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Some High Degree Gauss Legendre Quadrature Formulas for Triangles

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ABSTRACT: This paper presents a numerical integration formula for the evaluation of $II_{\Omega_N}(f) = \iint_{\Omega_N} f(x,y)dxdy$ where $f \epsilon C\,(\Omega_N)$ and Ω_N is any polygonal domain in $\mathbb{R}^2$. That is a closed domain with boundary composed of N oriented piecewise line segments $\ell_{i,k}$ $(k = i+1,\ i = 1,2,..N)$ with end points (x_i, y_i), (x_k, y_k) and $(x_1, y_1) = ((x_{N+1}, y_{N+1})$. We join each of these line segments $\ell_{i,k}$ to a reference point (x_r, y_r) interior to the domain Ω_N. This creates a coarse mesh of N triangles Δ_i $(i = 1, 2, \dots N)$ in Ω_N and each of these arbitrary triangles have three straight sides. These arbitrary triangles can be divided into n^2 arbitrary triangles $\Delta_{i,j}$ $,(i = 1,2,3, \dots .N; j = 1,2,3, \dots .n^2$). by using the triangular mesh generation scheme developed in this paper. We transform each $\Delta_{i,j}$=Δ (say) into a standard 1-square and then into a 2-square which can be integrated by using Gauss Legendre quadraure rules.We obtain enhanced accuracy by division of these arbitrary triangles Δ into four arbitrary triangles Δ^e (e=1,2,3,4) without refining the already generated triangular mesh.We first derive three different integration formulas for the integral $\iint_\Delta f(x,y)dxdy$ where $\Delta\ is\ an\ arbitrary\ triangle\ with\ vertices\ ((x_k\,, y_k), k = 1,2,3)$ Then we establish the necessary numerical integration formulas which use the well known Gauss Legendre quadrature rules. Proposed numerical integration formula is applied to integrals over triangular and polygonal domains with complicated integrands.

1.INTRODUCTION

The finite element method is one of the most powerful computational technique for approximate solution of a variety of " real world " engineering and applied science problems for over half a century since its inception in the mid 1960. Today, finite element analysis (FEA) has becomes an integral and major component in the design or modeling of a physical phenomenon in various disciplines. The triangular and quadrilateral elements with either straight sides or curved sides are very widely used in a variety of applications [1-3]. The basic problem of integrating a function of two variables over the surface of the triangle is the subject of extensive research by many authors [4-6] and the precision of these formulas is limited to degree 20 at most. Derivation of high precision formulas is now possible over the triangular region by application product formulas based only on the sampling points and weights of the well known Gauss Legendre quadrature rules [7-11]. The use of product formulas will remove the restrictions on the derivation of high precision numerical integration formulas. However a systematic derivtion of these formulas is not given so far in all these studies.This paper discusses the derivation in great detail and makes an attempt to suggest the precise use of Gauss legendre quadrature rules.This paper presents four different rules based on Gauss Legendre quadrature for the triangles. There are reasons which support the development of composite integration for practical applications. In several investigations composition integration is illustrated with reference to standard triangle [12].This paper systematic derivation of integration methods and also applies triangular mesh generation for composite numerical integration. In this paper it is also shown that for a given mesh of triangles a refinement of each triangle by four smaller triangles can be done without generating a new triangular mesh. In this paper, we have used composite integration for triangular domains which are fully discretised by triangles and the test function integrals are shown to agree with the exact values up to 16 significant digits for smooth functions, this implies that the absolute error is of the order 10^{-16} . This demonstrates the efficiency of the derived formulas. Our main purpose in this paper is to evolve practical and workable algorithms for efficient composite numerical integration over triangular surfaces by using the well known Gauss Legendre quadrature rules and triangular mesh generation scheme proposed recently by the authors. We have also appended the relevant and necessary computer codes. In section 2 ,we explain the basis of the derivation which can be traced to the bilinear trans formation of quadrilateral into a 1-square or 2-square.We have shown how to obtain bilinear transformations for a triangle by joining two vertices of a quadrilateral or by placing one of the vertices on a diagonal of the quadrilateral, In section 3 of this paper, these transformations are then applied to integrate a smooth function over a triangular surface. This results in the derivation of three integration formulas for a triangular surface. In section 4,we derive numerical schemes by application of the well known Gauss Legendre quadrature rules.

2. BILINEAR TRANSFORMATION OVER A TRIANGULAR SURFACE

The basic concept i the derivation is to establish a transformation between an arbitray linear triangle and a square region in the plane.Thus it is natural that we make use of the well known transformations

We can derive numerous bilinear transformations which map an arbitrary triangle into a standard square. We first describe three unique transformations for this purpose.

