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Common Fixed Point Theorem for Compatible Maps of Type (β)

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Abstract: *In this paper we prove a common fixed point theorem for compatible map of type* (β) *in Fuzzy 2-metric space.*

Keywords*: Fuzzy 2- metric space, G- Cauchy sequence, Weakly compatible, point of coincidence, complete metric space*

I. Introduction

In 1998, Vasuki [1] established a generalization of Grabiec's fuzzy contraction theorem wherein he proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Thereafter, Cho [2] extended the concept of compatible mappings of type (alpha) to fuzzy metric spaces and utilize the same to prove fixed point theorems in fuzzy metric spaces. Several years later, Singh and Chauhan [3] introduced the concept of compatible mappings and proved two common fixed point theorems in the fuzzy metric space with the minimum triangular norm. In 2002, Sharma [4] further extended some known results of fixed point theory for compatible mappings in fuzzy metric spaces. At the same time, Gregori and Sapena [5] introduced the notion of fuzzy contractive mapping and proved fixed point theorems in varied classes of complete fuzzy metric spaces in the senses of George and Veeramani [6], Kramosil and Michalek [7] and Grabiec's [8]. Soon after, Mihet [9] proposed a fuzzy fixed point theorem for (weak) Banach contraction in M-complete fuzzy metric spaces. In this continuation, Mihet [10, 11] further enriched the fixed point theory for various contraction mappings in fuzzy metric spaces besides introducing variants of some new contraction mappings such as: Edelstein fuzzy contractive mappings, fuzzy ψ−contraction of (ϵ, λ) type etc. In the same spirit, Qiu et al. [12, 13] also obtained some common fixed point theorems for fuzzy mappings under suitable conditions

Our objective of this paper is to prove a common fixed point theorem by removing the assumption of continuity, relaxing compatibility to compatible maps of type(β) weakly compatibility and replacing the completeness of the space with a set of alternative conditions for functions satisfying an implicit relation in FM-space.

II.Preliminaries

Definition 2.1: Let $I = [0,1]$,$*$ be a continuous t-norm and F be the set of all real continuous functions $F : I^6 \to R$ satisfying the following conditions

2.1 F is no increasing in the fifth and sixth variables,

2.2 if, for some constant $k \in (0,1)$ we have

2.2(a) $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1$, or

2.2(b) $F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1$

for any fixed $t > 0$ and any nondecreasing functions $u, v : (0, \infty) \to I$ with $0 \leq u(t), v(t) \leq 1$ then there exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$,

2.3 if, for some constant $k \in (0,1)$ we have

$$F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$$

for any fixed $t > 0$ and any nondecreasing function $u : (0, \infty) \to I$ then $u(kt) \geq u(t)$.

Beside this the concepts of Fuzzy 2-metric spaces are as follows,

Definition2.2: A triplet $(X, M, \star)$ is said to be a Fuzzy 2- metric space if X is an arbitrary set, $\star$ is a continuous $t-$ norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following condition for all $x, y, z, s, t > 0$,

2.2 $(FM-1)$ $M(x, y, \theta, t) > 0$

2.2 $(FM-2)$ $M(x, y, \theta, t) = 1$ if and only if $x = y = \theta$.

2.2 $(FM-3)$ $M(x, y, \theta, t) = M(y, \theta, x, t) = M(\theta, x, y, t)$

2.2 $(FM-4)$

$M(x, y, \theta, t) \star M(y, z, \theta, s) \star M(z, x, \theta, q) \leq M(x, y, z, t + s + q)$

2.2 (FM − 5) $M(x, y, \theta, \bullet) : (0, \infty) \to (0,1]$ is continuous.

Then M is called a Fuzzy 2- metric on X. The function $M(x, y, \theta, t)$ denote the degree of nearness between x , y and θ with respect to t.

Definition 2.3: A sequence $\{x_n\}$ in a Fuzzy 2- metric space $(X, M, \star)$ is said to be a converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_n, x, \theta, t) > 1 - \varepsilon$ for all $n \geq n_0$.

Definition 2.4: A sequence $\{x_n\}$ in a Fuzzy 2- metric space $(X, M, \star)$ is said to be a G- Cauchy sequence converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_m, x_n, \theta, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.

A Fuzzy 2- metric space $(X, M, \star)$ is said to be complete if every G- Cauchy sequence in it converges to a point in it.

