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A Common Fixed Point Theorem for A-Contraction Mapping and its Application

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ABSTRACT- In this paper, we prove analogues of some fixed point results for A-contraction mappings in nonlinear integral equation.

Keywords- Fixed points, A-contraction condition, Banach contraction principle, Metric space, Lesbesgue-integrable.

1.INTRODUCTION

The result of this note are inspired by a recent paper of B.E.Rhoades [3,4,5,6] in introduced the contractive type mapping and used it for solving fixed points problem in metric spaces. Fixed point theory plays a crucial part in nonlinear functional analysis and is useful for proving the existence theorems for nonlinear differential and integral equations. First important result on fixed points for contractive type mapping was given by S. Banach [12] in 1922.

The main aim of this paper is to prove the existence and uniqueness of common fixed point of mapping for a self map on a metric space by using A-contraction condition of integral type and its application.

Theorem1. (Banach contraction principle) Let (X; d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ be a mapping such that for each x, y ∈ X,

$$d(fx, fy) \le cd(x, y)$$

(1)

then f has a unique fixed point a ∈ X, such that for each x ∈ X, $\lim_{n\to\infty} f^n x = a$

An elementary account of the Banach contraction principle and some applications, including its role in solving nonlinear ordinary differential equations, is in [13]. The contraction mapping theorem is used to prove the inverse function theorem in [14]. A beautiful application of contraction mappings to the construction of fractals is in [8]. After the classical result by Banach, Kannan [11] gave a substantially new contractive mapping to prove the fixed point theorem. Since then there have been many theorems emerged as generalizations under various contractive conditions. Such conditions involve linear and nonlinear expressions (rational, irrational, and general type). The intrested reader who wants to know more about this matter is recommended to go deep into the survey articles by Rhoades [3,4,5,6] and Bianchini [10], and into the references therein.

1.1. Definition

A-contractions difined as follows:

Let a non-empty set A consisting of all functions $\alpha_+ : R_+^3 \to R_+$ satisfying: (A1): α is continuous on the set R_+^3 of all triplets of non-negative real (with respect to the Euclidean metric on R^3). (A2): $a \le kb$ for some $k \in [0,1)$ whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$) or $a \le \alpha(b,b,a)$ for all a, b.

1.2. Definition

A self-map T on a metric space X is said to be A-contraction, if it satisfies the condition

$$d(Tx,Ty) \le \alpha(d(x,y), d(x,Tx), d(y,Ty))$$

for all x; y ∈ X and some α ∈ A. In 2002, A.Branciari[1] analyzed the existence of fixed point for mapping T defined on a complete metric space (X, d) satisfying a general contractive condition of integral type in the following theorem.

Theorem 1.3. (Branciari) Let (X; d) be a complete metric space, c ∈ (0,1) and let $T : X \to X$ be a mapping such that for each x,y ∈ X,

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$, then T has a unique fixed point a ∈ X such that for each x ∈ X, $\lim_{n\to\infty} T^n x = a$.

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades[3] extending the result of Branciari.

In 2002, A.Branciari[1] analyzed the existence of fixed point for mapping T defined on a complete metric space (X, d) satisfying a general contractive condition of integral type in the following theorem.

In 2012, Saha and Dey, [9] analyzed the existence of fixed point for mapping T defined on a complete metric space (X, d) satisfying a A- contractive mapping of integral type in the following theorem.

Theorem 1.4. Let T be a self-mapping of a complete metric space (X; d) satisfying the following condition:

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \alpha\left(\int_0^{d(x,y)} \varphi(t)dt, \int_0^{d(x,Tx)} \varphi(t)dt, \int_0^{d(y,Ty)} \varphi(t)dt\right)$$

for each x; y ∈ X with some $\alpha \in A$, where $\varphi:[0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$,

then T has a unique fixed point z ∈ X such that for each x ∈ X, $\lim_{n\to\infty} T^n x = z$.