It is well known that there exists a unique bilinear transformation which maps an arbitrary quadrilateral into a standard square.We first use the simple concept of joining two of the vertices of an arbitraqry quadrilateral to obtain an arbitrary linear triangle.

Let us first consider the bilinear mapping of an arbitrary quadrilateral with vertices ($(x_i , y_i), i = 1,2,3\ 4$), into a standard unit square and a standard 2-square

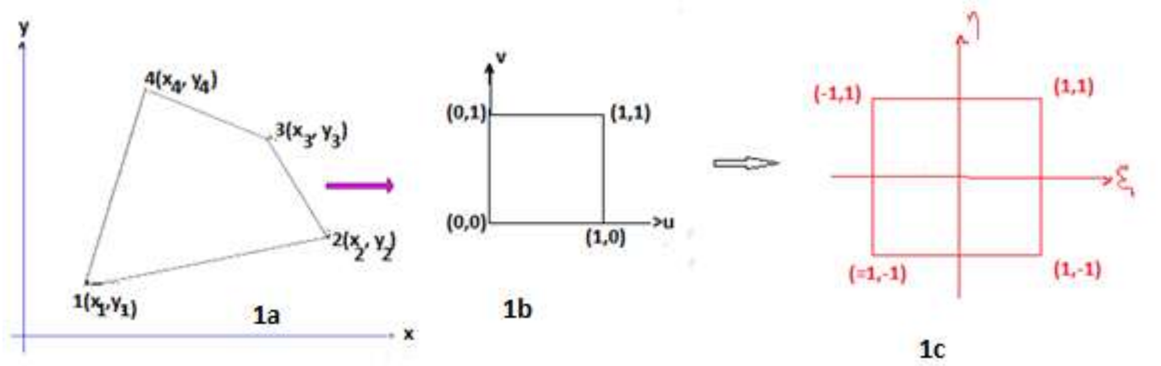

Fig.1a:convex quadrilateral in (x,y) space

Fig.1b:standard 1-square in (u,v) space

Fig.1c: standard 2-square in in $(\xi,\eta) space$

We can map an arbitrary qudrilateral with vertices ((x_i, y_i), $i = 1, 2, 3,4$) into a standard unit square in the (u, v) space and then into a 2-square as shown in Fig. 1a, 1b and 1c. The necessary transformation is given by the equations.

Let us consider an arbitrary four noded linear convex quadrilateral element in the global Cartesian space (x, y) as shown in Fig 1a which is mapped into a 1- square in the local parametric space (u, v) as shown in Fig 1b. The necessary bilinear transform is given by

$$\binom{x}{y} = \sum_{k=1}^{4} \binom{x_k}{y_k} N_k(u, v) \qquad (6)$$

Where (x_k, y_k) , $(k = 1,2,3,4)$ are the vertices of the quadrilateral element in (x, y) plane and $N_k(u, v)$ denotes the shape function of node k and they are expressed as in the standard texts [ ] ;

$$N_1 = (1 - u)(1 - v), N_2 = u(1 - v), N_3 = uv, N_4 = (1 - u)v \qquad (7a)$$

The 1-square in $(u, v) space$ of Fig.1b can be mapped into a 2-square in $(\xi, \eta) space$ of Fig.1c by using the transformation $u = (1 + \xi)/2, v = (1 + \eta)/2$ and in this case the equation for transformation from (x,y) space into the (ξ, η) space and the shape functions can be written as

$$\binom{x}{y} = \sum_{k=1}^{4} \binom{x_k}{y_k} N_k(\xi, \eta)$$

$$N_k(\xi, \eta) = \frac{1}{4}(1 + \xi\xi_k)(1 + \eta\eta_k) \qquad (7b)$$

$$\{(\xi_k, \eta_k), k = 1,2,3,4\} = \{(-1, 1), (1, -1), (1, 1), (-1, 1)\} \qquad (7c)$$

We can derive numerous bilinear transformations which map an arbitrary triangle into a standard square. We describe three unique transformations for this purpose

We first obtain two such transformations by joining two vertices of the quadrilateral and making the relevant changes in the transformation of eqs(6-7)

CASE(I)

