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The Odd Generalized Exponential Type-I Generalised Half Logistic Distribution: Properties and Application

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Abstract: We propose a new lifetime model, called the Odd Generalized Exponential Type-I Generalised Half Logistic Distribution (OGET-IGHLD).We obtain some of its mathematical properties. We made an attempt to study some structural properties of the new distribution. The Maximum Likelihood method is used for Estimating the model parameters and the Fisher's information matrix is also derived. We illustrate the applications with generalised data.

Keywords: Type-I Generalised Half Logistic Distribution, Hazard Function, Moments, Quantile, Maximum Likelihood, Estimation, Information Martix.

Introduction:

In life testing and reliability studies a combination of monotone and constant failure rates over various segments of the range of lifetime of a random variable is known as a bathtub or a non-monotone failure rate. In the biological and engineering sciences there are situations of non-monotone failure rates available to model such data; a comprehensive narration of the models is given in Rajarshi &Rajarshi (1988). Mudholkar, et al. (1995) presented an extension of the Weibull family that contains unimodel distributions with bathtub failure rates and also allows for a broader class of monotone hazard rates; they named their extended version the Exponentiated Weibull Family. Gupta and Kundu (1999) also proposed a new model called generalized exponential distribution. If θ is a positive real number and F(x) is the cumulative distribution function (cdf) of a continuous positive random variable, then $[F(x)]^\theta$ and the corresponding probability distribution may be termed exponentiated or generalized versions of F(x).This generalization is adapted to the half logistic distribution and the resulting model is considered in this study. A half logistic model obtained as the distribution of an absolute standard logistic variate is a probability model of recent origin (Balakrishnan, 1985). Some well- known generators are the Marshall- Olkin generated (MO- G) by Marshall and Olkin (1997), The properties of Exp- G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993).

There are always urge among the researchers for developing new and more flexible distributions. As a result, many new distributions have come up and studied. Recently, Tahir et al. (2015) propose a new class of distributions called the odd generalized exponential (OGE) family and study each of the OGE- Weibull (OGE-W) distribution, the OGE-Fréchet (OGE-Fr)distribution and the OGE-Normal (OGE-N) distribution. These models are flexible because of the hazard shapes: increasing, decreasing, bathtub and upside subset of down bathtub. A random variable X is said to have generalized exponential (GE) distribution with parameters α, β if the cumulative distribution function (CDF) is given by

$$F(x) = (1-e^{-\alpha x})^{\beta} \quad , \quad x > 0, \alpha >0, \beta>0 \quad (1.1)$$

The odd generalized exponential family suggested by Tahir et al. (2015) is defined as follows. If G(x; ε) is the CDF of any distribution and thus the survival function is $\bar{G}(x;\varepsilon)=1-G(x;\varepsilon)$, then the OGE-X is defined by replacing x in CDF of GE in equation (1.1) by $\frac{G(x;\in)}{\bar{G}(x;\in)}$ to get the CDF of the new distribution as follows:

$F(x;\alpha,\varepsilon,\beta)$ =(1-$e^{-\alpha\frac{G(x;\in)}{\bar{G}(x;\in)}}$) $^{\beta}$, x >0, α>0 *, ε>0, β>0 (1.2)*

In this paper, we define a new distribution using generalized exponential distribution and Type-I Generalised Half Logistic Distribution and named it as "The odd generalised exponential Type-I generalised half logistic distribution(OGET-IGHLD)" from a new family of distributions proposed by Tahir et al. (2015). The paper is organized as follows. The new distribution is developed in Section 2 and also we define the CDF, density function, survival function and hazard functions of the odd generalized exponential Type-I half logistic distribution(OGET-IGHLD). A comprehensive account of statistical properties of the new distribution is provided in Section 3.In Section 4, 5 we discuss the distribution of moment generating function and cumulative generating function for OGET-IGHLD. In section 6, maximum likelihood estimation and Fisher's information matrix are derived for the the parameters. A generalised data set has been analyzed the fitted distribution.

