k, y_k)$ and $(x_1, y_1) = (x_{N+1}, y_{N+1})$. We have assumed in the above eqn. (46) that P_N can be discretised into a set of M triangles, T_n $(n = 1, 2, 3, ..., M)$. In the numerical integration formula of section 6.3, we have $T_n = T_{pqr}^{xy} = \Delta PQR$, an arbitrary triangle with vertices $((x_\alpha, y_\alpha), \alpha = p, q, r)$ in Cartesian space (x, y). The numerical integration formula for $II_{T_n}(f) = II_{T_{pqr}^{xy}} = II_{\Delta PQR_n}$ is already explained in section 6.3. We can get higher accuracy for the integral $II_{P_N}(f)$ by using the refined triangular mesh of the polygonal domain.

We have written a computer code in MATLAB for this purpose. We first decompose the given polygonal domain into a coarse mesh of M triangles T_n (say), as expressed in eqn.(41). We then refine the mesh containing M triangles into a new mesh with $n^2 \times M$ triangles, satisfying the relation $T_n = \sum_{p=1}^{n^2} t_p$. This division can be carried by using the linear transformations connecting the Cartesian space (x, y) and the local space (u, v).We divide the standard triangle (right isosceles) in the (u, v) space into n^2 right isosceles triangles and then use the linear transformation to obtain the corresponding Cartesian nodal coordinates using the linear transformation. The nodal connectivity data in (u, v) space is also determined for the subdivisions. This is prepared for each subdivision and incorporated into the computer code. The computer code can obtain integral values $II_\Omega(f_i(x, y)),\ \Omega = P_N$ by dividing the P_N into meshes with refinements. The first mesh is the coarse mesh with M triangles, the subsequent meshes will have $n^2 \times M$ ($n^2 = 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, ...$) triangles.We know that the numerical algorithm of section 6.3 is for an arbitrary triangle which is further divided into twelve special quadrilaterals Thus the polygonal domain is now divided into $\left(12 \mathrm{x}\, n^2 \times M\right)$ quadrilaterals ·

7. Numerical Examples

In our earlier works the composite integration methods were applied to integrals over standard triangular region while in this paper, the proposed method is applied to a variety of regions including the triangular regions.

7.1 Integrals over Standard Triangular Domains

In this section, we consider some typical integrals which were experimented for the first time over the standard triangular domains in [13].

$$\mathrm{I}_1 = \int_0^1 \int_0^{1-y} (x+y)^{\frac{1}{2}}\, dxdy = 0.4$$

$$\mathrm{I}_2 = \int_0^1 \int_0^{1-y} (x+y)^{-\frac{1}{2}}\, dxdy = \frac{2}{3}$$

$$\mathrm{I}_3 = \int_0^1 \int_0^{y} (x^2+y^2)^{\frac{1}{2}}\, dxdy = \ln\left(\frac{1}{\sqrt{2}-1}\right)$$

$$\mathrm{I}_4 = \int_0^{\frac{\pi}{2}} \int_0^{y} \sin(x+y)\, dxdy = 1$$

$$\mathrm{I}_5 = \int_0^1 \int_0^{y} e^{|x+y-1|}\, dxdy = -2+e$$

We find the numerical solution to the above by joining the centroid of the standard triangle i.e, (1/3, 1/3) to the midpoints of sides which creates three special quadrilaterals This requires two equal divisions of each side of the standard triangle which actually creates a six node triangle.In general,by dividing the sides into 2n divisions and joining to opposite sides,we can create n^2 six node triangles.We can then divide each six node triangle into 3-special quadrilaterals.Each of these special quadrilaterals can be then divided into 4-smaller quadrilaterals by following the procedure explained in section 5.3.We have appended the first discretisation used in the numerical solution for above problems in Figs.4a-b

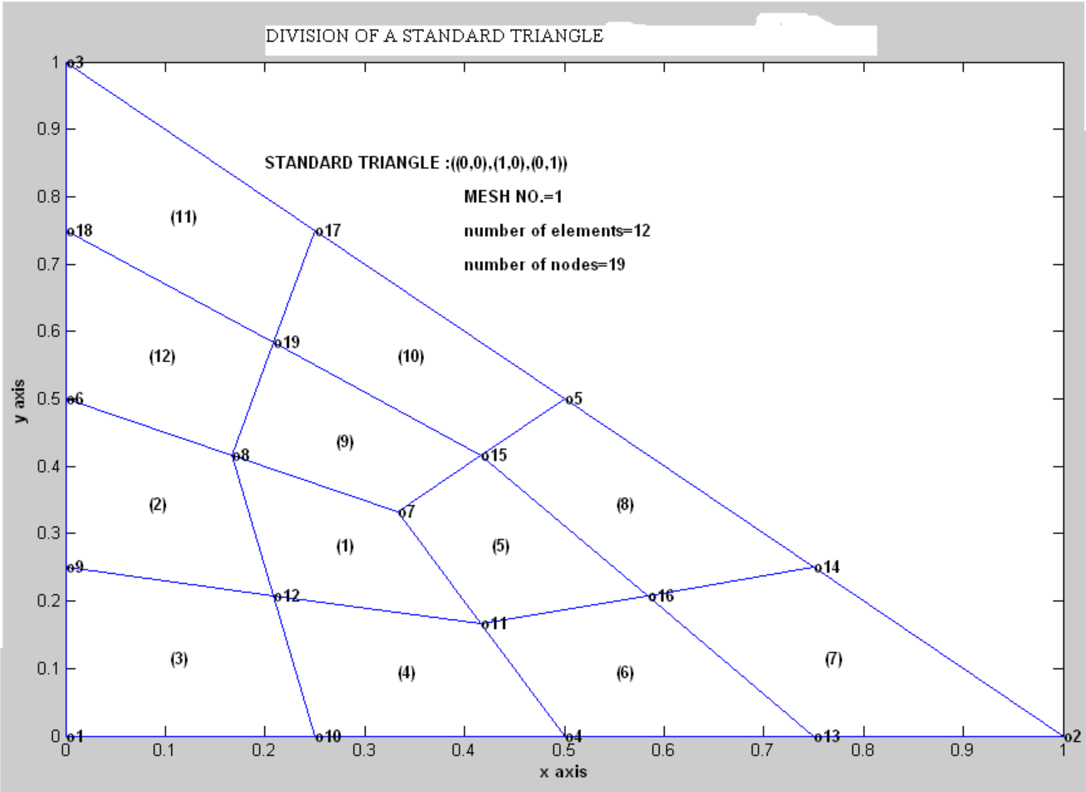


Fig.4a-first discritisation of a standard triangle

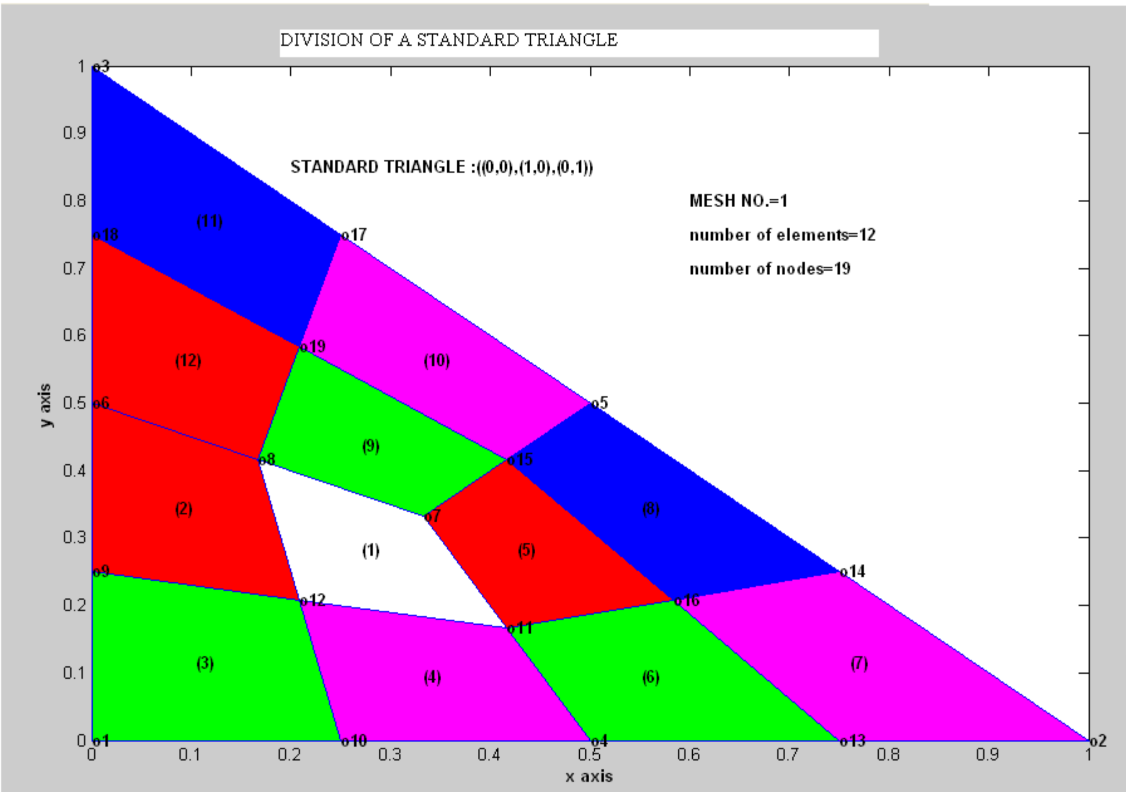


Fig.4b-first discretisation of a standard triangle

7.2 Integrals over Quadrilaterals and Standard 2-Squares

In this section, we consider two typical integrals which are considered in [14, 15] over the quadrilateral and standard 2-square.
We first consider [15]

$$I_6 = \int_0^{\frac{\pi}{4}} \int_0^{\sin y} \frac{dxdy}{\sqrt{1-x^2}} = \frac{\pi^2}{32} \approx 0.308425137533404243......$$

In order to solve I_6, by using the present method. We write:

$$\int_0^{\frac{\pi}{4}} \int_0^{\sin y} \frac{dxdy}{\sqrt{1-x^2}} = \int_{-1}^{1}\int_{-1}^{1} \frac{\frac{1}{2}\sin\left(\frac{\pi}{8}(1+s)\right)\frac{\pi}{8}\, dsdt}{\left\{1-\frac{1}{2}\left(\sin\left(\frac{\pi}{8}(1+s)\right)\right)^2(1+t)^2\right\}^{\frac{1}{2}}}$$

We find the numerical solution to the above by joining the centroid of the 2-square i.e, (0, 0) to the four vertices which creates four triangles.This is shown in Fig.5

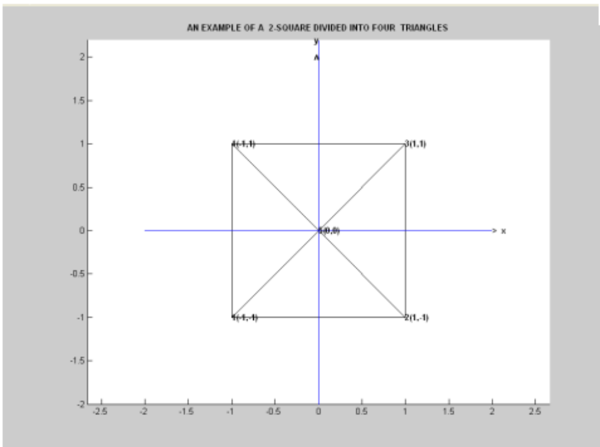


Fig.5-AN EXAMPLE OF A TWO SQUARE DIVIDED INTO FOUR TRIANGLES

$$I_7 = \iint_Q \frac{\mathbf{1}}{\boldsymbol{sqrt(x+y)}} dxdy = \sum_{n=1}^{4} \iint_{T_n} \frac{\mathbf{1}}{\boldsymbol{sqrt(x+y)}} dxdy,$$

$$=\frac{2}{3}(2-7\sqrt{3}\ -15\sqrt{5}\ +20\sqrt{6}\ )= 3.54961302678971$$

where $(T_{n\ ,}\ n = 1{:}\,4)$ *are the four triangles*
obtained by joining the centroid(5/4,5/2) of the quadrilateral Q to the four vertices
(-1,2),(2,1),(3,3),(1,4)

The computed values of integrals $I_N(N=1,\ 2,\ 3, 4,\ 5,\ 6, 7)$ are given in Table II which use the numerical scheme developed in sections 6.1 and 6.2.
This quadrilateral is shown in Fig.6.