In a very recent paper, Dey,Ganguly and Saha [7] proved some fixed point theorems for mixed type of contraction mappings of integral type in complete metric space. Motivated and inspired by these consequent works, we introduce the analogues of some fixed point results for A-contraction mappings in integral setting which in turn generalize several known results. Also we have analyzed the existence of fixed point of mapping over two related metrics due to [2] in integral setting. Our results substantially extend, improve, and generalize comparable results in the literature.

2.Main results

Theorem 2.1. Let(x,d) be a complete metric space and S,T,P:X→ X satisfying the following condition

(i)

$$\int_0^{d(SPx,TPy)} \varphi(t)dt \le \alpha\left(\int_0^{d(x,y)} \varphi(t)dt, \int_0^{d(x,SPx)} \varphi(t)dt, \int_0^{d(y,TPy)} \varphi(t)dt, \int_0^{d(SPx,TPy)} \varphi(t)dt\right)$$

(ii) One of three mapping S, T and P is continuous.

(iii) For each x; y ∈ X with some $\alpha \in A$, where $\varphi:[0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$,

Further assume that SP=PS or TP=PT,

then S,T and P have common unique fixed point in z ∈ X.

Proof:- Let $x_0 \in X$ be an arbitrary point. For each integral $n \ge 1$,We define a sequence $\{x_n\}$ as

$$x_{2n+1} = SPx_{2n}, n = 0,1,2,3,......$$ and

$$x_{2n+2} = TPx_{2n+1}, n = 0,1,2,3,......$$

Let $a_n = d(x_n, x_{n+1})$

(2)

Putting $x_n = x_{2n}$ and $y = x_{2n+1}$ for all $n \ge 1$

We get from (2.1)

$$\int_0^{d(SPx_{2n},TPx_{2n+1})} \varphi(t)dt \le \alpha\left(\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt, \int_0^{d(x_{2n},SPx_{2n})} \varphi(t)dt, \int_0^{d(x_{2n+1},TPx_{2n+1})} \varphi(t)dt, \int_0^{d(SPx_{2n},TPx_{2n+1})} \varphi(t)dt\right)$$

or,

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \le \alpha\left(\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt, \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt, \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt, \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\right)$$

Fom (2),we get,

$$\int_0^{a_{2n+1}} \varphi(t)dt \le \alpha\left(\int_0^{a_{2n}} \varphi(t)dt, \int_0^{a_{2n}} \varphi(t)dt, \int_0^{a_{2n+1}} \varphi(t)dt, \int_0^{a_{2n+1}} \varphi(t)dt\right)$$

(3)

If $a_{2n+1} > a_{2n}$,then

$$\int_0^{a_{2n+1}} \varphi(t)dt \le \alpha\left(\int_0^{a_{2n+1}} \varphi(t)dt, \int_0^{a_{2n+1}} \varphi(t)dt, \int_0^{a_{2n+1}} \varphi(t)dt, \int_0^{a_{2n+1}} \varphi(t)dt\right)$$

$$< \int_0^{a_{2n+1}} \varphi(t)dt \qquad (4)$$

Which is contracdition. Hence $a_{2n+1} \le a_{2n}, n \ge 1$

Similarly by putting $x = x_{2n}$,and $y = x_{2n-1}$ in (2.1)

We can show that $a_{2n+2} \le a_{2n+1}$, $n \ge 1$

Thus $a_{n+1} \le a_n$,

So that $\{a_n\}$ is a decreasing sequence of non negative real number and hence convergent to some $a \in R$. (5)

From (4) and (5) for all $n \ge 1$,obtain

$$\int_0^{a_{n+1}} \varphi(t)dt \le \int_0^{a_n} \varphi(t)dt, \int_0^{a_n} \varphi(t)dt, \int_0^{a_n} \varphi(t)dt, \int_0^{a_n} \varphi(t)dt$$