III. Main Result

Theorem 3.1 Let $(X, M, \star)$ be a complete Fuzzy 2- metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

3.1(a) $P(X) \subset ST(X)$ and $Q(X) \subset AB(X)$,

3.1(b) (P, AB) is compatible of type (β) and (Q, ST) is weak compatible,

3.1(c) there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$F\begin{pmatrix} M^2(Px, Qy, \theta, kt), M^2(ABx, STy, \theta, t), M^2(Px, ABx, \theta, t), \\ M^2(Qy, STy, \theta, t), M^2(Px, STy, \theta, t), M^2(ABx, Qy, \theta, t) \end{pmatrix} \geq 1$$

Then A, B,S,T, P and Q have a unique common fixed point in X.

Proof Let $x_0 \in X$, then from 3.1(a) we have $x_1, x_2 \in X$ such that

$$Px_0 = STx_1 \text{ and } Qx_1 = ABx_2$$

Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for $n \in N$

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$$Px_{2n-2} = STx_{2n-1} = y_{2n-1} \text{ and } Qx_{2n-1} = ABx_{2n} = y_{2n}$$

Put $x = x_{2n}$ and $y = x_{2n+1}$ in 3.1(c)then we have

$$F\begin{pmatrix} M^2(Px_{2n}, Qx_{2n+1}, \theta, kt), M^2(ABx_{2n}, STx_{2n+1}, \theta, t), \\ M^2(Px_{2n}, ABx_{2n}, \theta, t), M^2(Qx_{2n+1}, STx_{2n+1}, \theta, t)\ , \\ M^2(Px_{2n}, STx_{2n+1}, \theta, t), M^2(ABx_{2n}, Q_{2n+1}, \theta, t) \end{pmatrix} > 1$$

$$F\begin{pmatrix} M^2(y_{2n+1}, y_{2n+2}, \theta, kt), M^2(y_{2n}, y_{2n+1}, \theta, t), \\ M^2(y_{2n+1}, y_{2n}, \theta, t), M^2(y_{2n+2}, y_{2n+1}, \theta, t), \\ M^2(y_{2n+1}, y_{2n+1}, \theta, t), M^2(y_{2n}, y_{2n+2}, \theta, t) \end{pmatrix} > 1$$

$$F\begin{pmatrix} M^2(y_{2n+1}, y_{2n+2}, \theta, kt), M^2(y_{2n}, y_{2n+1}, \theta, t), \\ M^2(y_{2n+1}, y_{2n}, \theta, t), M^2(y_{2n+2}, y_{2n+1}, \theta, t), \\ M^2(y_{2n+1}, y_{2n+1}, \theta, t), \\ M^2\left(y_{2n}, y_{2n+1}, \theta, \frac{t}{2}\right) \star M^2\left(y_{2n+1}, y_{2n+2}, \theta, \frac{t}{2}\right) \end{pmatrix} > 1$$

From condition 3.2 (a) we have

$$M^2(y_{2n+1}, y_{2n+2}, \theta, kt) \geq M^2\left(y_{2n}, y_{2n+1}, \theta, \frac{t}{2}\right) \star M^2\left(y_{2n+2}, y_{2n+1}, \theta, \frac{t}{2}\right)$$

we have

$$M^2(y_{2n+1}, y_{2n+2}, \theta, kt) \geq M^2\left(y_{2n}, y_{2n+1}, \theta, \frac{t}{2}\right)$$

That is

$$M(y_{2n+1}, y_{2n+2}, \theta, kt) \geq M\left(y_{2n}, y_{2n+1}, \theta, \frac{t}{2}\right)$$

Similarly we have

$$M(y_{2n+2}, y_{2n+3}, \theta, kt) \geq M\left(y_{2n+1}, y_{2n+2}, \theta, \frac{t}{2}\right)$$

Thus we have

$$M(y_{n+1}, y_{n+2}, \theta, kt) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{2}\right)$$

$$M(y_{n+1}, y_{n+2}, \theta, t) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{2^k}\right)$$

$$M(y_n, y_{n+1}, \theta, t) \geq M\left(y_0, y_1, \theta, \frac{t}{2^{nk}}\right) \rightarrow 1 \ as \ n \rightarrow \infty,$$

and hence $M(y_n, y_{n+1}, \theta, t) \rightarrow 1 \ as\, n \rightarrow \infty \, for \ all \ t > 0$.

For each $\epsilon > 0 \ and \ t > 0$, we can choose $n_0 \in N$ such that

$$M(y_n, y_{n+1}, \theta, t) > 1 - \epsilon \ for \ all \ n > n_0.$$

For any $m, n \in N$ we suppose that $m \geq n$. Then we have

$$M(y_n, y_m, \theta, t) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{m-n}\right) \star M\left(y_{n+1}, y_{n+2}, \theta, \frac{t}{m-n}\right) \star$$

$$\dots . \star M\left(y_{m-1}, y_m, \theta, \frac{t}{m-n}\right)$$

$$M(y_n, y_m, \theta, t) \geq (1-\epsilon) \star (1-\epsilon) \star \dots \dots \star (1-\epsilon)(m-n) times$$

$$M(y_n, y_m, \theta, t) \geq (1-\epsilon)$$

And hance $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, M, \star)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converges to the same point $z \in X$.