2. The Probability Density and Distribution Function of the OGET-IGHLD

2.1. OGET-IGHLD Specifications

In this section we define new four parameters distribution called The odd generalised exponential Type-I generalised half logistic distribution with parameters α, β, θ and σ. The probability density function (pdf), cumulative distribution function (CDF), Survival Function S(x) and hazard function H(x) of the new model OGET-IGHLD are respectively defined as follows:

$$f(x;\Theta)=\frac{\alpha\beta\theta e^{\frac{x}{\sigma}}\left(e^{\frac{x}{\sigma}}+1\right)^{\theta-1}e^{-\alpha\left((\frac{e^{\frac{x}{\sigma}}+1}{2})^{\theta}-1\right)}\left(1-e^{-\alpha\left((\frac{e^{\frac{x}{\sigma}}+1}{2})^{\theta}-1\right)}\right)^{\beta-1}}{\sigma.2^{\theta}} \quad (2.1.1)$$

Where x>0, α, σ, β, θ >0.

A random variable x ⊔ OGET-IGHLD(Θ) has Cumulative Distribution Function in the form

$$F(x;\Theta)=\left[1-e^{-\alpha\left((\frac{e^{\frac{x}{\sigma}}+1}{2})^{\theta}-1\right)}\right]^{\beta}$$, x >0, α>0, β>0,θ >0, σ>0 (2.1.2)

A random variable x ⊔ OGET-IGHLD(Θ) has Survival Function in the form

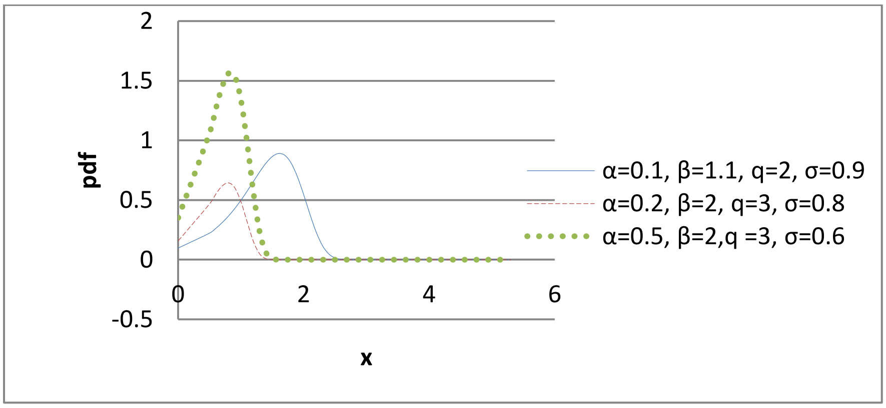
$$S(x)=1-\left[\left[1-e^{-\alpha\left((\frac{e^{\frac{x}{\sigma}}+1}{2})^{\theta}-1\right)}\right]^{\beta}\right] \quad (2.1.3)$$

A random variable x ⊔ OGET-IGHLD(Θ) has Hazard Function in the form

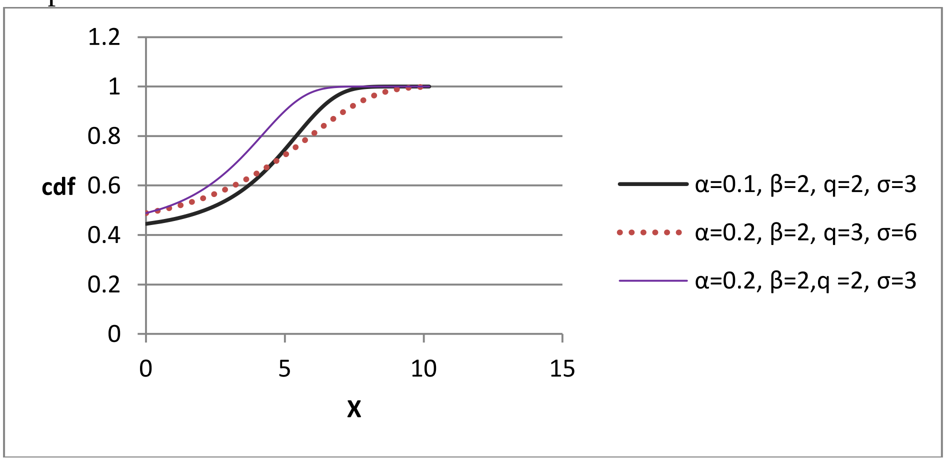
$$H(x)=\frac{f(x)}{S(x)}=\frac{\alpha\beta\theta e^{\frac{x}{\sigma}}\left(e^{\frac{x}{\sigma}}+1\right)^{\theta-1}e^{-\alpha\left(\left(\frac{e^{\frac{x}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\left(1-e^{-\alpha\left(\left(\frac{e^{\frac{x}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right)^{\beta-1}}{\sigma.2^{\theta}\left[1-\left[\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]^{\beta}\right]\right]} \quad (2.1.4)$$

Where α, σ are scale parameters and β, θ are shape parameters.
The graphs of f(x), F(x) and H(x) are given below for different values of the parameters are displayed in graph 1, graph 2 and graph 3.