Then $\int_0^{a_{n+1}} \varphi(t)dt \le \int_0^{a_n} \varphi(t)dt$

Summing up we obtain

$$\sum_{n=0}^{\infty} \int_0^{a_{n+1}} \varphi(t)dt \le \int_0^{a_0} \varphi(t)dt < \infty$$

Again from (5) $\{a_n\}$ is convergent and $a_n \to a$ as $n \to \infty$,Which implies that a=0, that is

$a = d(x_{n+1}, x_n) \to 0$ as $n \to \infty$

(6)

We next prove that $\{x_n\}$ is a Cauchy sequence , in view of (5) it is sufficient to prove that $\{x_{2r}\}_{r=1}^{\infty} \subset \{x_n\}$ is Cauchy sequence. If $\{x_{2r}\}_{r=1}^{\infty}$ is not Cauchy sequence of natural number {2mk,{2nk} such that nk > mk,

$d(x_{2mk}, x_{2nk}) \ge \varepsilon$

$d(x_{2mk}, x_{2nk-1}) < \varepsilon$

(7)

Then by (8)

$$\varepsilon < d(x_{2mk}, x_{2nk}) \le d(x_{2mk}, x_{2nk-1}) + d(x_{2mk}, x_{2nk-1})$$

$$< \varepsilon + d(x_{2nk}, x_{2nk-1})$$

Making $k \to \infty$ in the above inequality by virtue of (5)
We obtain

$$\lim_{n\to\infty} d(x_{2mk}, x_{2nk}) = \varepsilon$$

(08)

For all k=1,2,3……..

$$d(x_{2nk+1}, x_{2mk}) \le d(x_{2nk+1}, x_{2nk}) + d(x_{2nk}, x_{2mk})$$

(09)

Also for all k=1,2,3,……..

$$d(x_{2nk}, x_{2mk}) \le d(x_{2nk}, x_{2nk+1}) + d(x_{2nk+1}, x_{2mk})$$

(10)

Making $k \to \infty$ in (6) and (7) respectively, by using (4) and (5), we have

$\lim_{k\to\infty} d(x_{2nk+1}, x_{2m}) \le \varepsilon$ and $\varepsilon \le \lim_{k\to\infty} d(x_{2nk+1}, x_{2m})$

$\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) = \varepsilon$ for all k= 1,2,3,……

(11)

$$d(x_{2n(k)}, x_{2m(k)-1}) \le d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)-1})$$

$$d(x_{2n(k)}, x_{2m(k)}) \le d(x_{2n(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)})$$

Taking $k \to \infty$ in the above two inequalities and using (7) and (8) we obtain

$$\lim_{k\to\infty} d(x_{2nk}, x_{2m-1}) = \varepsilon$$

Putting $x = x_{2nk}$ and $y = x_{2mk-1}$ in (2.1), for all k=1, 2, 3,…….
We obtain

$$\int_0^{d(x_{2nk+1},x_{2mk})} \varphi(t)dt \le \int_0^{d(x_{2nk},x_{mk-1})} \varphi(t)dt, \int_0^{d(x_{2nk},x_{2mk-1})} \varphi(t)dt, \int_0^{d(x_{2mk-1},x_{2mk})} \varphi(t)dt, \int_0^{d(x_{2nk+1},x_{2mk})} \varphi(t)dt$$

Making $k \to \infty$ in the above inequality and taking into account the continuity of and by (5), (9), (10). WE have,

$$\int_0^{\varepsilon} \varphi(t)dt \le \int_0^{\varepsilon} \varphi(t)dt \text{ ,then}$$

The above inequality give a contradiction so that $\varepsilon = 0$. This establishes sequence and hence convergence in (X, d).