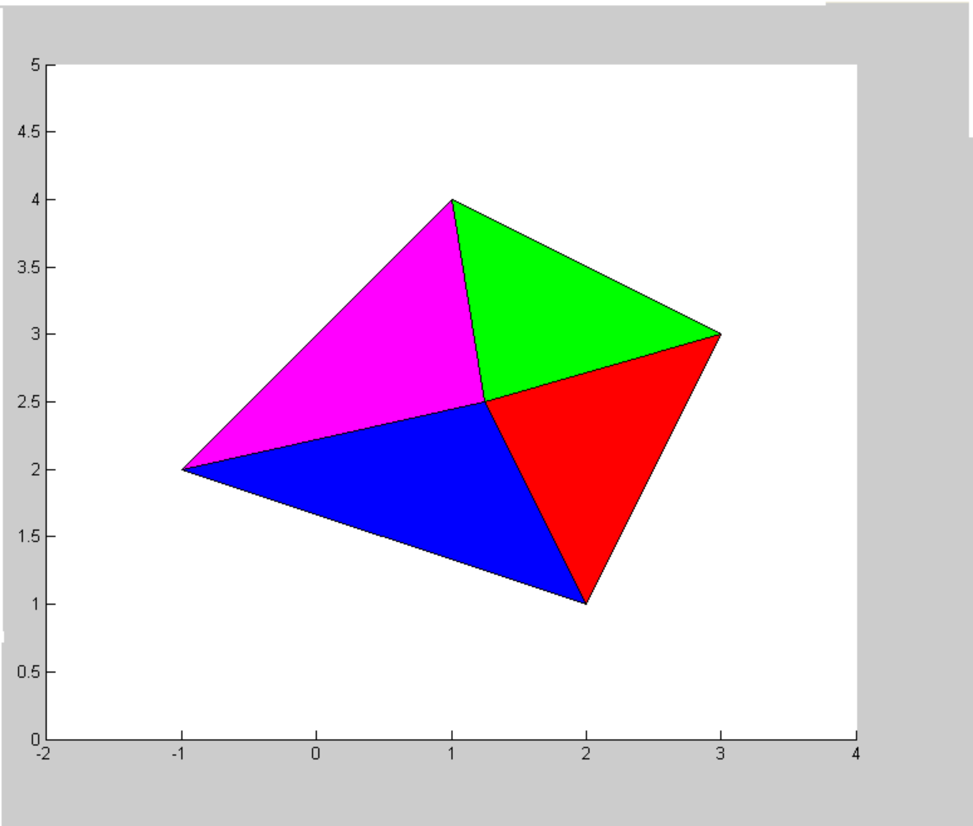


Fig.6-Quadrilateral region made up of four triangles

7.3 Some Integrals over the Polygonal Domains

In this section, integration of some typical examples is presented. The integration domains are the same as the ones considered in [11, 12]. Fig.7a-b shows the convex polygon with six sides which is discretised by seven composite triangles, whose coordinates of vertices are1(0.1,0),2(0.7,0.2),3(1,0.5),4(0.75,0.85),5(0.5,1),6(0.25,0.625),7(0,0.25) 2(0.7, 0.2), 3(1, 0.5), Fig.8a-b shows the nonconvex polygon with nine sides which is discretised by seven composite triangles whose coordinates of vertices are 1(0.25, 0), 2(0.75, 0.5), 3(0.75, 0), 4(1, 0.5), 5(0.75, 0.75), 6(0.75, 0.85), 7(0.5, 1), 8(0, 0.75), 9(0.25, 0.5).