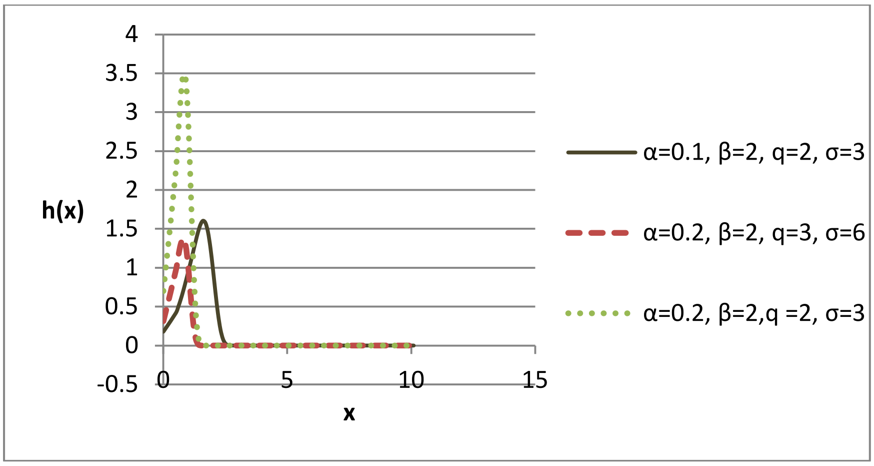
Graph 1



Graph 2



Graph 3



3. The statistical properties

In this section we study some statistical properties of OGET-IGHLD, especially quartile, median and moments.

3.1. Limit of Distribution Function:

Since the cdf of OGET-IGHLD is

$$F\ (x;\Theta) = \left[1 - e^{-\alpha\left(\left(\frac{e^{\frac{x}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]^{\beta}$$

$\lim_{n\to\infty} F\ (x;\Theta) = 0$

We have, $\lim_{n\to\infty} F\ (x;\Theta) = 0$ and $\lim_{n\to\infty} F\ (x;\Theta) = 1$

3.2. Quantile and Median of OGET-IGHLD

The 100_q percentile of X ⊔ OGET-IGHLD(Θ) is given by

$$x_q = \sigma\, ln\left[1 + 2\left[1 - \frac{1}{\alpha} ln(1 - q^{\frac{1}{\beta}})\right]^{\frac{1}{\theta}}\right] \quad 0<q<1 \quad (3.2.1)$$

Setting q=0.5 in (3.2.1), we obtain the median of x ⊔ OGET-IGHLD (Θ) distribution as follows:

$$\text{Median} = \sigma\, ln\left[1 + 2\left[1 - \frac{1}{\alpha} ln(1 - (0.5)^{\frac{1}{\beta}})\right]^{\frac{1}{\theta}}\right] \qquad (3.2.2)$$

3.3. Moments

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution. In this subsection, we will derive the r^{th} moments of the OGET-IGHLD (Θ) distribution as infinite series expansion.

Theorem 1. If X ⊔ OGET-IGHLD(Θ), where $\Theta = (\alpha, \beta, \theta, \sigma)$, then the r^{th} moment of X is given by

$$\mu_r^1 = \frac{\alpha\theta\beta}{\sigma^2 2^{\theta}} \sum_{i=0}^{(\beta-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{l=0}^{\theta(j+1)-1} \frac{(-1)^{i+k-r}}{j!\,2^{\theta j}[l(\theta(j+1)-1)+1]} \binom{\beta-1}{i} \binom{j}{k} \binom{\theta(j+1)-1}{l}$$

$\gamma(r+1, l(\theta(j+1)-1)+1)$

Proof: The rth moment of the random variable X with *pdf* $f(x, \Theta)$ *is defined by*

$$\mu_r^1 = \int_o^\infty x^r f(x,\Theta)dx \qquad (3.3.1)$$

Substituting from (3) into (11), we get

$$\mu_r^1 = \int_0^\infty x^r \frac{\alpha\beta\theta e^{\frac{x}{\sigma}}\left(e^{\frac{x}{\sigma}}+1\right)^{\theta-1} e^{-\alpha\left(\left(\frac{e^{\frac{x}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\left(1-e^{-\alpha\left(\left(\frac{e^{\frac{x}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right)^{\beta-1}}{\sigma.2^{\theta}} dx \ (3.3.2)$$