Let us join vertices 2 and 3 of the convex quadrilateral depicted in Fig.2a ,this gives us the new transformation by substituting $x_3 = x_2, y_3 = y_2 \;\; in\, eq(6) \;\; as$

$$\binom{x}{y} = \binom{x_1}{y_1} N_1 \;(u,v) + \binom{x_2}{y_2} N_2 \;(u,v) + \binom{x_2}{y_2} N_3(u,v) + \binom{x_4}{y_4} N_4(u,v) \;, \qquad (8a)$$

the above equation simplifies to

$$\binom{x}{y} = \binom{x_1}{y_1} N_1 \;(u,v) + \binom{x_2}{y_2} (N_2(u,v) + N_3(u,v)\;) + \binom{x_4}{y_4} N_4 \;(u,v)$$

$$= \binom{x_1}{y_1} N_1 \;(u,v) + \binom{x_2}{y_2} u + \binom{x_4}{y_4} N_4 \;(u,v)$$

$$= \binom{x_1}{y_1} + \binom{x_2 - x_1}{y_2 - y_1} \varphi(u,v) + \binom{x_4 - x_1}{y_4 - y_1} \psi(u,v) \qquad (8b)$$

Where,

$$\varphi(u,v) = u, \psi(u,v) = (1-u)v \qquad (8c)$$

OR

$$\binom{\boldsymbol{x}}{\boldsymbol{y}} = \binom{x_1}{y_1} + \binom{x_2 - x_1}{y_2 - y_1} \varphi(U,V) + \binom{x_3 - x_1}{y_3 - y_1} \psi(U,V\,) \qquad (8d)$$

$$\varphi(U,V) = 1 - U, \psi(U,V) = UV \qquad (8e)$$

Where , we have made the simple change of variable U=1-u , V=v and renamed

$(x_4\,, y_4)$ as $(x_3\,, y_3)$.

We note that transformations of eqs(8b-8c) or (8d-8e) map an arbitray triangle of (x,y) space into a standard unit square of (u,v) space. Wecan now use either $u = (1+\xi)/2, v = (1+\eta)/2$ or $U = (1+\xi)/2, V = (1+\eta)/2$

to map the standard unit square of (u,v) space into a standard 2-square of the $(\xi\,,\eta)$ space and facilitate the application of numerical integration over an arbitrary triangle in (x,y) space by using the Gauss Legendre quadrature rules.

We may further note that by settting $(x_1\,, y_1)$ =(0,0), $(x_2\,, y_2)$ =(1,0), $(x_3\,, y_3)$ =(0,1),we obtain right isosceles triangle in the (x,y) space and in this case the above transformation maps a standard triangle into a unit square in the (u,v) plane.This transformation is given by

$$\binom{x}{y} = \binom{\varphi(u,v)}{\psi(u,v)} \qquad (8f)$$

We can now use $u = \frac{1+\xi}{2}, v = \frac{1+\eta}{2}\, in\ eq(8f)$ to map the standard unit square of (u,v) space into a standard 2-square of the $(\xi\ ,\eta)$ space and facilitate the application of numerical integration over a standard triangle in (x,y) space by using the Gauss Legendre quadrature rules.

CASE(II)

Let us join vertices 1 and 4 of the convex quadrilateral depicted in Fig.2a ,this gives us the new transformation by substituting $x_4 = x_1\ , y_4 = y_1\ \ in\ eq(6)\ \ as$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} N_1\ (u,v) + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} N_2\ (u,v) + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} N_3(u,v) + \begin{pmatrix} x_4 \\ y_1 \end{pmatrix} N_4(u,v)\ , \qquad (9a)$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} (N_1\ (u,v) + N_4(u,v)) \ + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} N_2\ (u,v) + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} N_3(u,v)$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} (1-u) \ + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} N_2\ (u,v) + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} N_3(u,v)$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \varphi(u,v) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \psi(u,v) \qquad (9b)$$

Where,

$$\varphi(u,v) = u(1-v), \psi(u,v) = uv \qquad (9c)$$

Letting U=u,V=1-v,we obtain

$$\varphi(U,V) = UV, \psi(U,V) = U(1-V) \qquad (9d)$$

We note that transformations of eqs(9b-9c) map an arbitray triangle of (x,y) space into a standard unit square of (u,v) space.We can further use $u = (1+\xi)/2, v = (1+\eta)/2$ in **eqs(9b-9c)** to map the standard unit square of (u,v) space into a standard 2-square of the $(\xi\ ,\eta)$ space and facilitate the application of numerical integration over an arbitrary triangle in (x,y) space by using the Gauss Legendre quadrature rules.

We may further note that by settting $(x_1\ , y_1)$ =(0,0), $(x_2\ , y_2)$ =(1,0), $(x_3\ , y_3)$ =(0,1),we obtain right isosceles triangle in the (x,y) space and in this case the above transformation maps a standard triangle into a unit square in the (u,v) plane.This transformation is given by

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \varphi(u,v) \\ \psi(u,v) \end{pmatrix} \qquad (9d)$$

We can now use $u = \frac{1+\xi}{2}, v = \frac{1+\eta}{2}\, in\ eq(9d)$ to map the standard unit square of (u,v) space into a standard 2-square of the $(\xi\ ,\eta)$ space and facilitate the application of numerical integration over a standard triangle in (x,y) space by using the Gauss Legendre quadrature rules.