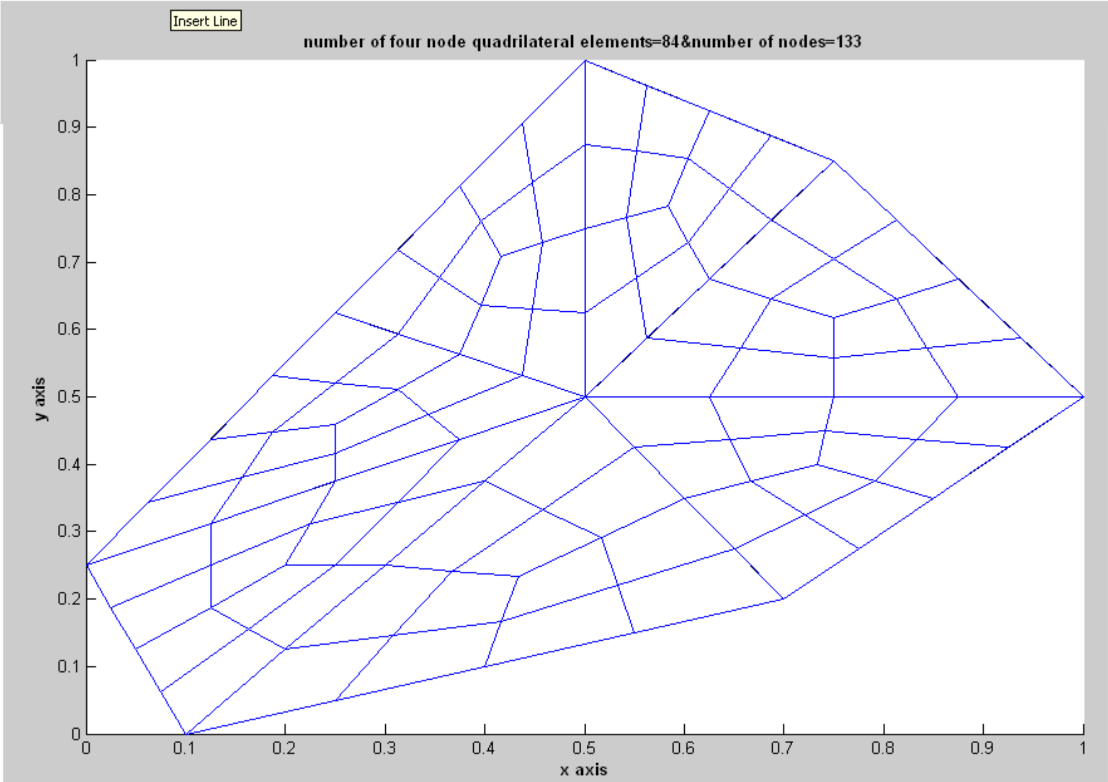


Fig.7a:convex polygon with six sides

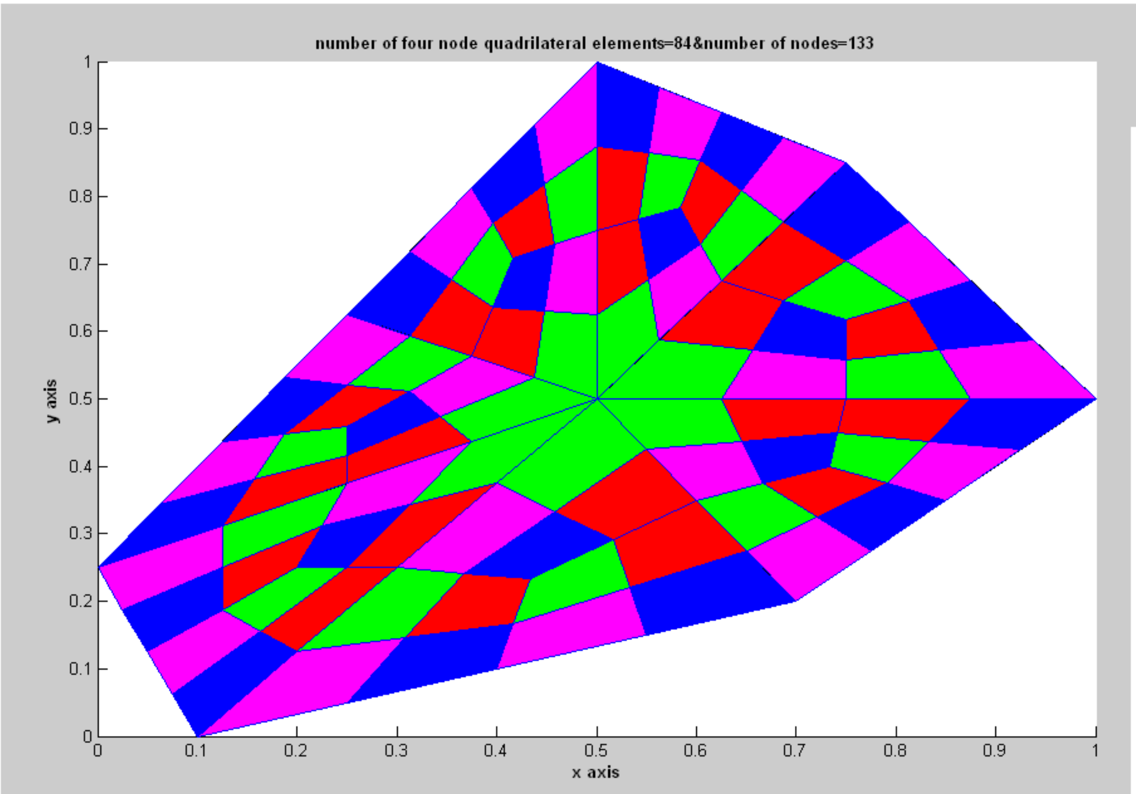


Fig.7b:convex polygon with sixsides

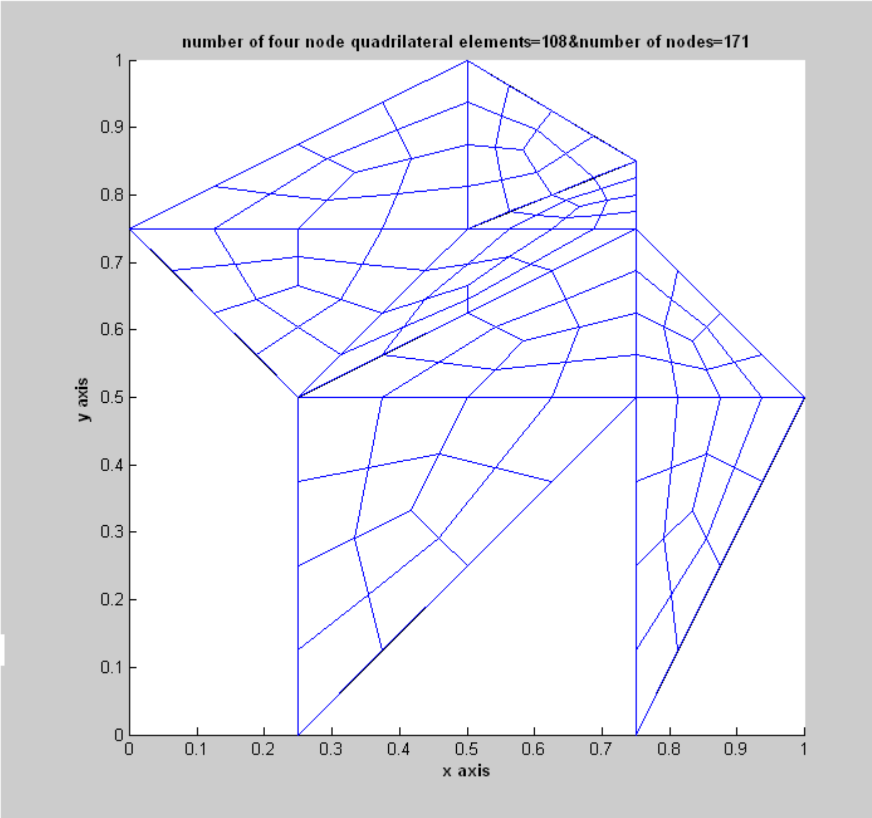


Fig.8a:nonconvex polygon with nine sides

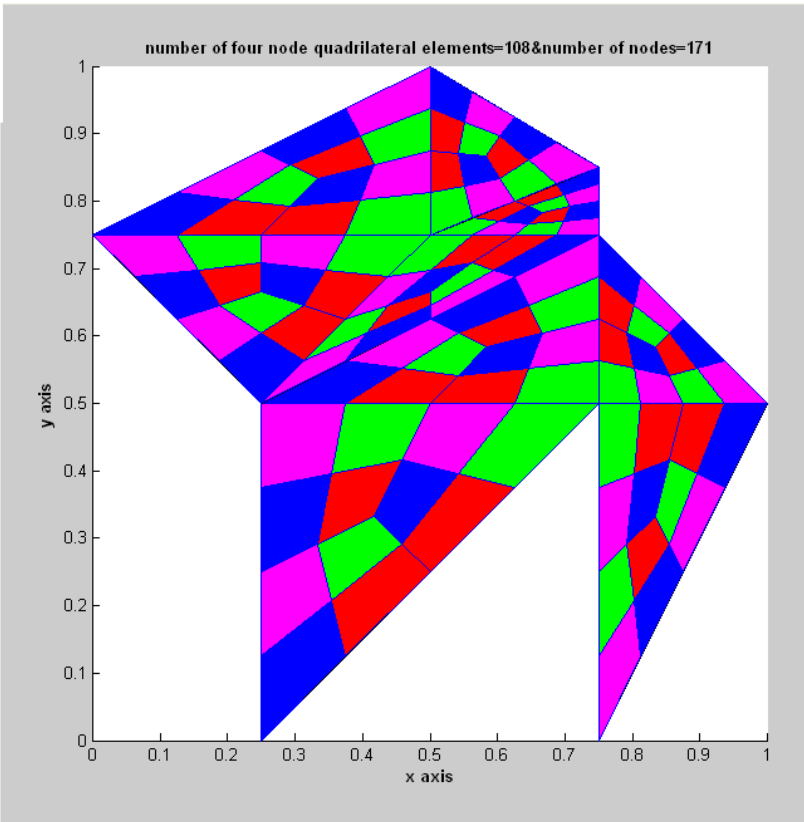


Fig.8b:nonconvex polygon with nine sides

We consider the evaluation of following integrals

$$II_{\Omega}(f_i) = \iint_{\Omega} f_i(x,y)dxdy,\ i = 1\,(1)9\ ,\quad \Omega = P_6\,, P_9$$

where, $f_1 = (x+y)^{19}$, f_2 =cos(30*(x+y))

$$f_3 = \sqrt{\left((x-1/2)^2 + (y-1/2)^2\right)},\ f_4 = \exp\left\{-(x-0.5)^2 - (y-0.5)^2\right\},$$

$$f_5 = \exp\left\{-100\,(x-0.5)^2 - 100\,(y-0.5)^2\right\},$$

$$f_6 = \frac{3}{4}\exp\left\{-\frac{1}{4}(9x-2)^2 + (9y-2)^2\right\} + \frac{3}{4}\exp\left\{-\frac{1}{49}(9x+1)^2 - \frac{1}{10}(9y+1)\right\} + \frac{1}{2}\exp\left\{-\frac{1}{4}(9x-7)^2 + (9y-3)^2\right\}$$

$$-\frac{1}{5}\exp\left\{-(9x-4)^2 + (9y-7)^2\right\},\ f_7 = \left|x^2 + y^2 - 1/4\right|,\ f_8 = \sqrt{\left|3-4x-4y\right|},$$

$$f_9 = \exp\left\{-(5-10x)^2/2\right\} + 0.75*\exp\left\{-(5-10x)^2/2 - (5-10y)^2/2\right\} + (x+y)^3(x-0.6)_+,$$

f_2 =cos(30*(x+y))

Table -1

Exact values of test integrals $= \iint_{P_N} f_i(x,y)dxdy = II_{P_N}(f_i),\ i = 1\,(1)9,\quad N = 6$ (convex polygon with sides), $N = 9$ (Non-convex polygon with nine sides). Using MATLAB symbolic method, Greens theorem and Boundary integration method.propsed in[16]

f_i	$II_{P_6}(f_i)$	$II_{P_9}(f_i)$
f_1	169.7043434031279086481893	130.8412349867964988121030
f_2	0.842118094148994776396486664e-2	0.14222050981512028804106645e-1
f_3	0.1568251255860885374289978	0.1393814567714511086304939
f_4	0.4850601470247113893333032	0.4374093366938112280464110
f_5	0.31414528632393338345376 69e-1	0.03122083897153926942995164
f_6	0.2663307419125152728165545	0.1829713239189687908913098
f_7	0.1990625494351890531 62	0.20842559601611674
f_8	0.5453868050054175481 57	0.4545305519051566
f_9	0.4492795032617593831499270	0.4115120322110287586271420

The computed values of integrals $II_{P_N}(f_i)$ $(N = 6{,}9),\ i = 1\,(1)9$ are given in Tables III- IV which use the numerical scheme developed in sections 6.1 and 6.2.

Conclusions:

The purpose of this paper is to develop efficient numerical integration schemes for arbitrary linear polygons in a 2-space which are very useful in finite element method, boundary integration method and mathematical modelling of several phenomena in science and engineering. The present study concentrates on those phenomena which may require integrating arbitrary functions over linear polygons which may be either convex or nonconvex. We can discretise these domains by using either triangles or quadrilaterals. In this paper, we propose to discretise the polygonal domain into triangles and these triangles are then divided into twelve special quadrilaterals,first by joining the centroid of the triangle to the midpoints of the three sides which creates three special quadrilaterals.

Then each of these special quadrilateral is further divided into four quadrilaterals by joining the centroid of special quadrilateral to the midpoints of four sides.This procedure creates 12-special quadrilaterals.This discretises the entire polygonal domain into a finite number of special quadrilaterals. The composite integration scheme is developed by discretising the arbitrary triangle into n^2,(n=1,2,3,4,5,……) triangles and then each of these triangles is divided further into twelve special quadrilaterals. The composite numerical integration is by performed by application of the well known Gauss Legendre Quadrature rules over the 2-square.Thus we are able to find sampling points and weight coefficients applicable for the entire polygonal domain. The composite integration scheme is tested on examples of integrals over convex and nonconvex polygons with complicated integrands.The present composite integration scheme performs better than the scheme propose in our earlier work[16]. The necessary and relevant MATLAB codes are also appended.The MATLAB codes are listed below:

special_convexquadrilateral_centroids_gausslegendrequadrature.m

fnxy.m

nodal_address_rtisosceles_triangle.m

coordinates_stdtriangle.m

glsampleptsweights.m

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TABLES:II,III AND IV

TABLE-IIa

COMPUTED VALUES OF INTEGRALS I_1

II(F_1)= I_1 , F_1 = sqrt(x+y),ST = standard triangle, = $\iint_{ST} \mathbf{sqrt(x+y)dxdy}$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
1^2	0.40000002194509	0.40000000076350	0.40000000010447	0.40000000002528
2^2	0.40000000387938	0.40000000013497	0.40000000001847	0.40000000000447
3^2	0.40000000140778	0.40000000004898	0.40000000000670	0.40000000000162
4^2	0.40000000068578	0.40000000002386	0.40000000000326	0.40000000000079
5^2	0.40000000039257	0.40000000001366	0.40000000000187	0.40000000000045
6^2	0.40000000024886	0.40000000000866	0.40000000000118	0.40000000000029
7^2	0.40000000016927	0.40000000000589	0.40000000000081	0.40000000000019
8^2	0.40000000012123	0.40000000000422	0.40000000000058	0.40000000000014
9^2	0.40000000009031	0.40000000000314	0.40000000000043	0.40000000000010
10^2	0.40000000006940	0.40000000000241	0.40000000000033	0.40000000000008
11^2	0.40000000005468	0.40000000000190	0.40000000000026	0.40000000000006
12^2	0.40000000004399	0.40000000000153	0.40000000000021	0.40000000000005
13^2	0.40000000003601	0.40000000000125	0.40000000000017	0.40000000000004
14^2	0.40000000002992	0.40000000000104	0.40000000000014	0.40000000000003
15^2	0.40000000002518	0.40000000000088	0.40000000000012	0.40000000000003
16^2	0.40000000002143	0.40000000000075	0.40000000000010	0.40000000000002
17^2	0.40000000001842	0.40000000000064	0.40000000000009	0.40000000000002
18^2	0.40000000001596	0.40000000000056	0.40000000000008	0.40000000000002
19^2	0.40000000001395	0.40000000000048	0.40000000000007	0.40000000000002
20^2	0.40000000001227	0.40000000000043	0.40000000000006	0.40000000000001

TABLE-IIb

COMPUTED VALUES OF INTEGRALS I_2

II(F_2)=I_2, F_2= sqrt(x+y),ST = standard triangle, I_2 = $\iint_{ST} (\frac{1}{\mathbf{sqrt(x+y)}})\mathbf{dxdy}$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40

1^2	0.66663553908536	0.66666249840522	0.66666540185815	0.66666612659056
2^2	0.66665566140475	0.66666519296370	0.66666621948933	0.66666647572093
3^2	0.66666067616085	0.66666586448438	0.66666642325415	0.66666656272897
4^2	0.66666277571900	0.66666614563399	0.66666650856560	0.66666659915715
5^2	0.66666388253116	0.66666629384603	0.66666655353875	0.66666661836079
6^2	0.66666454870302	0.66666638305240	0.66666658060735	0.66666662991914
7^2	0.66666498593526	0.66666644160170	0.66666659837343	0.66666663750530
8^2	0.66666529100893	0.66666648245380	0.66666661076950	0.66666664279845
9^2	0.66666551379328	0.66666651228661	0.66666661982191	0.66666664666385
10^2	0.66666568232612	0.66666653485467	0.66666662666991	0.66666664958796
11^2	0.66666581345586	0.66666655241411	0.66666663199811	0.66666665186312
12^2	0.66666591785344	0.66666656639388	0.66666663624010	0.66666665367445
13^2	0.66666600257146	0.66666657773838	0.66666663968245	0.66666665514435
14^2	0.66666607243838	0.66666658709419	0.66666664252136	0.66666665635657
15^2	0.66666613085954	0.66666659491730	0.66666664489519	0.66666665737020
16^2	0.66666618029821	0.66666660153758	0.66666664690403	0.66666665822798
17^2	0.66666622257564	0.66666660719891	0.66666664862189	0.66666665896151
18^2	0.66666625906437	0.66666661208508	0.66666665010454	0.66666665959460
19^2	0.66666629081630	0.66666661633694	0.66666665139472	0.66666666014551
20^2	0.66666631864973	0.66666662006409	0.66666665252568	0.66666666062843

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TABLE-IIc

COMPUTED VALUES OF INTEGRALS I_3

II(F_3)=I_3, F_3= 1/(sqrt(x^2+(1-y)^2)),ST = standard triangle, $I_3 = \iint_{ST}(\mathbf{1/sqrt(x^\wedge 2} + (\mathbf{1}-\mathbf{y})^\wedge \mathbf{2}))\,\mathbf{dxdy}$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=12*NTRIAS