Let $x_n \to z$ as $n \to \infty$

Putting $x = x_{2n}$ and x=y in (2.1) for all n=1,2,3,………

$$\int_0^{d(x_{2n+1},TPz)} \varphi(t)dt \le \int_0^{d(x_{2n},z)} \varphi(t)dt, \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt, \int_0^{d(z,TPz)} \varphi(t)dt, \int_0^{d(x_{2n+1},TPz}$$

Making $n \to \infty$ in the above inequality, by using (5) and (11), we obtain

$$\int_0^{d(z,TPz)} \varphi(t)dt \le \int_0^{d(z,TPz)} \varphi(t)dt$$

If $d(z, TPz) \ne 0$,

Which is a contradiction. Hence, we obtain
d(z, TPz) = 0,or z = TPz in an exactly similarly way to prove z = SPz

Thus SPz = z = TPz

(12)

Also fallows that z is the common fixed point of SP and TP.

Suppose S is continuous,then

$S^2Px_{2n} = Sz$, $Sx_{2n} = Sz$ .If SP = PS.

Then putting $x = Sx_{2n}, y = x_{2n+1}$ in (2.1),

$$\int_0^{d(s^2Px_{2n},TPx_{2n+1})} \varphi(t)dt \le \int_0^{d(Sx_{2n},x_{2n+1})} \varphi(t)dt, \int_0^{d(Sx_{2n},S^2Px_{2n})} \varphi(t)dt, \int_0^{d(x_{2n+1},TPx_{2n+1})} \varphi(t)dt, \int_0^{d(S^2Px_{2n},TP}$$

$$\Rightarrow$$

$$\int_0^{d(sz,z)} \varphi(t)dt \le \int_0^{d(Sz,z)} \varphi(t)dt, \int_0^{d(Sz,Sz)} \varphi(t)dt, \int_0^{d(z,z)} \varphi(t)dt, \int_0^{d(Sz,z)} \varphi(t)dt$$

$$\Rightarrow$$

$$\int_0^{d(sz,z)} \varphi(t)dt \le \int_0^{d(Sz,z)} \varphi(t)dt, \int_0^{d(Sz,z)} \varphi(t)dt$$

$$\Rightarrow \int_0^{d(sz,z)} \varphi(t)dt \le \int_0^{d(Sz,z)} \varphi(t)dt$$

It is a contradiction, if $Sz \ne z$,hence Sz = z.
Hence by (12) Tz = z = Sz.
Similarly if TP = PT, then also Pz = z = Sz = Tz.
Hence this z is a common fixed point of S,T,P.Thus the theorem is proved

Application

Let X={0,1,2,3,4}and d be the usual metric of real. Let $T : X \to X$ be a given by

Tx = 2, if x=0

= 1, otherwise

Again let $\varphi : R_+ \to R_+$ be given by $\varphi(t) = 1$ for all $t \in R_+$.

Then $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$,

Example

A self-map T satisfying

$$d(T_x, T_y) \le \beta \max\{ d(T_x, x) + d(T_y, y), d(T_y, y) + d(x, x), d(T_x, x) + d(x, x)\}$$

For all x ,y $\in$ X and some $\beta \in \left[0, \frac{1}{2}\right)$, is an A-contraction; we have

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \alpha\left(\int_0^{d(x,y)} \varphi(t)dt, \int_0^{d(x,Tx)} \varphi(t)dt, \int_0^{d(y,Ty)} \varphi(t)dt \right)$$

$$= \beta \max\left\{ \int_0^{d(T_x,x)+d(x,x)} \varphi(t)dt, \int_0^{T_x,x)+d(T_y,y} \varphi(t)dt, \int_0^{d(T_y,y)+d(x,x)} \varphi(t)dt \right\}$$

Which is satisfied for all x, y$\in$ X and some $\beta \in \left[0, \frac{1}{2}\right)$.

So all the axioms of Theorem 2.1 are satisfied and 1, is of course a unique fixed point of T.

We also can show the clear distinction between our result and that of Branciari (contractive condition1.3)

Let us take x = 0, y = 1. Then from condition 1.3, we have

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt \text{ implies } c \ge 1$$

which is not true. So T does not satisfy the condition 1.3 of Branciari.

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