We may further note that by settting $(x_1\ , y_1)$ =(0,0), $(x_2\ , y_2)$ =(1,0), $(x_3\ , y_3)$ =(0,1),we obtain right isosceles triangle in the (x,y) space and in this case the above transformation maps a standard triangle into a unit square in the (u,v) plane.This transformation is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varphi(u,v) \\ \psi(u,v) \end{pmatrix} \qquad (9e)$$

We can now use $u = \frac{1+\xi}{2}, v = \frac{1+\eta}{2}\, in\, eq(9d)$ to map the standard unit square of (u,v) space into a standard 2-square of the (ξ , η) space and facilitate the application of numerical integration over a standard triangle in (x,y) space by using the Gauss Legendre quadrature rules.

Similar transformations can be obtained by joining adjacent vertices of the quadrilateral and none of them will be different from the ones which are already presrnted in CASE(I} and CASE(II) above from computational point of view. At this stage,it is necessary to state that the above formulas generate asymmetric (eqs(8b-8f) and symmetric (eq(9b-9e) Gauss Legendre Quadrature rules for numerical integration over a standard triangle.

Now,we experiment with another method of creating a triangle from the convex quadrilateral of Fig,1a.

CASE(III)

In this new scheme ,one of the vertex node is placed on a diagonal of the quadrilateral.The convex quadrilateral of Fig.2a has two diagonals.These diagonals can be obtained by joinig vertices 1 and 3 and vertices 2 and 4.By placing vertex 3 on the diagonal joining vertices 2 and 4 ,we obtain the triangle spanned by vertices 1,2 and 4.Similarly by placing vertex 4 on the diagonal joining vertices 1 and 3,we obtain the triangle spanned by vertices 1,2,and 3

We select the diagonal joining the vertices 2 and 4 and place vertex 3 at some point on the diagonal,this will create a triangle spanning vertices 1,2, and 4. There are several ways to place the vertex 3 .We choose the vertex at the mid point of diagonal joining vertices 2 and 4,that is done by substituting $x_3 = (x_2 + x_4)/2, y_3 = (y_2 + y_4)/2$, in the bilinear transformation given in eq(6).Thus ,we obtain:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} N_1\ (u, v) + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} N_2\ (u, v) + \begin{pmatrix} (x_2 + x_4)/2 \\ (y_2 + y_4)/2 \end{pmatrix} N_3(u, v) + \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} N_4(u, v) \qquad (10a)$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} N_1\ (u, v) + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} (N_2(u, v) + N_3(u, v)\ /2\ ) + \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} (N_4(u, v) + N_3(u, v)\ /2\ )$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} N_1\ (u, v) + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \left(u - \frac{uv}{2}\right) + \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} (v - \frac{uv}{2}) \qquad (10b)$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \varphi(u, v) + \begin{pmatrix} x_4 - x_1 \\ y_4 - y_1 \end{pmatrix} \psi(u, v) \qquad (10c)$$

Where

$$\varphi(u, v) = \left(u - \frac{uv}{2}\right) = u\left(1 - \frac{v}{2}\right),\ \psi(u, v) = (v - \frac{uv}{2}) = v(1 - \frac{u}{2}) \qquad (10d)$$

OR

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \varphi(U, V) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \psi(U, V\ )$$

Where,

$$\varphi(U, V) = U(1 - \frac{V}{2}) \quad , \psi(U, V) = V(1 - \frac{U}{2}) \qquad (10e)$$

Where , we have made the simple change of variable U=u , V=v and renamed

(x_4 , y_4) as (x_3 , y_3) .

We note that transformations of eqs(10b-10c) map an arbitray triangle of (x,y) space into a standard unit square of (u,v) space.We can further use $u = (1+\xi)/2, v = (1+\eta)/2$ in **eqs(10b-10c)** to map the standard unit square of (u,v) space into a standard 2-square of the $(\xi\ ,\eta)$ space and facilitate the application of numerical integration over an arbitrary triangle in (x,y) space by using the Gauss Legendre quadrature rules.