==

ORDGLQ/ NTRIAS	10	20	30	40
1^2	0.87987874288470	0.88098206516840	0.88119676962379	0.88127331841190
2^2	0.88062616495212	0.88117782609397	0.88128517832167	0.88132345271572
3^2	0.88087530564126	0.88124307973583	0.88131464788763	0.88134016415033
4^2	0.88099987598583	0.88127570655676	0.88132938267061	0.88134851986763
5^2	0.88107461819257	0.88129528264932	0.88133822354039	0.88135353329801
6^2	0.88112444633040	0.88130833337769	0.88134411745358	0.88135687558494
7^2	0.88116003785742	0.88131765532652	0.88134832739158	0.88135926293274
8^2	0.88118673150269	0.88132464678815	0.88135148484507	0.88136105344359
9^2	0.88120749322678	0.88133008459164	0.88135394064224	0.88136244606314
10^2	0.88122410260606	0.88133443483443	0.88135590527997	0.88136356015878
11^2	0.88123769209819	0.88133799412398	0.88135751271084	0.88136447169158
12^2	0.88124901667497	0.88134096019861	0.88135885223656	0.88136523130224
13^2	0.88125859900917	0.88134346995407	0.88135998568141	0.88136587404972
14^2	0.88126681243848	0.88134562117303	0.88136095720556	0.88136642497614
15^2	0.88127393074389	0.88134748556280	0.88136179919316	0.88136690244570
16^2	0.88128015926112	0.88134911690385	0.88136253593231	0.88136732023157
17^2	0.88128565501161	0.88135055632242	0.88136318599626	0.88136768886615
18^2	0.88129054012316	0.88135183580559	0.88136376383089	0.88136801654134
19^2	0.88129491101245	0.88135298060632	0.88136428084082	0.88136830972440
20^2	0.88129884481280	0.88135401092699	0.88136474614976	0.88136857358916
21^2	0.88130240396550	0.88135494312187	0.88136516714355	0.88136881232394
22^2	0.88130563955887	0.88135579057176	0.88136554986519	0.88136902935556
23^2	0.88130859379629	0.88135656433036	0.88136589930668	0.88136922751486
24^2	0.88131130184726	0.88135727360908	0.88136621962805	0.88136940916089

25^2	0.88131379325415	0.88135792614550	0.88136651432371	0.88136957627524
26^2	0.88131609301436	0.88135852848681	0.88136678635048	0.88136973053463
27^2	0.88131822242196	0.88135908621024	0.88136703822711	0.88136987336741
28^2	0.88132019972901	0.88135960409629	0.88136727211255	0.88137000599784
29^2	0.88132204067007	0.88136008626606	0.88136748986797	0.88137012948135
30^2	0.88132375888171	0.88136053629117	0.88136769310635	0.88137024473262

TABLE-IId

COMPUTED VALUES OF INTEGRALS I_4

II(F_4)=I_4, F_4= pi^2/4*sin(((pi*(x-y+1)))/2),ST = standard triangle, $I_4 = \iint_{ST} \mathbf{pi^{\wedge}2/4 * sin(((pi * (x-y+1)))/2)dxdy}$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
1^2	1.00000000000000	1.00000000000000	1.00000000000000	0.99999999999999
2^2	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
3^2	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
4^2	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
5^2	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000

TABLE-IIe

COMPUTED VALUES OF INTEGRALS I_5

II(F_5)= I_5 , F_5 = exp(abs(x-y)),ST = standard triangle, $I_5 = \iint_{ST} \mathbf{exp(abs(x-y))dxdy}$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
1^2	0.71823000157704	0.71826823632684	0.71827568839041	0.71827834624597
2^2	0.71826887155499	0.71827843041348	0.71828029343934	0.71828095790496
3^2	0.71827606981991	0.71828031821554	0.71828114622785	0.71828144154605
4^2	0.71827858922156	0.71828097894687	0.71828144470396	0.71828161082047
5^2	0.71827975534617	0.71828128477119	0.71828158285578	0.71828168917035
6^2	0.71828038879700	0.71828145089802	0.71828165790122	0.71828173173079
7^2	0.71828077074800	0.71828155106726	0.71828170315125	0.71828175739338
8^2	0.71828101864896	0.71828161608095	0.71828173252026	0.71828177404940
9^2	0.71828118860906	0.71828166065413	0.71828175265556	0.71828178546871
10^2	0.71828131018053	0.71828169253706	0.71828176705822	0.71828179363687
11^2	0.71828140012968	0.71828171612683	0.71828177771456	0.71828179968039
12^2	0.71828146854339	0.71828173406878	0.71828178581959	0.71828180427698
13^2	0.71828152178535	0.71828174803183	0.71828179212720	0.71828180785421
14^2	0.71828156403121	0.71828175911109	0.71828179713210	0.71828181069263
15^2	0.71828159811301	0.71828176804927	0.71828180116979	0.71828181298252
16^2	0.71828162600648	0.71828177536452	0.71828180447435	0.71828181485663
17^2	0.71828164912390	0.71828178142722	0.71828180721309	0.71828181640985
18^2	0.71828166849652	0.71828178650782	0.71828180950817	0.71828181771146

$\mathbf{19^2}$	0.71828168489157	0.71828179080752	0.71828181145050	0.71828181881301
$\mathbf{20^2}$	0.71828169888940	0.71828179447855	0.71828181310884	0.71828181975350
$\mathbf{21^2}$	0.71828171093555	0.71828179763773	0.71828181453595	0.71828182056286
$\mathbf{22^2}$	0.71828172137669	0.71828180037599	0.71828181577292	0.71828182126438
$\mathbf{23^2}$	0.71828173048578	0.71828180276491	0.71828181685208	0.71828182187640
$\mathbf{24^2}$	0.71828173848012	0.71828180486148	0.71828181779918	0.71828182241353
$\mathbf{25^2}$	0.71828174553447	0.71828180671153	0.71828181863491	0.71828182288750

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TABLE-IIf

COMPUTED VALUES OF INTEGRALS $\boldsymbol{I_6}$

II($\boldsymbol{F_6}$)=$\boldsymbol{I_6}$, $\boldsymbol{F_6}$= 1/sqrt(1-$\boldsymbol{x^2}$),D ={(x,y)|0<=y<=π/4,0<=x<=sin(y)} $\boldsymbol{I_6} = \iint_D \left(\mathbf{1/sqrt}(\mathbf{1}-\boldsymbol{x^2})\right)\mathbf{dxdy}$

$$\boldsymbol{I_6} = \int_{-1}^{1}\int_{-1}^{1} \frac{\frac{1}{2}sin\left(\frac{\pi}{8}(1+s)\right)\frac{\pi}{8}}{\left\{1-\left(\frac{sin\left(\frac{\pi}{8}(1+s)\right)}{2}\right)^2(1+t)^2\right\}}\,dsdt = \sum_{n=1}^{4}\iint_{T_n} \frac{\frac{1}{2}sin\left(\frac{\pi}{8}(1+s)\right)\frac{\pi}{8}}{\left\{1-\left(\frac{sin\left(\frac{\pi}{8}(1+s)\right)}{2}\right)^2(1+t)^2\right\}}\,dsdt$$

, where $(\boldsymbol{T_n}\,,\ \boldsymbol{n}=\mathbf{1}:\mathbf{4})$ are the four triangle obtained by joining the centroid (0,0) of the 2-square to the four vertices (-1,-1),(-1,1),(1,1),(-1,1)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=4*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
$\mathbf{1^2}$	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
$\mathbf{2^2}$	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
$\mathbf{3^2}$	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
$\mathbf{4^2}$	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
$\mathbf{5^2}$	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404

TABLE-IIg

COMPUTED VALUES OF INTEGRALS $\boldsymbol{I_7}$

II($\boldsymbol{F_7}$)=$\boldsymbol{I_7}$,$\boldsymbol{F_7} = \frac{1}{sqrt(x+y)}$,Q=quadrilateral connecting the points{(-1,2),(2,1),(3,3),(1,4)}

$\boldsymbol{I_7} = \iint_Q \frac{1}{sqrt(x+y)}\,dxdy = \sum_{n=1}^{4}\iint_{T_n}\frac{1}{sqrt(x+y)}\,dxdy$, where $(\boldsymbol{T_n}\,,\ \boldsymbol{n}=\mathbf{1}:\mathbf{4})$ *are the four triangles* obtained by joining the centroid(5/4,5/2) of the quadrilateral Q to the four vertices (-1,2),(2,1),(3,3),(1,4)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=4*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
$\mathbf{1^2}$	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678971
$\mathbf{2^2}$	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678972
$\mathbf{3^2}$	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678971
$\mathbf{4^2}$	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678971
$\mathbf{5^2}$	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678972

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TABLE-III=a

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES $\boldsymbol{P_6}$:
$\boldsymbol{10^{-2}\, II_{P_6}(f_1), f_1 = (x+y)^{19}}$
ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE
NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{10^{-2} * II_{P_6}(f_1)}$, $\boldsymbol{f_1 = (x+y)^{19}}$			
1^2	1.69704343403128	1.69704343403128	1.69704343403128	1.69704343403127
2^2	1.69704343403128	1.69704343403128	1.69704343403128	1.69704343403128
3^2	1.69704343403128	1.69704343403128	1.69704343403128	1.69704343403128
4^2	1.69704343403128	1.69704343403128	1.69704343403128	1.69704343403128
5^2	1.69704343403128	1.69704343403128	1.69704343403128	1.69704343403128

TABLE-III=b

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES $\boldsymbol{P_6 : II_{P_6}(f_2), f_2 = cos(30*(x+y))}$
ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE
NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{II_{P_6}(f_2), f_2 = cos(30*(x+y))}$			
1^2	0.00842118094148974	0.00842118094148994	0.00842118094148994	0.00842118094148994
2^3	0.00842118094148995	0.00842118094148996	0.00842118094148996	0.00842118094149
3^2	0.00842118094148997	0.00842118094148997	0.00842118094148997	0.00842118094148997
4^2	0.00842118094148997	0.00842118094148999	0.00842118094149001	0.00842118094148996
5^2	0.00842118094148995	0.00842118094148996	0.00842118094149	0.00842118094148999

TABLE-III=c

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES $\boldsymbol{P_6}$:
$\boldsymbol{II_{P_6}(f_3), f_3 = \sqrt{(x-1/2)^2 + (y-1/2)^2}}$
ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE
NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{II_{P_6}(f_3), f_3 = \sqrt{(x-1/2)^2 + (y-1/2)^2}}$			
1^2	0.156825123965559	0.156825125557227	0.156825125583435	0.156825125585605
2^2	0.156825125383522	0.156825125582481	0.156825125585757	0.156825125586028
3^2	0.156825125526069	0.15682512558502	0.15682512558599	0.156825125586071
4^2	0.156825125560768	0.156825125585638	0.156825125586047	0.156825125586081
5^2	0.156825125573124	0.156825125585858	0.156825125586067	0.156825125586085

TABLE-III=d

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES $\boldsymbol{P_6}$
$\boldsymbol{II_{P_6}(f_4)}$, f_4=exp(-((x-1/2)^2+(y-1/2)^2))
ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE
NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/	10	20	30	40

NTRIAS COMPUTED VALUES OF $II_{P_6}(f_4)$, f_4 =exp(-((x-1/2)^2+(y-1/2)^2))

1^2	0.485060147024712	0.485060147024711	0.485060147024711	0.485060147024711
2^2	0.485060147024711	0.485060147024711	0.485060147024711	0.485060147024711
3^2	0.485060147024711	0.485060147024711	0.485060147024711	0.485060147024711
4^2	0.485060147024711	0.485060147024711	0.485060147024711	0.485060147024711
5^2	0.485060147024711	0.485060147024711	0.485060147024711	0.485060147024711

TABLE-III=e

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES P_6

$II_{P_6}(f_5)$, f_5 =exp(-100*((x-1/2)^2+(y-1/2)^2))

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_6}(f_5)$, f_5 =exp(-100*((x-1/2)^2+(y-1/2)^2))			
1^2	0.0314145286323933	0.0314145286323934	0.0314145286323934	0.0314145286323933
2^2	0.0314145286323934	0.0314145286323934	0.0314145286323933	0.0314145286323933
3^2	0.0314145286323933	0.0314145286323933	0.0314145286323933	0.0314145286323933
4^2	0.0314145286323933	0.0314145286323933	0.0314145286323933	0.0314145286323933
5^2	0.0314145286323933	0.0314145286323934	0.0314145286323933	0.0314145286323933