Since $0<(1-e^{-\alpha(\frac{e^{\frac{x}{\sigma}}-1}{2})^{\theta}})^{(\beta-1)}<1$ for x>0, we have

$$\left(1-e^{-\alpha\left(\left(\frac{e^{\frac{x}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right)^{\beta-1} = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^i.e^{-\alpha i\left[\left[\frac{1+e^{\frac{x}{\sigma}}}{2}\right]^{\theta}-1\right]} \ (3.3.3)$$

and

$$e^{-\alpha(i+1)\left[\left[\frac{1+e^{\frac{x}{\sigma}}}{2}\right]^{\theta}-1\right]} = \frac{\left(\alpha(i+1)\left[\left[\frac{1+e^{\frac{x}{\sigma}}}{2}\right]^{\theta}-1\right]\right)^j}{j!} \quad (3.3.4)$$

For that, we obtained

$$\mu_r^1 = \frac{\alpha\theta\beta}{\sigma^2 2^{\theta}}\sum_{i=0}^{(\beta-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\theta(j+1)-1}\frac{(-1)^{i+k-r}}{2^{\theta j}}\binom{\beta-1}{i}\binom{j}{k}\binom{\theta(j+1)-1}{l}\int_0^\infty x^r e^{\frac{x[l(\theta(j+1)-1)+1]}{\sigma}} dx \ (3.3.5)$$

By using the definition of gamma function in the form

$$\int_0^\infty e^x x^{ax}\, dx = \frac{\gamma(r+1,ax)}{(-1)^r a^{r+1}} \ (3.3.6)$$

Thus we obtain moment of OGET-IGHLD (Θ) as follows

$$\mu_r^1 = \frac{\alpha\theta\beta}{\sigma^2 2^{\theta}}\sum_{i=0}^{(\beta-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\theta(j+1)-1}\frac{(-1)^{i+k-r}}{j!2^{\theta j}[l(\theta(j+1)-1)+1]}\binom{\beta-1}{i}\binom{j}{k}\binom{\theta(j+1)-1}{l}$$
$$\gamma(r+1, l(\theta(j+1)-1)+1) \qquad (3.3.7)$$

4. **Moment Generating Function**

$$M_x^t = \mathrm{E}(e^{xt})$$
$$= \int_o^\infty e^{tx} f(x,\Theta)dx$$
$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^1$$
$$= \frac{\alpha\theta\beta}{\sigma^2 2^{\theta}}\sum_{r=0}^{\infty}\sum_{i=0}^{(\beta-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\theta(j+1)-1}\frac{(-1)^{i+k-r}}{r!j!2^{\theta j}[l(\theta(j+1)-1)+1]}\binom{\beta-1}{i}\binom{j}{k}\binom{\theta(j+1)-1}{l}$$
$$\gamma(r+1, l(\theta(j+1)-1)+1) \qquad (4.1)$$

5. Cummulative Generating Function

$K_x^t = \log(M_x^t)$

$$=\log\Big(\frac{\alpha\theta\beta}{\sigma^2 2^\theta}\sum_{r=0}^{\infty}\sum_{i=0}^{(\beta-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\theta(j+1)-1}\frac{(-1)^{i+k-r}}{r!j!2^{\theta j}[l(\theta(j+1)-1)+1]}\binom{\beta-1}{i}\binom{j}{k}\binom{\theta(j+1)-1}{l}\gamma(r+1,l(\theta(j+1)-1)+1)\Big) \quad (5.1)$$

6. Estimation Inference

Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from OGET-IGHLD (Θ), where Θ =(α, β, θ, σ), then the likely hood function L of this sample is defined as

$$l_n = ln(f(x,\Theta))$$

$$= ln\left(\frac{\alpha\beta\theta}{\sigma.2^\theta}\right) + \frac{\sum_{i=1}^{n} x_i}{\sigma} + (\theta-1)\sum_{i=1}^{n}\ln\left(1+e^{\frac{x_i}{\sigma}}\right) - \alpha\sum_{i=1}^{n}\left[\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]^\theta - 1\right]$$

$$+(\beta-1)\sum_{i=1}^{n} ln\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^\theta-1\right)}\right] \quad (6.1)$$