We may further note that by settting $(x_1\ ,y_1)$ =(0,0), $(x_2\ ,y_2)$ =(1,0), $(x_3\ ,y_3)$ =(0,1),we obtain right isosceles triangle in the (x,y) space and in this case the above transformation maps a standard triangle into a unit square in the (u,v) plane.This transformation is given by

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \varphi(u,v) \\ \psi(u,v) \end{pmatrix} \qquad (10f)$$

We can now use $u = \frac{1+\xi}{2}, v = \frac{1+\eta}{2}\, in\ eq(10d)$ to map the standard unit square of (u,v) space into a standard 2-square of the $(\xi\ ,\eta)$ space and facilitate the application of numerical integration over a standard triangle in (x,y) space by using the Gauss Legendre quadrature rules.

3.NUMERICAL INTEGRATION FORMULAS

In some physical applications, we are required to compute integrals of some functions which are expressed in explicit form. In finite element and boundary element method, evaluation of two dimensional integrals with explicit function as integrands is of great importance. This is the subject matter of several investigations [ ]. We now consider the evaluation of the integral

$$II_{\Omega_N}(f) = \iint_{\Omega_N} f(x,y)dxdy, \qquad (11)$$

where Ω_N is a : polygonal domain

$II_{\Omega_N}(f)$ can be computed as finite sum of linear integrals and this can be expressed as

$$II_{\Omega_N}(f) = \sum_i \iint_{\Delta_i} f(x,y)dxdy \qquad (12)$$

where it is assumed that $\Omega_N = \bigcup_{i=1}^{N},\ \Delta_i,\ \Delta_i$ = an arbitrary triangle of the domain Ω_N .

In the previous section,the derivation of integration formulas is explained.Now,we discuss the numerical implementation of these formulas proposed in CASE(I),CASE(II) and CASE(III).

We consider the numerical integration of a function $f(x,y)$ of two variable which is sufficiently smooth over a given polygonal domain Ω_N and denote this as $\iint_{\Omega_N} f(x,y)dxdy$.The region Ω can triangulated or quadrangulated and so it is sufficient for all practical purposes to implement numerical integration schemes over the triangles.Let us now consider the integration $of\ a\ smooth\ function\ f(x,y)$ over an arbitrary triangular region,say, Δ,and denote this integral as $\iint_{\Delta} f(x,y)dxdy$

We assume , Δ as an arbitrary triangle spanned by vertices $((x_i, y_i)$,i=1.2.3) Then using the derivations integration formulas of the prewvious section we can write the numerical schemes for each of them by applying Gauss Legendre quadrature rules

The transformation which maps an arbitrary triangle first into a 1-square and the into a 2-square are clearly defined in the previous section and they are all expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \varphi(u,v) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \psi(u,v)$$

Using this transformation,we now proceed to establish necessary formulas,

$$\iint_\Delta f(x,y)dxdy = \int_0^1 \int_0^1 f\big(x(u,v), y(u,v)\big) \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

$$= \mathbf{2\Delta_{123}} \int_0^1 \int_0^1 f\big(x(u,v), y(u,v)\big) \left|\frac{\partial(\varphi,\psi,)}{\partial(u,v)}\right| du\, dv \qquad (13a)$$

Where,

$$2\Delta_{123} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = 2 \text{ X area of the triangle } \Delta \qquad (13b)$$

4. Numerical Schemes

The derivation numerical schemes is now fully dependent on the application of quadrature rule and the explicit form of the transformation $\big(x(u,v), y(u,v)\big)$ and the Jacobian of transformation $\frac{\partial(x,y,)}{\partial(u,v)}$.

CASE (I)

From eq(8a-8f),the transformation in this case is given by

$$\begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \varphi(u,v) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \psi(u,v)$$

Where,

$$\varphi(u,v) = 1 - u, \psi(u,v) = uv$$

Further using the simple transformation $u = u(\xi) = \frac{1+\xi}{2}, v = v(\eta) = \frac{1+\eta}{2}$,we obtain from

eq(13a-b)

$$\left|\frac{\partial(\varphi,\psi,)}{\partial(u,v)}\right| = u = \frac{1+\xi}{2}, dudv = \frac{1}{4} d\,\xi d\eta, \qquad (14a)$$

$$\iint_\Delta f(x,y)dxdy = \mathbf{2\Delta_{123}} \int_{-1}^1 \int_{-1}^1 f\big(x(u,v), y(u,v)\big)\left(\frac{1+\xi}{2}\right)\frac{1}{4} d\,\xi d\eta \qquad (14b)$$

where

$$\begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \varphi(u,v) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \psi(u,v)$$

$$\varphi(u,v)=\frac{1-\xi}{2},\psi(u,v)=(\frac{1+\xi}{2})(\frac{1+\eta}{2}) \qquad (14c)$$