TABLE-III=f

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES P_6

$II_{P_6}(f_6)$, f_6 =0.75*exp(-0.25*(9*x-2)^2-0.25*(9*y-2)^2)
+0.75*exp((-1/49)*(9*x+1)^2-0.1*(9*y+1))
+0.5*exp(-0.25*(9*x-7)^2-0.25*(9*y-3)^2)
-0.2*exp(-(9*y-4)^2-(9*y-7)^2)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_6}(f_6)$			
1^2	0.266330741912515	0.266330741912515	0.266330741912515	0.266330741912516
2^2	0.266330741912515	0.266330741912515	0.266330741912515	0.266330741912515
3^2	0.266330741912515	0.266330741912515	0.266330741912515	0.266330741912515
4^2	0.266330741912515	0.266330741912515	0.266330741912515	0.266330741912515
5^2	0.266330741912515	0.266330741912515	0.266330741912515	0.266330741912515

TABLE-III=g

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES P_6

$II_{P_6}(f_7)$, f_7 =abs(x^2+y^2-1/4)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/	10	20	30	40

NTRIAS	COMPUTED VALUES OF $II_{P_6}(f_7)$			
1^2	0.199063682003122	0.199062417721252	0.199062568197312	0.199062565340202
2^2	0.199062675029936	0.199062561904209	0.199062543498971	0.199062548569028
3^2	0.19906253604323	0.19906253569576	0.199062548533577	0.199062549384845
4^2	0.19906257991409	0.199062551322415	0.199062551535734	0.199062548282495
5^2	0.19906256246502 4	0.19906255078575	0.199062550080506	0.199062548262593

TABLE-III-h

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES P_6

$II_{P_6}(f_8)$, f_8 =sqrt(abs(3-4*x-3*y))

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_6}(f_8)$			
1^2	0.545352368148517	0.5453872385192	0.545388747334642	0.545387028851558
2^2	0.545395232433075	0.545384864774656	0.545387316084602	0.545386891072513
3^2	0.545385962838818	0.545386323432861	0.545386991731237	0.54538687870392
4^2	0.545387913563219	0.545386771887475	0.545386762414189	0.545386794302575
5^2	0.545385437760795	0.545386324026092	0.545386696285851	0.545386684194642

TABLE-III-i

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 6-SIDES P_6: $II_{P_6}(f_9)$

f_9 =exp(-((5-10*x)^2)/2)+0.75*exp(-((5-10*y)^2)/2)+0.75*(exp(-((5-10*x)^2)/2-((5-10*y)^2)/2))+((x+y)^3)* max((x-0.6),0)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=7*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_6}(f_9)$			
1^2	0.449279871932761	0.449279545229348	0.449279524605234	0.449279514622101
2^2	0.449279250502769	0.44927950533792	0.449279494434926	0.449279501458188
3^2	0.449279493949995	0.449279506719873	0.449279502772062	0.449279503603932
4^2	0.449279491361516	0.449279500398301	0.449279502621436	0.449279502495234
5^2	0.449279469355417	0.449279496903507	0.449279499913728	0.449279501256445

TABLE-IV-a

COMPUTED VALUES OF INTEGRALS OVER NONCONVEX POLYGON WITH 9-SIDES P_9 :

$II_{P_9}(f_1)$, $f_1 = (x + y)^{19}$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_9}(f_1)$, $f_1 = (x + y)^{19}$			
1^2	130.841234986797	130.841234986797	130.841234986797	130.841234986796
2^2	130.841234986796	130.841234986796	130.841234986796	130.841234986796
3^2	130.841234986797	130.841234986797	130.841234986796	130.841234986797
4^2	130.841234986797	130.841234986797	130.841234986797	130.841234986796
5^2	130.841234986796	130.841234986796	130.841234986796	130.841234986796

TABLE-IV-b

COMPUTED VALUES OF INTEGRALS OVER NONCONVEX POLYGON WITH 9-SIDES P_9 :

$II_{P_9}(f_2)$,$f_2 = cos(30 * (x + y))$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_9}(f_2)$,$f_2 = cos(30 * (x + y))$			
1^2	0.0142220509815109	0.014222050981512	0.014222050981512	0.014222050981512
2^2	0.014222050981512	0.014222050981512	0.0142220509815121	0.0142220509815122
3^2	0.0142220509815121	0.014222050981512	0.014222050981512	0.014222050981512
4^2	0.014222050981512	0.014222050981512	0.0142220509815121	0.0142220509815121
5^2	0.014222050981512	0.014222050981512	0.0142220509815119	0.0142220509815122

TABLE-IV-c

COMPUTED VALUES OF INTEGRALS OVER NONCONVEX POLYGON WITH 9-SIDES P_9 :

$II_{P_9}(f_3)$, $f_3 = \sqrt{(x - 1/2)^2 + (y - 1/2)^2}$

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_9}(f_3)$, $f_3 = \sqrt{(x - 1/2)^2 + (y - 1/2)^2}$			
1^2	0.139381455990486	0.13938145675778	0.139381456770207	0.139381456771225
2^2	0.139381456496385	0.139381456766523	0.139381456770997	0.139381456771368
3^2	0.139381456742526	0.139381456770945	0.139381456771405	0.139381456771443
4^2	0.139381456737068	0.139381456770835	0.139381456771394	0.139381456771441
5^2	0.139381456765203	0.139381456771342	0.139381456771441	0.139381456771449

TABLE-IV-d

COMPUTED VALUES OF INTEGRALS OVER NONCONVEX POLYGON WITH 9-SIDES P_9 :

$II_{P_9}(f_4)$, f_4=exp(-((x-1/2)^2+(y-1/2)^2))

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $II_{P_9}(f_4)$, f_4=exp(-((x-1/2)^2+(y-1/2)^2))			

1^2	0.437409336693811	0.437409336693811	0.437409336693811	0.437409336693812
2^2	0.437409336693811	0.437409336693811	0.437409336693811	0.437409336693811
3^2	0.437409336693811	0.437409336693811	0.437409336693811	0.437409336693811
4^2	0.437409336693811	0.437409336693811	0.437409336693811	0.437409336693811
5^2	0.437409336693811	0.437409336693811	0.437409336693811	0.437409336693811

TABLE-IV-e

COMPUTED VALUES OF INTEGRALS OVER NONCONVEX POLYGON WITH 9-SIDES $\boldsymbol{P_9}$:

$\boldsymbol{II_{P_9}(f_5)}$, $\boldsymbol{f_5}$ =exp(-100*((x-1/2)^2+(y-1/2)^2))

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{II_{P_9}(f_5)}$, $\boldsymbol{f_5}$ =exp(-100*((x-1/2)^2+(y-1/2)^2))			
1^2	0.0312208389715392	0.0312208389715393	0.0312208389715392	0.0312208389715393
2^2	0.0312208389715393	0.0312208389715393	0.0312208389715392	0.0312208389715393
3^2	0.0312208389715393	0.0312208389715393	0.0312208389715392	0.0312208389715393
4^2	0.0312208389715393	0.0312208389715393	0.0312208389715393	0.0312208389715392
5^2	0.0312208389715392	0.0312208389715392	0.0312208389715393	0.0312208389715393

TABLE-IV-f

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 9-SIDES $\boldsymbol{P_9}$

$\boldsymbol{II_{P_9}(f_6)}$, $\boldsymbol{f_6}$ =0.75*exp(-0.25*(9*x-2)^2-0.25*(9*y-2)^2)
+0.75*exp((-1/49)*(9*x+1)^2-0.1*(9*y+1))
+0.5*exp(-0.25*(9*x-7)^2-0.25*(9*y-3)^2)
-0.2*exp(-(9*y-4)^2-(9*y-7)^2)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{II_{P_9}(f_6)}$			
1^2	0.182971323918969	0.182971323918969	0.182971323918968	0.182971323918969
2^2	0.182971323918969	0.182971323918969	0.182971323918969	0.182971323918969
3^2	0.182971323918969	0.182971323918969	0.182971323918969	0.182971323918969
4^2	0.182971323918969	0.182971323918969	0.182971323918969	0.182971323918969
5^2	0.182971323918969	0.182971323918969	0.182971323918969	0.182971323918969

TABLE-IV-g

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 9-SIDES $\boldsymbol{P_9}$

$\boldsymbol{II_{P_9}(f_7)}$, f_7 =abs(x^2+y^2-1/4)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{II_{P_9}(f_7)}$			
1^2	0.208428425039323	0.208426062371752	0.208425558303571	0.208425516193425
2^2	0.208425720811087	0.208425607607683	0.208425609219724	0.208425599826904

3^2	0.208425603158739	0.208425608167868	0.208425595222261	0.20842559463275
4^2	0.208425611489742	0.208425593387142	0.208425596454439	0.208425595571175
5^2	0.208425594454964	0.208425595723138	0.208425596148378	0.208425596320229

TABLE-IV-h

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 9-SIDES $\boldsymbol{P_9}$

$\boldsymbol{II_{P_9}(f_8)}$, f_8 =sqrt(abs(3-4*x-3*y))

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{II_9(f_8)}$			
1^2	0.454552840350322	0.454536698753457	0.454534754796649	0.454530878187737
2^2	0.45452810060636	0.454531637953984	0.454531691197886	0.454530687718435
3^2	0.454530035292255	0.454531135439704	0.454531015455077	0.454530529688537
4^2	0.454533919479444	0.454531446213546	0.454531173336192	0.454530534829894
5^2	0.454531807046796	0.454531122646137	0.454531151798593	0.454530505517146

TABLE-IV-i

COMPUTED VALUES OF INTEGRALS OVER CONVEX POLYGON WITH 9-SIDES $\boldsymbol{P_9}$: $\boldsymbol{II_{P_9}(f_9)}$,

f_9 =exp(-((5-10*x)^2)/2)+0.75*exp(-((5-10*y)^2)/2)+0.75*(exp(-((5-10*x)^2)/2-((5-10*y)^2)/2))+((x+y)^3)* max((x-0.6),0)

ORDGLQ=ORDER OF GAUSS LEGENDRE QUADRATURE

NTRIAS=NUMBER OF TRIANGLES ,NQUADS=NUMBER OF QUADRILATERALS=9*12*NTRIAS

ORDGLQ/ NTRIAS	10	20	30	40
	COMPUTED VALUES OF $\boldsymbol{II_9(f_9)}$			
1^2	0.411512598792815	0.411511997727817	0.41151201569197	0.411512048669476
2^2	0.41151204293932	0.411512026537537	0.41151202403989	0.411512033286248
3^2	0.411511874131632	0.411512059922242	0.411512026618889	0.411512030840718
4^2	0.411512002721366	0.411512030493165	0.411512032225334	0.411512032580411
5^2	0.411511964014124	0.411512013576217	0.41151202344099	0.411512026891239