Differentiate (6.1) w. r. t to α, β, θ and σ we get

$$\frac{\partial}{\partial\alpha} l_n =$$

$$\frac{n}{\alpha} - \sum_{i=1}^{n}\left[\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]^\theta - 1\right] + (\beta-1)\sum_{i=0}^{n}\frac{\left[\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]^\theta - 1\right]\left[e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^\theta-1\right)}\right]}{\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^\theta-1\right)}\right]} \quad (6.2)$$

$$\frac{\partial}{\partial\theta} l_n =$$

$$-\frac{n}{\theta} - \frac{n\theta}{2} + \sum_{i=1}^{n} ln(1+e^{\frac{x_i}{\sigma}}) - \alpha\sum_{i=1}^{n}\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]^\theta \ln\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right] \quad +$$

$$\alpha(\beta-1)\sum_{i=0}^{n}\frac{\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta} ln\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}}{\left[1-e^{-\alpha\left((\frac{e^{\frac{x_i}{\sigma}}+1}{2})^{\theta}-1\right)}\right]} \quad (6.3)$$

$$\frac{\partial}{\partial\sigma}l_n =$$

$$-\frac{n}{\sigma}-\frac{(\theta-1)}{\sigma^2}\sum_{i=1}^{n}\frac{x_i e^{\frac{x_i}{\sigma}}}{1+e^{\frac{x_i}{\sigma}}}-\frac{1}{\sigma^2}\sum_{i=1}^{n}x_i-\frac{\alpha\theta}{2\sigma^2}\sum_{i=1}^{n}e^{\frac{x_i}{\sigma}}\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta-1}+$$

$$\frac{\alpha\theta(\beta-1)}{2\sigma^2}\sum_{i=1}^{n}\frac{x_i e^{\frac{x_i}{\sigma}}\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta-1}\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}{\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]} \quad (6.4)$$

$$\frac{\partial}{\partial\beta}l_n =$$

$$\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right] \quad (6.5)$$

The normal equations can be obtained by above equations (6.2) to (6.5) equating to zero. Then the non-linear equations becomes

$$=> \frac{n}{\alpha}-\sum_{i=1}^{n}\left[\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]^{\theta}-1\right]+(\beta-1)\sum_{i=0}^{n}\frac{\left[\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]^{\theta}-1\right]\left[e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]}{\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]}=0 \quad (6.6)$$

$$=> -\frac{n}{\theta}-\frac{n\theta}{2}+\sum_{i=1}^{n}ln(1+e^{\frac{x_i}{\sigma}})-\alpha\sum_{i=1}^{n}\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]^{\theta}\ln\left[\frac{1+e^{\frac{x_i}{\sigma}}}{2}\right]+$$

$$\alpha(\beta-1)\sum_{i=0}^{n}\frac{\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta} ln\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right) e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}}{\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]} = 0 \quad (6.7)$$

$$=> -\frac{n}{\sigma}-\frac{(\theta-1)}{\sigma^2}\sum_{i=1}^{n}\frac{x_i e^{\frac{x_i}{\sigma}}}{1+e^{\frac{x_i}{\sigma}}}-\frac{1}{\sigma^2}\sum_{i=1}^{n}x_i-\frac{\alpha\theta}{2\sigma^2}\sum_{i=1}^{n}e^{\frac{x_i}{\sigma}}\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta-1}$$

$$+\frac{\alpha\theta(\beta-1)}{2\sigma^2}\sum_{i=1}^{n}\frac{x_i e^{\frac{x_i}{\sigma}}\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta-1}\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}{\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]} = 0 \quad (6.8)$$

$$=> \frac{n}{\beta}+\sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right] = 0 \quad (6.9)$$

From equation (6.9) the MLE of β can be obtained as follows

$$\hat{\beta} = -\frac{n}{\sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]} \quad (6.10)$$