Now,the integral of eq(14b) can be approximated to any degree of accuracy by applying the well known Gauss Legendre quadrature rule of order m and n along ξ and η axes respectively.Thus, we obtain

$$\iint_{\Delta} f(x,y)dxdy \sim \mathbf{2\Delta_{123}} \sum_{i=1}^{m}\sum_{j=1}^{n} \boldsymbol{f(x(u_i^m,v_j^n),y(u_i^m,v_j^n))\, w_i^m\, w_j^n}\ (\frac{\mathbf{1}+\xi_i^m}{\mathbf{8}}) \qquad \textbf{(15a)}$$

Where,

$$\boldsymbol{u_i^m} = (\frac{\mathbf{1}+\xi_i^m}{\mathbf{2}}), \boldsymbol{v_j^n} = (\frac{\mathbf{1}+\eta_j^n}{\mathbf{2}}), \qquad \textbf{(15b)}$$

and $((\xi_i^m, w_i^m), i = 1,2,\dots,m)$, $((\eta_j^n, w_j^n), j = 1,2,\dots,n)$ are the Gauss Legendre

sampling points and weights of order m and n respectively

We may note that in this case the preferred order of the Gauss Legendre quadrature rule is m=n+1 and n=n , this is suggestive on seeing the integrand in eq(14b). It may be further noted that for a standard triangle,the transformation rule is given by eq(8f) viz

$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}=\begin{pmatrix}\varphi(u,v)\\ \psi(u,v)\end{pmatrix}$ and this gives the sampling points

$$\begin{pmatrix}\boldsymbol{x(u_i^m,v_j^n)}\\ \boldsymbol{y(u_i^m,v_j^n)}\end{pmatrix}=\begin{pmatrix}\varphi(\boldsymbol{u_i^m,v_j^n})\\ \psi(\boldsymbol{u_i^m,v_j^n})\end{pmatrix} ,(i=1,2,....m;j=1,2,....n)$$

$$=\begin{pmatrix}(\frac{\mathbf{1}-\xi_i^m}{\mathbf{2}})\\ (\frac{\mathbf{1}+\xi_i^m}{\mathbf{2}})(\frac{\mathbf{1}+\eta_j^n}{\mathbf{2}})\end{pmatrix} \qquad (15c)$$

and the corresponding weights

$$\boldsymbol{w_i^m\, w_j^n}\ (\frac{\mathbf{1}+\xi_i^m}{\mathbf{8}})\ \mathbf{,}(i=1,2,....m;j=1,2,....n) \qquad (15d)$$

Eqs{15c,15d) generate an **asymmetric** table of values

$$[\varphi(\boldsymbol{u_i^m,v_j^n}) \quad \psi(\boldsymbol{u_i^m,v_j^n}) \quad w_i^m\, w_j^n \ \left(\frac{\mathbf{1}+\xi_i^m}{\mathbf{8}}\right)] \qquad (15e)$$

of sampling points and the corresponding weights for the standard triangle

W e have presented a sample of this for m=3,n=2 in Appendix-1,this table consists of a triplet of six values

CASE (II)

From eq(9a-9e),the transformation in this case is given by

$$\begin{pmatrix}x(u,v)\\ y(u,v)\end{pmatrix} = \begin{pmatrix}x_1\\ y_1\end{pmatrix}+\begin{pmatrix}x_2-x_1\\ y_2-y_1\end{pmatrix}\varphi(u,v)+\begin{pmatrix}x_3-x_1\\ y_3-y_1\end{pmatrix}\psi(u,v)$$

Where,

$\varphi(u,v) = uv, \psi(u,v) = u(1-v)$

Further using the simple transformation $u = u(\xi) = \frac{1+\xi}{2}, v = v(\eta) = \frac{1+\eta}{2}$,we obtain from

eq(13a-b)

$\left|\frac{\partial(\varphi,\psi,)}{\partial(u,v)}\right|$=u=$\frac{1+\xi}{2}$,dudv=$\frac{1}{4}$d $\xi d\eta$, (16a)

$\iint_\Delta f(x,y)dxdy$=$\mathbf{2}\Delta_{\mathbf{123}} \int_{-1}^{1}\int_{-1}^{1} f\big(x(u,v),y(u,v)\big)(\frac{1+\xi}{2})\frac{1}{4}$d $\xi d\eta$ (16b)

where

$$\begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \varphi(u,v) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \psi(u,v)$$

$$\varphi(u,v) = \left(\frac{1+\xi}{2}\right)(\frac{1+\eta}{2}), \psi(u,v) = (\frac{1+\xi}{2})(\frac{1-\eta}{2}) \qquad (16c)$$