COMPUTER PROGRAMS

(1) special_convexquadrilateral_centroids_gausslegendrequadrature

```
function[]=special_convexquadrilateral_centroids_gausslegendrequadrature(ga,gz,m,mesh,mdiv,ndiv)
%NUMERICAL INTEGRATION USING LEMMA2
%special_convexquadrilateral_centroids_gausslegendrequadrature(5,40,38,10,1,2)
%gauss legendre quadrature rules choosen:5,10,15,20,25,30,35,40
syms Ui Vi Wi
switch mesh
case 1%convex polygon with 6  sides for this paper
mst_tri=[8 1 2;...%1
       8 2 3;...%2
       8 3 4;...%3
       8 4 5;...%4
       8 5 6;...%5
       8 6 7;...%6
       8 7 1];%7
gcoord=[0.1 0.0;...%1
      0.7 0.2;...%2
      1.0 0.5;...%3
      .75 .85;...%4
      0.5 1.0;...%5
      0.25 0.625;...%6
      0.0 0.25;...%7
      0.5 0.5];%8
[mst_elm,dimension]=size(mst_tri);
case 2%nonconvex polygon with 9 sides for this paper

mst_tri=[2 9 1;...%1-3
       4 2 3;...%4-6
       5 2 4;...%7-9
       5 9 2;...%10-12
       5 7 9;...%13-15
       6 7 5;...%16-18
       7 8 9];%19-21
gcoord=[0.25 0.00;...%1
      0.75 0.50;...%2
      0.75 0.00;...%3
      1.00 0.50;...%4
      0.75 0.75;...%5
      0.75 0.85;...%6
      0.50 1.00;...%7
      0.00 0.75;...%8
      0.25 0.50];%9
[mst_elm,dimension]=size(mst_tri);
case 3%convex poygon:new mesh
mst_tri=[3 7 8;...%1-3
       4 7 3;...%4-6
       5 7 4;...%7-9
       9 7 5;...%10-12
       9 6 7;...%13-15
       1 7 6;...%16-18
       2 7 1;...%19-21
       8 7 2];%22-24
gcoord=[0.1 0.0;...%1
      0.7 0.2;...%2
      1.0 0.5;...%3
      .75 .85;...%4
      0.5 1.0;...%5
      0.0 .25;...%6
      0.5 0.5;...%7
      .75 .25;...%8
      0.3 0.7];%9
[mst_elm,dimension]=size(mst_tri);
case 4%standard triangle functions at cases 34-38
   mst_tri=[1 2 3];
```

```
   gcoord=[0.0 0.0;...%1
         1.0 0.0;...%2
         0.0 1.0];%3
   [mst_elm,dimension]=size(mst_tri);
case 5%quadrilateral function at case 39
       mst_tri=[1 2 5;...%1-3
              2 3 5;...%4-6
              3 4 5;...%7-9
              4 1 5];%10-12
       gcoord=[-1 2;...%1
              2 1;...%2
              3 3;...%3
              1 4;...%4
             5/4 5/2];%5
      [mst_elm,dimension]=size(mst_tri);
 case 6%standard square-function at case 50
   mst_tri=[1 2 5;...%1-3
          2 3 5;...%4-6
          3 4 5;...%7-9
          4 1 5];%10-12
  gcoord=[-1 -1;...%1
         1 -1;...%2
         1  1;...%3
        -1  1;...%4
         0  0];%5
 [mst_elm,dimension]=size(mst_tri);
%[nel,nnel]=size(nodes);
[nnode,dimension]=size(gcoord);
%if necessary include gauss legendre quadrature


for div=mdiv:ndiv
[Mst_tri]=nodal_address_rtisosceles_triangle(div);
 mst=double(vpa(Mst_tri));
%compute element cartesian/global coordinates
for L=1:mst_elm
if div==1
   for M=1:3
     LM=mst_tri(L,M);
     xx(L,M)=gcoord(LM,1);
     yy(L,M)=gcoord(LM,2);
    end
else
  for M=1:3
     LM=mst_tri(L,M);
     xx(L,M)=gcoord(LM,1);
     yy(L,M)=gcoord(LM,2);
   end
[Ui,Vi,Wi]=coordinates_stdtriangle(div);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
%wi=double(vpa(Wi));
xxL1=xx(L,1);
xxL2=xx(L,2)-xxL1;
xxL3=xx(L,3)-xxL1;
yyL1=yy(L,1);
yyL2=yy(L,2)-yyL1;
yyL3=yy(L,3)-yyL1;

%cartesian/global coordinates
 for i=4:(div+1)*(div+2)/2
  uii1=ui(i,1);vii1=vi(i,1);
  xx(L,i)=xxL1+xxL2*uii1+xxL3*vii1;
  yy(L,i)=yyL1+yyL2*uii1+yyL3*vii1;
 end
end%if div
end%for L
%_________________________________________________________________
format long
ggg=0;
for ng=ga:10:gz
 ggg=ggg+1;
[s,wwww]=glsampleptsweights(ng);
nn=ng^2;
kkk=0;
for i=1:ng
   for j=1:ng
       kkk=kkk+1;
```

```
      si=s(i,1);sj=s(j,1);
      wiwj=wwww(i,1)*wwww(j,1);
      uij1(kkk,1)=(-3/16+1/48*sj)*(-3/2-1/2*si);vij1(kkk,1)=(1/16+1/48*sj)*(9/2-1/2*si);jacij1=1/128-1/768*si-1/768*sj;
      uij2(kkk,1)=(-3/16+1/48*si)*(-1/2+1/2*sj);vij2(kkk,1)=(1/16+1/48*si)*(11/2+1/2*sj);jacij2=1/96+1/768*sj-1/768*si;
      uij3(kkk,1)=(-11/48-1/48*sj)*(-1/2+1/2*si);vij3(kkk,1)=(1/48-1/48*sj)*(11/2+1/2*si);jacij3=5/384+1/768*si+1/768*sj;
      uij4(kkk,1)=(-11/48-1/48*si)*(-3/2-1/2*sj);vij4(kkk,1)=(1/48-1/48*si)*(9/2-1/2*sj);jacij4=1/96-1/768*sj+1/768*si;
      wtij1(kkk,1)=wiwj*jacij1;wtij2(kkk,1)=wiwj*jacij2;wtij3(kkk,1)=wiwj*jacij3;wtij4(kkk,1)=wiwj*jacij4;
  end
end

   no(ggg,1)=ng;
   integralvalue(ggg,div)=0;
for iel=1:mst_elm
for eldiv=1:div*div
funxy=0;
n1=mst(eldiv,1);
n2=mst(eldiv,2);
n3=mst(eldiv,3);
x1=xx(iel,n1);
x2=xx(iel,n2);
x3=xx(iel,n3);
y1=yy(iel,n1);
y2=yy(iel,n2);
y3=yy(iel,n3);
delabc=(x2-x1)*(y3-y1)-(x3-x1)*(y2-y1);
for kkkk=1:nn
UIJ1=uij1(kkkk,1);UIJ2=uij2(kkkk,1);UIJ3=uij3(kkkk,1);UIJ4=uij4(kkkk,1);
VIJ1=vij1(kkkk,1);VIJ2=vij2(kkkk,1);VIJ3=vij3(kkkk,1);VIJ4=vij4(kkkk,1);
WIJ1=1-UIJ1-VIJ1;WIJ2=1-UIJ2-VIJ2;WIJ3=1-UIJ3-VIJ3;WIJ4=1-UIJ4-VIJ4;
wc1=wtij1(kkkk,1);wc2=wtij2(kkkk,1);wc3=wtij3(kkkk,1);wc4=wtij4(kkkk,1);
xx11=x1*WIJ1+x2*UIJ1+x3*VIJ1;yy11=y1*WIJ1+y2*UIJ1+y3*VIJ1;
xx12=x1*WIJ2+x2*UIJ2+x3*VIJ2;yy12=y1*WIJ2+y2*UIJ2+y3*VIJ2;
xx13=x1*WIJ3+x2*UIJ3+x3*VIJ3;yy13=y1*WIJ3+y2*UIJ3+y3*VIJ3;
xx14=x1*WIJ4+x2*UIJ4+x3*VIJ4;yy14=y1*WIJ4+y2*UIJ4+y3*VIJ4;
xx21=x2*WIJ1+x3*UIJ1+x1*VIJ1;yy21=y2*WIJ1+y3*UIJ1+y1*VIJ1;
xx22=x2*WIJ2+x3*UIJ2+x1*VIJ2;yy22=y2*WIJ2+y3*UIJ2+y1*VIJ2;
xx23=x2*WIJ3+x3*UIJ3+x1*VIJ3;yy23=y2*WIJ3+y3*UIJ3+y1*VIJ3;
xx24=x2*WIJ4+x3*UIJ4+x1*VIJ4;yy24=y2*WIJ4+y3*UIJ4+y1*VIJ4;
xx31=x3*WIJ1+x1*UIJ1+x2*VIJ1;yy31=y3*WIJ1+y1*UIJ1+y2*VIJ1;
xx32=x3*WIJ2+x1*UIJ2+x2*VIJ2;yy32=y3*WIJ2+y1*UIJ2+y2*VIJ2;
xx33=x3*WIJ3+x1*UIJ3+x2*VIJ3;yy33=y3*WIJ3+y1*UIJ3+y2*VIJ3;
xx34=x3*WIJ4+x1*UIJ4+x2*VIJ4;yy34=y3*WIJ4+y1*UIJ4+y2*VIJ4;
funxy=funxy+(fnxy(m,xx11,yy11)+fnxy(m,xx21,yy21)+fnxy(m,xx31,yy31))*wc1...
        +(fnxy(m,xx12,yy12)+fnxy(m,xx22,yy22)+fnxy(m,xx32,yy32))*wc2...
        +(fnxy(m,xx13,yy13)+fnxy(m,xx23,yy23)+fnxy(m,xx33,yy33))*wc3...
        +(fnxy(m,xx14,yy14)+fnxy(m,xx24,yy24)+fnxy(m,xx34,yy34))*wc4;
end
fff=funxy*delabc;
integralvalue(ggg,div)=integralvalue(ggg,div)+fff;
end%eldiv
end%iel
%disp([ ggg div integralvalue(ggg,div)])
end%ng
end%div


switch m
  case 16%polygonal domain
     disp('fn=(x+y)^19');

  case 17 %polygonal domain
     disp('fn=cos(30*(x+y))');
  case 18%polygonal domain
     disp('fn=sqrt((x-1/2)^2+(y-1/2)^2)');
  case 19%polygonal domain
     disp('fn=exp(-((x-1/2)^2+(y-1/2)^2))');
  case 20%polygonal domain
    disp(' fn=exp(-100*((x-1/2)^2+(y-1/2)^2))');
  case 21%polygonal domain
     disp('f1=0.75*exp(-0.25*(9*x-2)^2-0.25*(9*y-2)^2)');
      disp('f2=0.75*exp((-1/49)*(9*x+1)^2-0.1*(9*y+1))');
      disp('f3=0.5*exp(-0.25*(9*x-7)^2-0.25*(9*y-3)^2)');
       disp('f4=-0.2*exp(-(9*y-4)^2-(9*y-7)^2)');
       disp('fn=f1+f2+f3+f4');

   case 34%EX-1 standard triangle
     disp('fn=sqrt(x+y)');
  case 35%EX-2 standard triangle
```

```
  disp('fn=1/sqrt(x+y)');
case 36%EX-3 standard triangle
  disp('fn=1/sqrt(x^2+(1-y)^2)');
case 37%EX-4 standard triangle
  disp('fn=pi^2/4*sin(((pi*(x-y+1)))/2)');
case 38%EX-5 standard triangle
  disp('EX-5 standard triangle')
 disp(' fn=exp(abs(x-y))');
case 39
  disp('fn=exp(-100*((x-1/2)^2+(y-1/2)^2))');
case 40
disp(' fn=sqrt((x-1/2)^2+(y-1/2)^2)');
case 41%polygonal domain
 disp('fn=abs(x^2+y^2-1/4)');
case 42%polygonal domain
 disp('fn=sqrt(abs(3-4*x-3*y))');
case 43%polygonal domain
disp('fm=double((x-0.6))');
disp('(if fm<=0')
  disp(' fm=0');
disp('end')
disp('fn=exp(-((5-10*x)^2)/2)+0.75*exp(-((5-10*y)^2)/2)+0.75*(exp(-((5-10*x)^2)/2-((5-10*y)^2)/2))+((x+y)^3)*fm');
case 44%polygonal domain
 disp('fm=double((x+y-1))');
disp('if fm<=0')
   disp('fm=0');
disp('end')
%
disp(' fn=((1/9*sqrt(64-81*((x-.5)^2+(y-.5)^2)))-.5)*fm');
case 50%EX-7 standard 2_square
   disp('fx=(1/2)*sin(pi*(1+x)/8)');
   disp('f1=fx*pi/8');
   disp('f2=sqrt(1-(fx*(1+y))^2)');
   disp('fn=f1/f2');
case 51%EX-6 arbitrary quadrilateral
   disp(' fn=1/sqrt(x+y)');
otherwise
   disp('something wrong')
end
format long
%numdiv=ndiv-mdiv+1;
%for k=1:numdiv
 %   tridiv(k+1,1)=mdiv+k-1;
 %end
%tridiv(1,1)=0
table(:,1)=no;
%TABLE(:,1)=tridiv;
TABLE(:,1)=no;
TABLE(:,2:ndiv-mdiv+2)=integralvalue(:,mdiv:ndiv);
format long g
disp([TABLE'])
```