Substitute (6.10) into (6.6),(6.7) and (6.8) we get the maximum likelihood estimators of α,θ, σ by solving the non-linear equations

$$\frac{n}{\hat{\alpha}}-\sum_{i=1}^{n}\left[\left[\frac{1+e^{\frac{x_i}{\hat{\sigma}}}}{2}\right]^{\hat{\theta}}-1\right]+(\hat{\beta}-1)\sum_{i=0}^{n}\frac{\left[\left[\frac{1+e^{\frac{x_i}{\hat{\sigma}}}}{2}\right]^{\hat{\theta}}-1\right]\left[e^{-\hat{\alpha}\left(\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}-1\right)}\right]}{\left[1-e^{-\hat{\alpha}\left(\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}-1\right)}\right]} = 0 \quad (6.11)$$

$$-\frac{n}{\hat{\theta}}-\frac{n\hat{\theta}}{2}+\sum_{i=1}^{n}ln(1+e^{\frac{x_i}{\hat{\sigma}}})-\hat{\alpha}\sum_{i=1}^{n}\left[\frac{1+e^{\frac{x_i}{\hat{\sigma}}}}{2}\right]^{\theta}\ln\left[\frac{1+e^{\frac{x_i}{\hat{\sigma}}}}{2}\right]$$

$$+\hat{\alpha}(\hat{\beta}-1)\sum_{i=0}^{n}\frac{\left(\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}-1\right)\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}ln\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)e^{-\hat{\alpha}\left(\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}-1\right)}}{\left[1-e^{-\hat{\alpha}\left(\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}-1\right)}\right]}=0 \quad (6.12)$$

$$-\frac{n}{\hat{\sigma}}-\frac{(\hat{\theta}-1)}{\hat{\sigma}^2}\sum_{i=1}^{n}\frac{x_ie^{\frac{x_i}{\hat{\sigma}}}}{1+e^{\frac{x_i}{\hat{\sigma}}}}-\frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}x_i-\frac{\hat{\alpha}\hat{\theta}}{2\hat{\sigma}^2}\sum_{i=1}^{n}e^{\frac{x_i}{\hat{\sigma}}}\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}-1}$$

$$+\frac{\hat{\alpha}\hat{\theta}(\hat{\beta}-1)}{2\sigma^2}\sum_{i=1}^{n}\frac{x_ie^{\frac{x_i}{\hat{\sigma}}}\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}-1}\left(\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}-1\right)}{\left[1-e^{-\hat{\alpha}\left(\left(\frac{e^{\frac{x_i}{\hat{\sigma}}}+1}{2}\right)^{\hat{\theta}}-1\right)}\right]}=0 \quad (6.13)$$

These equations cannot solve directly and statistical Software can be used to solve the equations numerically. We use iterative technique can be used to solve the $\hat{\beta}$

Asymptotic Confidence bounds:

Here we derive the asymptotic confidence bounds for unknown parameters α, β, θ and σ *when α>0, β>0, θ>0 and σ>0* The simplest large sample approach is to assume that the MLEs (α, β, θ, σ) are approximately normal with mean (α, β, θ, σ) and covariance matrix I_0^{-1}, where I_0^{-1} is the inverse of the observed information matrix which defined as follows

$$I_0^{-1}=\begin{bmatrix}\frac{\partial^2 l_n}{\partial\alpha} & \frac{\partial^2 l_n}{\partial\alpha\partial\beta} & \frac{\partial^2 l_n}{\partial\alpha\partial\theta} & \frac{\partial^2 l_n}{\partial\alpha\partial\sigma}\\ \frac{\partial^2 l_n}{\partial\alpha\partial\beta} & \frac{\partial^2 l_n}{\partial\beta^2} & \frac{\partial^2 l_n}{\partial\beta\partial\theta} & \frac{\partial^2 l_n}{\partial\beta\partial\sigma}\\ \frac{\partial^2 l_n}{\partial\alpha\partial\theta} & \frac{\partial^2 l_n}{\partial\theta\partial\beta} & \frac{\partial^2 l_n}{\partial\theta^2} & \frac{\partial^2 l_n}{\partial\theta\partial\sigma}\\ \frac{\partial^2 l_n}{\partial\alpha\partial\sigma} & \frac{\partial^2 l_n}{\partial\beta\partial\sigma} & \frac{\partial^2 l_n}{\partial\sigma\partial\theta} & \frac{\partial^2 l_n}{\partial\sigma^2}\end{bmatrix}$$