That is

$$\{Px_{2n+2}\} \rightarrow z \ and \ \{STx_{2n+1}\} \rightarrow z$$

$$\{Qx_{2n+1}\} \rightarrow z \ and \ \{ABx_{2n}\} \rightarrow z$$

As (P, AB) is compatible pair of type (β), we have

$$M(PPx_{2n}, (AB)(AB)x_{2n}, \theta, t) = 1, \ for \, all \, t > 0$$

Or $\qquad M(PPx_{2n}, ABz, \theta, t) = 1$

Therefore, $PPx_{2n} \to ABz.$

Put $x = (AB)x_{2n}$ and $y = x_{2n+1}$ in 3.1(c)we have

$$F\begin{pmatrix} M^2(P(AB)\mathrm{x}_{2n}, Qy, \theta, kt), M^2(AB(AB)x_{2n}, STx_{2n+1}, \theta, t), \\ M^2(P(AB)x_{2n}, AB(AB)x_{2n}, \theta, t), M^2(Qx_{2n+1}, STx_{2n+1}, \theta, t) \\ M^2(P(AB)x_{2n}, STx_{2n+1}, \theta, t), M^2(AB(AB)x_{2n}, Qx_{2n+1}, \theta, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ and 3.1(a) we get

$$M^2\big((AB)z, z, \theta, kt\big) \geq M^2\big((AB)z, z, \theta, t\big)$$

That is $M\big((AB)z, z, \theta, kt\big) \;\geq\; \mathrm{M}\big((AB)z, z, \theta, t\big)$

so we have

$ABz = z$.

Put $x = z$ and $y = x_{2n+1}$ in 3.1(c)we have

$$F\begin{pmatrix} M^2(Pz, Q\,x_{2n+1}, \theta, kt), M^2(Q\,x_{2n+1}, ST\,x_{2n+1}, \theta, t), \\ M^2(ABz, ST\,x_{2n+1}, \theta, t) \star M^2(Pz, ABz, \theta, t) \\ M^2(Pz, ST\,x_{2n+1}, \theta, t), M^2(ABz, Q\,x_{2n+1}, \theta, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ 3.2 (a) we have

That is $M^2(Pz, z, \theta, kt) \geq M^2(Pz, z, \theta, t)$

And hence $M(Pz, z, \theta, kt) \geq M(Pz, z, \theta, t)$

we get $Pz = z$

So we have $ABz = Pz = z$.

Putting $x = Bz$ and $y = x_{2n+1}$ in $3.1(c)$, we get

$$F\begin{pmatrix} M^2(PBz, Qx_{2n+1}, kt), M^2(ABBz, STx_{2n+1}, t), \\ M^2(PBz, ABBz, t), M^2(Qx_{2n+1}, STx_{2n+1}, t), \\ M^2(PBz, STx_{2n+1}, t), M^2(ABBz, Qx_{2n+1}, t) \end{pmatrix} > 1$$

Taking $n \to \infty$, 3.2(a) we get

$$M^2(Bz, z, \theta, kt) \geq M^2(Bz, z, \theta, t)$$

That is $M(Bz, z, \theta, kt) \geq M(Bz, z, \theta, t)$

we have $Bz = z$

And also we have $ABz = z \ implies \ Az = z$

Therefore $Az = Bz = Pz = z$.

As $P(X) \subset ST(X)$ there exists $u \in X$ such that

$$z = Pz = STu$$

Putting $x = x_{2n} \ and \ y = u$ in 3.1(c)we get

$$F\begin{pmatrix} M^2(Px_{2n}, Qu, \theta, kt), M^2(ABx_{2n}, STu, \theta, t), \\ M^2(Px_{2n}, ABx_{2n}, \theta, t), M^2(Qu, STu, \theta, t), \\ M^2(Px_{2n}, STu, \theta, t), M^2(ABx_{2n}, Qu, \theta, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ we get

$$F\begin{pmatrix} M^2(z, Qu, \theta, kt), M^2(z, STu, \theta, t), M^2(z, z, \theta, t) \\ M^2(Qu, STu, \theta, t), M^2(z, STu, \theta, t), M^2(z, Qu, \theta, t) \end{pmatrix} > 1$$

$$M^2(z, Qu, \theta, kt) \geq M^2(z, Qu, \theta, t)$$

That is $M(z, Qu, \theta, kt) \geq M(z, Qu, \theta, t)$

we have $Qu = z$

Hence $STu = z = Qu$.

Hence (Q, ST) is weak compatible, therefore, we have

$$QSTu = STQu$$

Thus $Qz = STz$.