(2)fnxy.m

```
function[fn]=fnxy(n,x,y)
switch n
  case 16
     fn=(x+y)^19;

  case 17
     fn=cos(30*(x+y));
  case 18
     fn=sqrt((x-1/2)^2+(y-1/2)^2);
  case 19
     fn=exp(-((x-1/2)^2+(y-1/2)^2));
  case 20
     fn=exp(-100*((x-1/2)^2+(y-1/2)^2));
  case 21
     f1=0.75*exp(-0.25*(9*x-2)^2-0.25*(9*y-2)^2);
      f2=0.75*exp((-1/49)*(9*x+1)^2-0.1*(9*y+1));
      f3=0.5*exp(-0.25*(9*x-7)^2-0.25*(9*y-3)^2);
        f4=-0.2*exp(-(9*y-4)^2-(9*y-7)^2);
        fn=f1+f2+f3+f4;
  case 34
     fn=sqrt(x+y);
  case 35
   fn=1/sqrt(x+y);
case 36
```

```
 fn=1/sqrt(x^2+(1-y)^2);
case 37
 fn=pi^2/4*sin(((pi*(x-y+1)))/2);
case 38
 fn=exp(abs(x-y));
case 41
 fn=abs(x^2+y^2-1/4);
case 42
 fn=sqrt(abs(3-4*x-3*y));
case 43
fm=double((x-0.6));
if fm<=0
   fm=0;
end
fn=exp(-((5-10*x)^2)/2)+0.75*exp(-((5-10*y)^2)/2)+0.75*(exp(-((5-10*x)^2)/2-((5-10*y)^2)/2))+((x+y)^3)*fm;
case 44
 fm=double((x+y-1));
if fm<=0
   fm=0;
end
%
 fn=((1/9*sqrt(64-81*((x-.5)^2+(y-.5)^2)))-.5)*fm;
case 50%EX-7:A 2-SQUARE DIVIDED INTO FOUR TRIANGLES

fx=(1/2)*sin(pi*(1+x)/8);
f1=fx*pi/8;
f2=sqrt(1-(fx*(1+y))^2);
fn=f1/f2;
 case 51%EX-6 arbitrary quadrilateral
    fn=1/sqrt(x+y);

otherwise
   disp('something wrong')
end
```

(3) nodal_address_rtisosceles_triangle.m

```
function[mst_tri]=nodal_address_rtisosceles_triangle(n)
syms mst_tri x
%disp('address and node number')
%disp('triangle vertices')
%disp([1  1;(n+1)  2;(n+1)*(n+2)/2  3])
%disp('triangle base nodes')
%disp([(2:n)' (4:(n+2))'])
%disp('left edge')
nni=1;
for i=0:(n-2)
   nni=nni+(n-i)+1;
  %disp([nni 3*n-i])
end
%disp('right edge')
nni=n+1;
for i=0:(n-2)
   nni=nni+n-i;
   %disp([nni n+3+i])
end
%disp('interior nodes')
nni=1;jj=0;
for i=0:n-3
   nni=nni+(n-i)+1;
   for j=1:n-2-i
      jj=jj+1;
      nnj=nni+j;
      %disp([nnj 3*n+jj])
   end
end
%disp('triangle  nodal vertices')
elm(1,1)=1;elm(n+1,1)=2;elm((n+1)*(n+2)/2)=3;
%disp('triangle base nodes')
kk=3;
for k=2:n
   kk=kk+1;
   elm(k,1)=kk;
end
%disp('left edge nodes')
nni=1;
for i=0:(n-2)
```

```
    nni=nni+(n-i)+1;
    elm(nni,1)=3*n-i;
end
%disp('right edge nodes')
nni=n+1;
for i=0:(n-2)
    nni=nni+n-i;
  elm(nni,1)= n+3+i;
end
%disp('interior nodes')
nni=1;jj=0;
for i=0:n-3
    nni=nni+(n-i)+1;
    for j=1:n-2-i
        jj=jj+1;
        nnj=nni+j;
        elm(nnj,1)= 3*n+jj;
    end
end
%disp(elm)
%disp(length(elm))
%to find elements of elm array as n-rows
jj=0;kk=0;
for j=0:n-1
    jj=j+1;
    for k=1:(n+1)-j
        kk=kk+1;
        row_nodes(jj,k)=elm(kk,1);
    end
end
row_nodes(n+1,1)=3;
for jj=(n+1):-1:1
%disp(row_nodes(jj,:))
end
kk=0;
for i=1:n
    for k=1:(n+1)-i
        kk=kk+1;
        mst_tri(kk,1)=row_nodes(i,k);
        mst_tri(kk,2)=row_nodes(i,k+1);
        mst_tri(kk,3)=row_nodes(i+1,k);
        %mst_tri(kk,4)=x;
    end
       for k=1:(n)-i
        kk=kk+1;
        mst_tri(kk,1)=row_nodes(i+1,k+1);
        mst_tri(kk,2)=row_nodes(i+1,k);
        mst_tri(kk,3)=row_nodes(i,k+1);
        %mst_tri(kk,4)=x;
    end


end%for i
%disp([mst_tri])
%disp(length(mst_tri))
%([mst_tri])
```

(4) coordinates_stdtriangle.m

```
function[ui,vi,wi]=coordinates_stdtriangle(n)
%divides the standard triangle into n^2 right isoscles triangles
% each of side length 1/n
syms ui vi wi table
%corner nodes
ui=sym([0;1;0]);
vi=sym([0;0;1]);
wi=sym([1;0;0]);
%nodes along v=0
if (n-1)>0
k1=3;
for i1=1:n-1
    k1=k1+1;
    ui(k1,1)=sym(i1/n);
    vi(k1,1)=sym(0);
    wi(k1,1)=sym(1-ui(k1,1));
end
%nodes along w=0
```

```
k2=k1;
for i2=1:n-1
   k2=k2+1;
   ui(k2,1)=sym((n-i2)/n);
   vi(k2,1)=sym(1-ui(k2,1));
   wi(k2,1)=0;
end
%nodes along u=0
k3=k2;
for i3=1:n-1
   k3=k3+1;
   wi(k3,1)=sym(i3/n);
    vi(k3,1)=sym(1-wi(k3,1));
    ui(k3,1)=sym(0);
 end
end
 if (n-2)>0
 k4=k3;
  for i4=1:(n-2)
      for j4=1:(n-1)-i4
          k4=k4+1;
          ui(k4,1)=sym(j4/n);
          vi(k4,1)=sym(i4/n);
          wi(k4,1)=sym(1-ui(k4,1)-vi(k4,1));
      end
  end

 end
 %N=length(ui)
 %num=(1:N)';
%table(:,1)=ui(:,1);
%table(:,2)=vi(:,1);
%table(:,3)=wi(:,1);
%disp(table)
%[num ui vi wi]
%[(1:N); ui';vi';wi']
```

(5) glsampleptsweights.m

```
function [s,wwww]=glsampleptsweights(n)
%  n must be in multiples of 10,i.e.10,20,30,40
switch n

  case 10
      table=[  -.14887433898163121088482600112972,    .29552422471475287017389299465132
             .14887433898163121088482600112972,    .29552422471475287017389299465132
            -.43339539412924719079926594316579,    .26926671930999635509122692156937
             .43339539412924719079926594316579,    .26926671930999635509122692156937
            -.67940956829902440623432736511485,    .21908636251598204399553493422951
             .67940956829902440623432736511485,    .21908636251598204399553493422951
            -.86506336668898451073209668842350,    .14945134915058059314577633965488
             .86506336668898451073209668842350,    .14945134915058059314577633965488
            -.97390652851717172007796401208445, .66671344308688137593568809896211e-1
             .97390652851717172007796401208445, .66671344308688137593568809896211e-1];

           s=table(:,1);wwww=table(:,2);
      case 20

 table=[ -.76526521133497333754640409398840e-1,     .15275338713072585069808433195511
        .76526521133497333754640409398840e-1,     .15275338713072585069808433195511
         -.22778585114164507808049619536857,     .14917298647260374678782873700183
          .22778585114164507808049619536857,     .14917298647260374678782873700183
         -.37370608871541956067254817702493,     .14209610931838205132929832506179
          .37370608871541956067254817702493,     .14209610931838205132929832506179
         -.51086700195082709800436405095525,     .13168863844917662689849449974692
          .51086700195082709800436405095525,     .13168863844917662689849449974692
         -.63605368072651502545283669622630,     .11819453196151841731237737774560
          .63605368072651502545283669622630,     .11819453196151841731237737774560
         -.74633190646015079261430507035565,     .10193011981724043503675013591012
          .74633190646015079261430507035565,     .10193011981724043503675013591012
         -.83911697182221882339452906170150,  .83276741576704748724758149344510e-1
          .83911697182221882339452906170150,  .83276741576704748724758149344510e-1
         -.91223442825132590586775244120330,  .62672048334109063569506532377206e-1
          .91223442825132590586775244120330,  .62672048334109063569506532377206e-1
         -.96397192727791379126766613119730,  .40601429800386941331039956506228e-1
          .96397192727791379126766613119730,  .40601429800386941331039956506228e-1
         -.99312859918509492478612238847130,  .17614007139152118311861976122751e-1
          .99312859918509492478612238847130,  .17614007139152118311861976122751e-1];
```

```
s=table(:,1);wwww=table(:,2);

case 30
   table=[ -.5147184255531769583302521316672e-1,    .10285265289355882523892690167866
         .51471842555317695833025213166720e-1,    .10285265289355882523892690167866
         -.15386991360858354696379467274326,    .10176238974840548965415889420028
          .15386991360858354696379467274326,    .10176238974840548965415889420028
         -.25463692616788984643980512981781,  .99593420586795252438990588447572e-1
          .25463692616788984643980512981781,  .99593420586795252438990588447572e-1
         -.35270472553087811347103720708938,  .96368737174644245489174990606085e-1
          .35270472553087811347103720708938,  .96368737174644245489174990606085e-1
         -.44703376953808917678060990032285,  .92122522237786115190831605601158e-1
          .44703376953808917678060990032285,  .92122522237786115190831605601158e-1
         -.53662414814201989926416979331110,  .86899787201082967042466189352565e-1
          .53662414814201989926416979331110,  .86899787201082967042466189352565e-1
         -.62052618298924286114047755643120,  .80755895229420203496911291451305e-1
          .62052618298924286114047755643120,  .80755895229420203496911291451305e-1
         -.69785049479331579693229238802665,  .73755974737705195438293294379393e-1
          .69785049479331579693229238802665,  .73755974737705195438293294379393e-1
         -.76777743210482619491797734097450,  .65974229882180485440811801154796e-1
          .76777743210482619491797734097450,  .65974229882180485440811801154796e-1
         -.82956576238276839744289811973250,  .57493156217619058039725903636851e-1
          .82956576238276839744289811973250,  .57493156217619058039725903636851e-1
         -.88256053579205268154311646253025,  .484026728305940545795742219106848e-1
          .88256053579205268154311646253025,  .484026728305940545795742219106848e-1
         -.92620004742927432587932427708045,  .38799192569627043899699995479347e-1
          .92620004742927432587932427708045,  .38799192569627043899699995479347e-1
         -.96002186496830751221687102558180,  .28784707883323365123125596530187e-1
          .96002186496830751221687102558180,  .28784707883323365123125596530187e-1
         -.98366812327974720997003258160565,  .18466468311090956430751785735094e-1
          .98366812327974720997003258160565,  .18466468311090956430751785735094e-1
         -.99689348407464954027163005091870,  .79681924961666044453614922773122e-2
          .99689348407464954027163005091870,  .79681924961666044453614922773122e-2];

   s=table(:,1);wwww=table(:,2);

case 40
 table=[ -.38772417506050821933193444024624e-1,  .77505947978424796396831052966037e-1
  .38772417506050821933193444024624e-1,  .77505947978424796396831052966037e-1
   -.11608407067525520848345128440802,  .77039818164247950810825852852202e-1
    .11608407067525520848345128440802,  .77039818164247950810825852852202e-1
   -.19269758070137109971551685206515,  .76110361900626227772361120968616e-1
    .19269758070137109971551685206515,  .76110361900626227772361120968616e-1
   -.26815218500725368114118434480860,  .74723169057968249867078378271926e-1
    .26815218500725368114118434480860,  .74723169057968249867078378271926e-1
   -.34199409082575847300749248117920,  .72886582395804045079686716752540e-1
    .34199409082575847300749248117920,  .72886582395804045079686716752540e-1
   -.41377920437160500152487974580371,  .70611647391286766151028945995058e-1
    .41377920437160500152487974580371,  .70611647391286766151028945995058e-1
   -.48307580168617871290856657424482,  .67912045815233890799062536542532e-1
    .48307580168617871290856657424482,  .67912045815233890799062536542532e-1
   -.54946712509512820207593130552950,  .64804013456601025644096819401890e-1
    .54946712509512820207593130552950,  .64804013456601025644096819401890e-1
   -.61255388966798023795261245023070,  .61306242492928927407006960905878e-1
    .61255388966798023795261245023070,  .61306242492928927407006960905878e-1
   -.67195668461417954837935451496150,  .57439769099391540348879646916123e-1
    .67195668461417954837935451496150,  .57439769099391540348879646916123e-1
   -.72731825518992710328099645175495,  .53227846983936814145029242551666e-1
    .72731825518992710328099645175495,  .53227846983936814145029242551666e-1
   -.77830565142651938769497154550650,  .4869580763507222721976504885192e-1
    .77830565142651938769497154550650,  .4869580763507222721976504885192e-1
   -.82461223083331166319632023066610,  .43870908185673263581207213110889e-1
    .82461223083331166319632023066610,  .43870908185673263581207213110889e-1
   -.86595950321225950382078180835460,  .38782167974472010160626968740686e-1
    .86595950321225950382078180835460,  .38782167974472010160626968740686e-1
   -.90209880696887429672825333086850,  .33460195282547841077564588554743e-1
    .90209880696887429672825333086850,  .33460195282547841077564588554743e-1
   -.93281280827867653336085216684520,  .27937006980023395834739366432647e-1
    .93281280827867653336085216684520,  .27937006980023395834739366432647e-1
   -.95791681921379165580454099945275,  .22245849194166952990288087942050e-1
    .95791681921379165580454099945275,  .22245849194166952990288087942050e-1
   -.97725994998377426266337028371290,  .16421058381907885512705412731658e-1
    .97725994998377426266337028371290,  .16421058381907885512705412731658e-1
   -.99072623869945700645305435222135,  .10498284531152811265876265047737e-1
    .99072623869945700645305435222135,  .10498284531152811265876265047737e-1
```

```
  -.99823770971055920034962270242060,  .45212770985331899801771377849908e-2
  .99823770971055920034962270242060,  .45212770985331899801771377849908e-2];


   s=table(:,1);wwww=table(:,2);
end
```