Second derivatives information matrix are given by

$$\frac{\partial^2 l_n}{\partial \alpha} = \frac{-n}{\alpha^2} + (\beta - 1) \sum_{i=1}^{n} F_i B_i$$

$$\frac{\partial^2 l_n}{\partial \beta^2} = \frac{-n}{\beta^2}$$

$$\frac{\partial^2 l_n}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{n}{2} + \alpha \left\{ \sum_{i=1}^{n} A_i \left[ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right) \right]^2 \right\} + (\beta - 1) \sum_{i=1}^{n} \left\{ \left[\frac{1}{c_i} \right] [E_i \left[\left[\left[ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right) \right]^2 D_i B_i \right] + F_i A_i B_i \left[ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right) \right] - \alpha F_i^2 A_i B_i \left[ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right) \right]^2 \right] - \alpha F_i^2 A_i B_i \left[ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right) \right]^2 \right]$$

$$\frac{\partial^2 l_n}{\partial \sigma^2} = \frac{n}{\sigma^2} + \frac{2}{\sigma^3} + \sum_{i=1}^{n} x_i G_i + \frac{1}{\sigma^4} \sum_{i=1}^{n} x_i^2 e^{\frac{x_i}{\sigma}} + \frac{2 \sum_{i=1}^{n} x_i}{\sigma^3} - \frac{\alpha\theta}{\sigma^3} \sum_{i=1}^{n} x_i A_i \left[\frac{2}{1+e^{\frac{x_i}{\sigma}}} \right] - \frac{2\alpha\theta}{\sigma^4} \{ \sum_{i=1}^{n} x_i^2 G_i A_i [1 + G_i] \} + \frac{4\alpha\theta(\beta-1)}{\sigma^3} \sum_{i=1}^{n} x_i G_i A_i - \frac{2\alpha\theta(\beta-1)}{\sigma^4} \sum_{i=1}^{n} \left\{ \left[\frac{1}{c_i} \right] [E_i [x_i^2 B_i G_i A_i + \frac{(\theta-1)}{2} x_i I_i + \alpha\theta x_i G_i A_i B_i] - 2\alpha\theta x_i^2 e^{\frac{x_i}{\sigma}} G_i A_i^2 B_i F_i \right\}$$

$$\frac{\partial^2 l_n}{\partial \alpha \partial \beta} = \alpha \sum_{i=1}^{n} \frac{B_i F_i}{E_i}$$

$$\frac{\partial^2 l_n}{\partial \alpha \partial \theta} = \alpha \sum_{i=1}^{n} A_i ln + (\beta - 1) \sum_{i=1}^{n} \frac{ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right) B_i A_i F_i}{E_i} + \alpha^2 (\beta - 1) \sum_{i=1}^{n} B_i F_i A_i ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right)$$

$$\frac{\partial^2 l_n}{\partial \alpha \partial \sigma} = \frac{\theta}{\sigma^2} \sum_{i=1}^{n} x_i G_i A_i - \frac{\alpha\theta(\beta-1)}{\sigma^2} \sum_{i=1}^{n} \frac{x_i G_i F_i A_i B_i}{E_i}$$

$$\frac{\partial^2 l_n}{\partial \beta \partial \theta} = \alpha \sum_{i=1}^{n} \frac{ln \left(\frac{1+e^{\frac{x_i}{\sigma}}}{2} \right) F_i A_i B_i}{C_i}$$

$$\frac{\partial^2 l_n}{\partial \beta \partial \sigma} = \frac{\alpha\theta}{2\sigma^2} \sum_{i=1}^{n} \frac{x_i G_i A_i B_i}{E_i}$$

$$\frac{\partial^2 l_n}{\partial \theta \partial \sigma} = \sum_{i=1}^{n} x_i G_i + \frac{\alpha}{\sigma^2} \sum_{i=1}^{n} x_i G_i A_i + \frac{2\alpha\theta(\theta-1)}{\sigma^2} \sum_{i=1}^{n} A_i \left[\frac{x_i}{1+e^{\frac{x_i}{\sigma}}} \right] - \frac{\alpha\theta(\beta-1)[\theta ln2+1]}{2^{\theta}\sigma^2} \left[\sum_{i=1}^{n} A_i \left[\frac{x_i}{1+e^{\frac{x_i}{\sigma}}} \right] B_i E_i - \sum_{i=1}^{n} x_i B_i A_i \left[E_i [\left(\frac{\theta-1}{1+e^{\frac{x_i}{\sigma}}} \right) - \theta\alpha F_i] \right] - \alpha\theta G_i F_i A_i \left(1 + e^{\frac{x_i}{\sigma}} \right)^{\theta-1}$$