Putting $x = x_{2n}$ and $y = z$ in 3.2(c)we get

$$F\begin{pmatrix} M^2(Px_{2n}, Qz, \theta, kt), M^2(ABx_{2n}, STz, \theta, t), \\ M^2(Px_{2n}, ABx_{2n}, \theta, t), M^2(Qz, STz, \theta, t), \\ M^2(Px_{2n}, STz, \theta, t), M^2(ABx_{2n}, Qz, \theta, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ we get

$$F\begin{pmatrix} M^2(z, Qz, \theta, kt), M^2(z, STz, \theta, t), M^2(z, z, \theta, t) \\ M^2(Qz, STz, \theta, t), M^2(z, STz, \theta, t), M^2(z, Qz, \theta, t) \end{pmatrix} > 1$$

$$M^2(z, Qz, \theta, kt) \geq M^2(z, Qz, \theta, t)$$

And hence $M(z, Qz, \theta, kt) \geq M(z, Qz, \theta, t)$

we get $Qz = z$.

Putting $x = x_{2n}$ and $y = Tz$ in 3.2(c)we get

$$F\begin{pmatrix} M^2(Px_{2n}, QTz, \theta, kt), M^2(ABx_{2n}, STTz, \theta, t), \\ M^2(Px_{2n}, ABx_{2n}, \theta, t), M^2(QTz, STTz, \theta, t), \\ M^2(Px_{2n}, STTz, \theta, t), M^2(ABx_{2n}, QTz, \theta, t) \end{pmatrix} > 1$$

As $QT = TQ$ and $ST = TS$ we have

$$QTz = TQz = Tz$$

And $ST(Tz) = T(STz) = TQz = Tz$.

Taking $n \to \infty$ we get

$$F\begin{pmatrix} M^2(z, Tz, \theta, kt), M^2(z, Tz, \theta, t), M^2(z, z, \theta, t) \\ M^2(Tz, Tz, \theta, t), M^2(z, Tz, \theta, t), M^2(z, Tz, \theta, t) \end{pmatrix} > 1$$

$$M^2(z, Tz, \theta, kt) \geq M^2(z, Tz, \theta, t)$$

Therefore $M(z, Tz, \theta, kt) \geq M(z, Tz, \theta, t)$

we have $Tz = z$

Now $STz = Tz = z$ $implies$ $Sz = z$.

Hence $Sz = Tz = Qz = z$

we have

$$Az = Bz = Pz = Sz = Tz = Qz = z$$

Hence z is the common fixed point of A, B, S, T, P and Q.

Uniqueness Let u be another common fixed point of A,B,S,T,P and Q. Then

$$Au = Bu = Su = Tu = Pu = Qu = u$$

Putting $x = u$ and $y = z$ in 3.2(c)then we get

$$F\begin{pmatrix} M^2(Pu, Qz, \theta, kt), M^2(ABu, STz, \theta, t), \\ M^2(Pu, ABu, \theta, t), M^2(Qz, STz, \theta, t), \\ M^2(Pu, STz, \theta, t), M^2(ABu, Qz, \theta, t) \end{pmatrix} > 1$$

Taking limit both side then we get

$$F\begin{pmatrix} M^2(u, z, \theta, kt), M^2(u, z, \theta, t), M^2(u, u, \theta, t) \\ M^2(z, z, \theta, t), M^2(u, z, \theta, t), M^2(u, z, \theta, t) \end{pmatrix} > 1$$

$$M^2(u, z, \theta, kt) \geq M^2(u, z, \theta, t)$$

And hence $M(u, z, \theta, kt) \geq M(u, z, \theta, t)$

we get $z = u$.

That is z is a unique common fixed point of A,B, S, T, P and Q in X.

Remark If we take $B = T = I$ identity map on X in Theorem then we get following Corollary

Corollary 3.2 Let $(X, M,\star)$ be a complete Fuzzy 2- metric space and let $A, S, P \; and \; Q$ be mappings from X into itself such that the following conditions are satisfied:

(a) $P(X) \subset S(X) \; and \; Q(X) \subset A(X),$

(b) (P, A) is compatible of type (β) and (Q, S) is weak compatible,

(c) there exists $k \in (0,1)$ such that for every $x, y \in X \; and \; t > 0$

$$F\begin{pmatrix} M^2(Px, Qy, \theta, kt), M^2(Ax, Sy, \theta, t), M^2(Px, Ax, \theta, t), \\ M^2(Qy, Sy, \theta, t), M^2(Px, Sy, \theta, t), M^2(Ax, Qy, \theta, t) \end{pmatrix} \geq 1$$

Then A, S, P and Q have a unique common fixed point in X.

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