FIGURES

Fig.9a

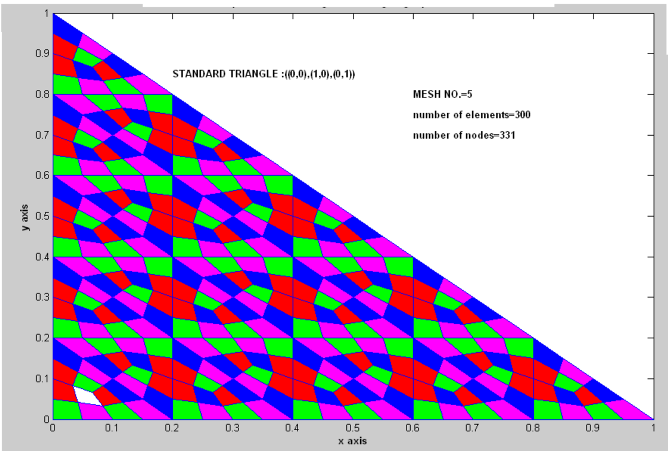

Fig.9b

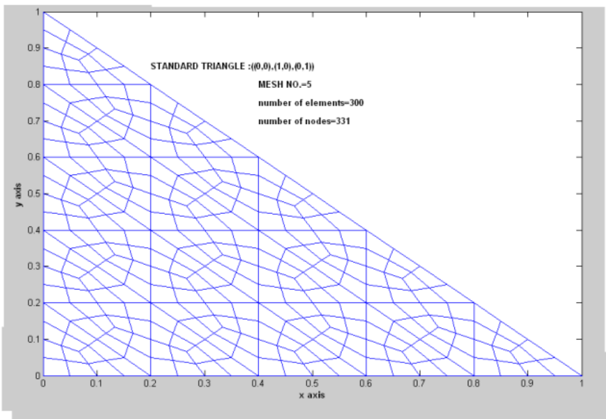

Fig.10

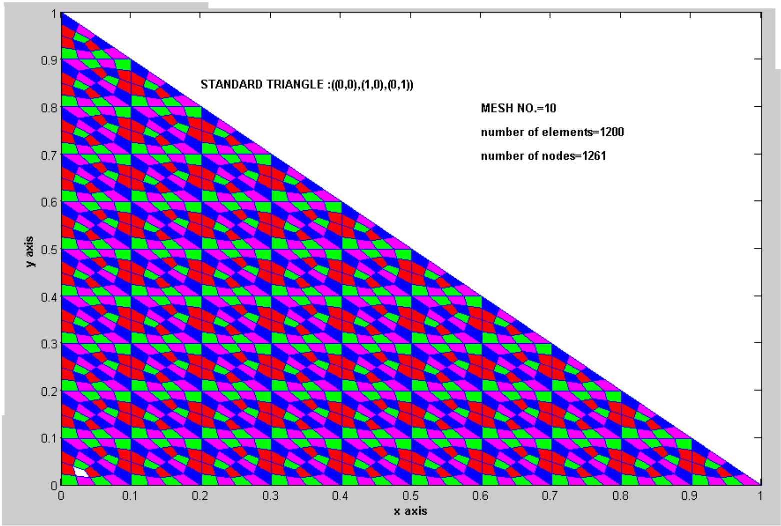

Fig.10b

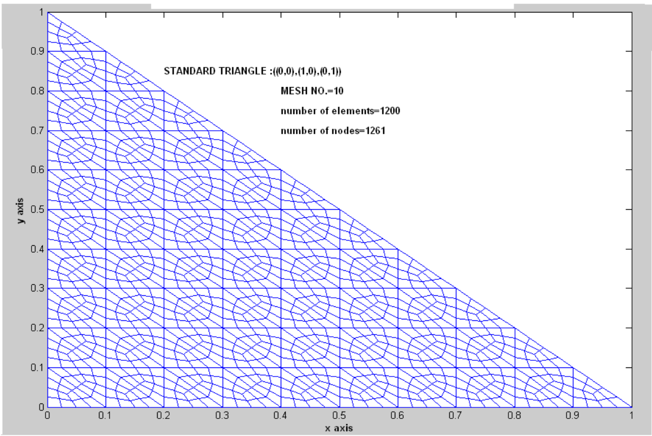

Fig.11a

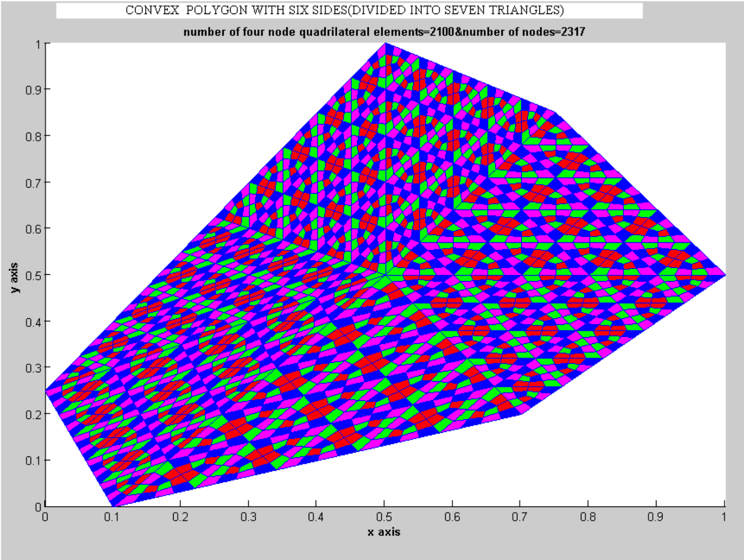
CONVEX POLYGON WITH SIX SIDES(DIVIDED INTO SEVEN TRIANGLES)
number of four node quadrilateral elements=2100&number of nodes=2317
y axis
x axis

Fig.11b

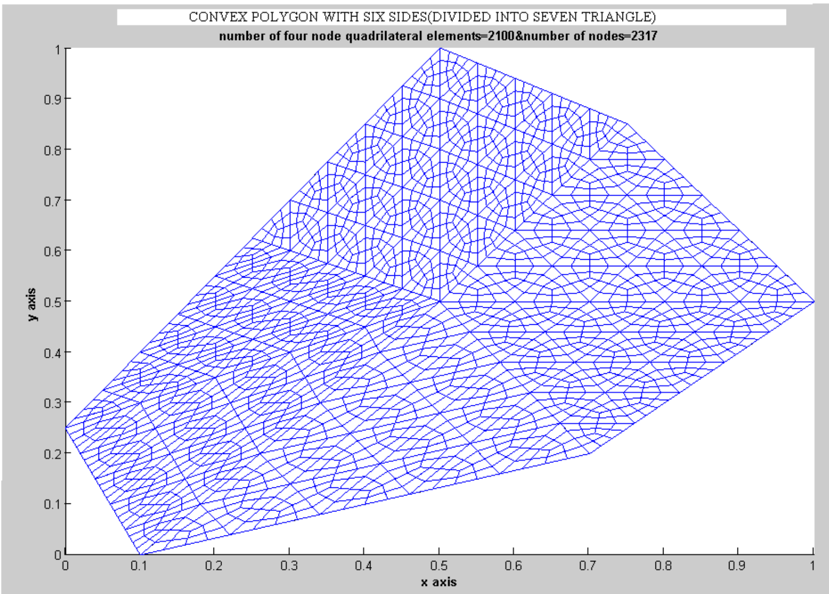
CONVEX POLYGON WITH SIX SIDES(DIVIDED INTO SEVEN TRIANGLE)
number of four node quadrilateral elements=2100&number of nodes=2317
y axis
x axis

Fig.12a

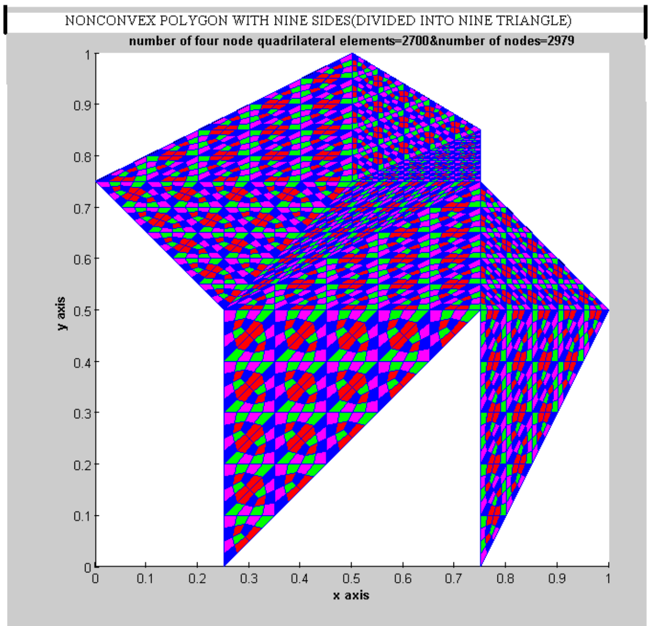

Fig.12b