Here

$$A_i = \left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta},\ B_i = e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)},\ C_i = \left[1 - e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)}\right]^2$$

$$D_i = \left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{2\theta},\ E_i = 1 - e^{-\alpha\left(\left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta}-1\right)},\ F_i = \left(\frac{e^{\frac{x_i}{\sigma}}+1}{2}\right)^{\theta} - 1,\ G_i = \frac{e^{\frac{x_i}{\sigma}}}{1+e^{\frac{x_i}{\sigma}}}$$

The Asymptotic (1-α)100% Confident intervals for estimated parameters are as follows
$\hat{\alpha} + z_{\frac{\alpha}{2}}[var(\hat{\alpha})], \hat{\beta} + z_{\frac{\alpha}{2}}[var(\hat{\beta})], \hat{\theta} + z_{\frac{\alpha}{2}}[var(\hat{\theta})]\ and\ \hat{\sigma} + z_{\frac{\alpha}{2}}[var(\hat{\sigma})]$

Application:

We evaluate the performance of the maximum likelihood method for estimating the OGET-IGHLD (Θ), where Θ = (α, β, θ, σ) parameters using Monte Carlo simulation for a four parameter combinations and the process is repeated 200 times. Two different sample sizes *n*=100 and 200 are considered. The MLEs and their standard deviations of the parameters are listed in Table.

Table 1

Sape Size	Actual values				Estimated values				Standard deviations			
n	α	β	θ	σ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$
100	0.5	1	2	2	0.5264	1.3568	2.2244	2.4546	0.0002	0.0028	0.0022	0.004
	0.5	2	1	1	0.5782	1.9586	1.5689	1.1245	0.0005	0.0004	0.0056	0.0012
	0.5	1	2	1	0.4578	1.5684	2.4871	1.2454	0.0004	0.0056	0.0048	0.0024
	2	1	1	2	2.4698	1.3560	1.5842	2.0457	0.0038	0.0025	0.0025	0.0026
	2	2	2	1	1.4569	2.4523	2.2546	1.2548	0.0046	0.0045	0.0025	0.0038
	2	1	1	1	2.3486	1.5689	1.2543	1.3546	0.0051	0.0056	0.0014	0.0026
	2.5	1	2	2	2.8956	1.6894	2.1457	2.2345	0.0028	0.0064	0.0078	0.0017
	2.5	2	1	1	2.5678	2.1535	1.7846	1.1784	0.0006	0.0015	0.0054	0.0001
	2.5	1	2	1	2.3256	1.5684	2.5477	1.0147	0.0017	0.0056	0.0068	0.00001
	3	1	1	2	3.5469	1.4865	1.6874	2.0014	0.0046	0.0039	0.0035	0.002
	3	2	2	1	3.2625	2.5684	2.3581	1.1124	0.0022	0.0006	0.0045	0.0003
	3	1	1	1	2.9856	1.3568	1.4578	1.0245	0.0004	0.00005	0.0045	0.0009
200	0.5	1	2	2	0.5004	1.0752	2.3564	2.1245	0.000004	0.0026	0.0013	0.0002
	0.5	2	1	1	0.4589	2.0054	1.1450	1.1004	0.0004	0.001	0.0010	0.0012
	0.5	1	2	1	0.4965	1.2485	2.1004	1.1165	0.0003	0.00007	0.0005	0.0016
	2	1	1	2	2.0065	1.1458	1.0042	2.0045	0.00006	0.00005	0.00001	0.00004
	2	2	2	1	1.9865	2.0078	2.0008	1.0047	0.0001	0.0014	0.0002	0.00004
	2	1	1	1	1.995	1.0058	1.024	1.0145	.0000	0.0015	0.001	.00014

					6		5		51		0	
	2.5	1	2	2	2.589 5	1.0568	2.104 5	2.0036	.0008	0.0016	0.001	.00003
	2.5	2	1	1	2.500 4	2.1457	1.012 1	1.1245	.0000 04	0.0000 5	.0000 4	.0012
	2.5	1	2	1	2.458 9	1.1457	2.000 3	1.0048	0.000 4	0.0000 7	.0000 03	.00004
	3	1	1	2	3.004 8	1.1562	1.047 5	2.2356	0.000 04	0.0000 4	.0001 04	.0023
	3	2	2	1	2.945 8	2.0045	2.010 4	1.1478	.0005 42	0.0000 1	.0000 21	.0014
	3	1	1	1	3.012	1.0072	1.004 5	1.0078	0.000 12	0.0000 2	.0000 03	.00007

Conclusion:

In the observation of above simulated data Sample size increases biases and the standard deviations of the MLEs decrease as expected.

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