Now,the integral of eq(16b) can be approximated to any degree of accuracy by applying the well known Gauss Legendre quadrature rule of order m and n along ξ and η axes respectively.Thus, we obtain

$$\iint_\Delta f(x,y)dxdy \sim \mathbf{2}\Delta_{\mathbf{123}} \sum_{i=1}^{m}\sum_{j=1}^{n} \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{u}_i^m, \boldsymbol{v}_j^n), \boldsymbol{y}(\boldsymbol{u}_i^m, \boldsymbol{v}_j^n))\, \boldsymbol{w}_i^m \boldsymbol{w}_j^n \left(\frac{\mathbf{1}+\xi_i^m}{\mathbf{8}}\right) \qquad \textbf{(17a)}$$

Where,

$$\boldsymbol{u}_i^m = \left(\frac{\mathbf{1}+\xi_i^m}{\mathbf{2}}\right), \boldsymbol{v}_j^n = \left(\frac{\mathbf{1}+\eta_j^n}{\mathbf{2}}\right), \qquad \textbf{(17b)}$$

and $((\xi_i^m, w_i^m), i = 1,2,\dots,m)$, $((\eta_j^n, w_j^n), j = 1,2,\dots,n)$ are the Gauss Legendre

sampling points and weights of order m and n respectively

We may note that in this case the preferred order of the Gauss Legendre quadrature rule is one of m=n,n+1; this is suggestive on seeing the integrand in eq(16b). It may be further noted that for a standard triangle,the transformation rule is given by eq(9e) viz

$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \varphi(u,v) \\ \psi(u,v) \end{pmatrix}$ and this gives the sampling points

,

$$\begin{pmatrix} \boldsymbol{x}(\boldsymbol{u}_i^m, \boldsymbol{v}_j^n) \\ \boldsymbol{y}(\boldsymbol{u}_i^m, \boldsymbol{v}_j^n) \end{pmatrix} = \begin{pmatrix} \varphi(\boldsymbol{u}_i^m, \boldsymbol{v}_j^n) \\ \psi(\boldsymbol{u}_i^m, \boldsymbol{v}_j^n) \end{pmatrix}$$,(i=1,2,....m;j=1,2,....n)

$$=\begin{pmatrix} (\frac{1+\xi_i^m}{2})(\frac{1+\eta_j^n}{2}) \\ (\frac{1+\xi_i^m}{2})(\frac{1-\eta_j^n}{2}) \end{pmatrix} \quad (17c)$$

and the corresponding weights

$$\boldsymbol{w_i^m\, w_j^n}\ (\frac{1+\xi_i^m}{8})\ ,(i=1,2,....m;j=1,2,....n) \quad (17d)$$

Eqs{17c,17d) generate **symmetric** table of values

$$[\varphi(\boldsymbol{u_i^m}, \boldsymbol{v_j^n}) \quad \psi(\boldsymbol{u_i^m}, \boldsymbol{v_j^n}) \quad w_i^m\, w_j^n \ \left(\frac{1+\xi_i^m}{8}\right)] \quad (17e)$$

of sampling points and the corresponding weights for the standard triangle

W e have presented a sample of this for m=n=2;m=3,n=2 in Appendix-2,these tables consists of a triplet of four and six values

CASE (III)

From eq(9a-9e),the transformation in this case is given by

$$\begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}\varphi(u,v) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}\psi(u,v)$$

Where,

$$\varphi(u,v) = \left(1 - \frac{v}{2}\right),\ \psi(u,v) == v(1 - \frac{u}{2})$$

Further using the simple transformation $u = u(\xi) = \frac{1+\xi}{2}, v = v(\eta) = \frac{1+\eta}{2}$,we obtain from

eq(13a-b)

$$\left|\frac{\partial(\varphi,\psi,)}{\partial(u,v)}\right| = 1 - \left(\frac{u+v}{2}\right) = 1 - (\frac{2+\xi+\eta}{4}\ ),\ ,dudv=\frac{1}{4}d\ \xi d\eta, \quad (18a)$$

$$\iint_{\Delta}\ f(x,y)dxdy = \mathbf{2\Delta_{123}} \int_{-1}^{1}\int_{-1}^{1} f\big(x(u,v), y(u,v)\big)(1 - (\frac{2+\xi+\eta}{4}\ )\ )\,\frac{1}{4}d\ \xi d\eta \quad (18b)$$

where

$$\begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}\varphi(u,v) + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}\psi(u,v)$$

$$\varphi(u,v) = \left(1 - \left(\frac{1+\eta}{4}\right)\right)(\frac{1+\xi}{2}), \psi(u,v) = \left(1 - \left(\frac{1+\xi}{4}\right)\right)(\frac{1+\eta}{2}) \quad (18c)$$

Now,the integral of eq(16b) can be approximated to any degree of accuracy by applying the well known Gauss Legendre quadrature rule of order m and n along ξ and η axes respectively.

Thus, we obtain

$$\iint_{\Delta} f(x,y)dxdy$$

$$\sim 2\Delta_{123} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x(u_i^m, v_j^n), y(u_i^m, v_j^n))\,(1-\left(\frac{2+\xi_i^m+\eta_j^n}{4}\right))\frac{1}{4} \qquad \textbf{(19a)}$$

Where,

$$u_i^m = \left(\frac{1+\xi_i^m}{2}\right),\ v_j^n = \left(\frac{1+\eta_j^n}{2}\right), \qquad \textbf{(19b)}$$

and $((\xi_i^m, w_i^m), i = 1,2,\ldots,m)$, $((\eta_j^n, w_j^n), j = 1,2,\ldots,n)$ are the Gauss Legendre

sampling points and weights of order m and n respectively

We may note that in this case the preferred order of the Gauss Legendre quadrature rule is one of m=n,n+1; this is suggestive on seeing the integrand in eq(19b). It may be further noted that for a standard triangle,the transformation rule is given by eq(10e) viz

$\binom{x}{y} = \binom{\varphi(u,v)}{\psi(u,v)}$ and this gives the sampling points

,

$$\begin{pmatrix} x(u_i^m, v_j^n) \\ y(u_i^m, v_j^n) \end{pmatrix} = \begin{pmatrix} \varphi(u_i^m, v_j^n) \\ \psi(u_i^m, v_j^n) \end{pmatrix} ,\text{(i=1,2,....m;j=1,2,....n)}$$

$$= \begin{pmatrix} \left(1-\left(\frac{1+\eta_j^n}{4}\right)\right)\left(\frac{1+\xi_i^m}{2}\right) \\ \left(1-\left(\frac{1+\xi_i^m}{4}\right)\right)\left(\frac{1+\eta_j^n}{2}\right) \end{pmatrix} \qquad (19c)$$

and the corresponding weights

$$w_i^m\, w_j^n\, (1-\left(\frac{2+\xi_i^m+\eta_j^n}{4}\right))\frac{1}{4} ,\text{(i=1,2,....m;j=1,2,....n)} \qquad (19d)$$

Eqs{19c,19d) generate **symmetric** table of values

$$[\varphi(u_i^m, v_j^n), \psi(u_i^m, v_j^n),\ w_i^m\, w_j^n\, (1-\left(\frac{2+\xi_i^m+\eta_j^n}{4}\right))\frac{1}{4}] \qquad (19e)$$

of sampling points and the corresponding weights for the standard triangle

We may note that the third column in eq(19e) contains both variables ξ and η , it may be suggested that the order of Gauss Legendre quadrature are taken as same along both axes,i.e m=n

W e have presented a sample of this for m=n=2 in Appendix-3,these tables consists of a triplet of four values

5. Division of an Arbitrary Triangle

We can map an arbitrary triangle with vertices ((x_i, y_i), $i = 1, 2, 3$) into a right isosceles triangle in the (u, v) space as shown in Fig. 1a, 1b. The necessary transformation is given by the equations.

$$x = x_1 + (x_2 - x_1)p + (x_3 - x_1)q,$$

$$y = y_1 + (y_2 - y_1)p + (y_3 - y_1)q \tag{20}$$

The mapping of eqn.(20) describes a unique relation between the coordinate systems. This is illustrated by using the area coordinates and division of each side into three equal parts in Fig. 2a Fig. 2b. It is clear that all the coordinates of this division can be determined by knowing the coordinates ((x_i, y_i), $i = 1, 2, 3$) of the vertices for the arbitrary triangle. In general , it is well known that by making 'n' equal divisions on all sides and the concept of area coordinates, we can divide an arbitrary triangle into n^2 smaller triangles having the same area which equals Δ/n^2 where Δ is the area of a linear arbitrary triangle with vertices ((x_i, y_i), $i = 1, 2, 3$) in the Cartesian space.

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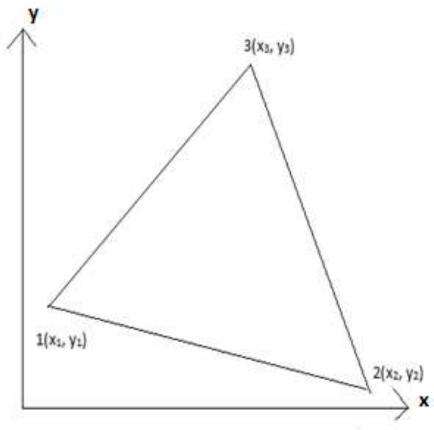

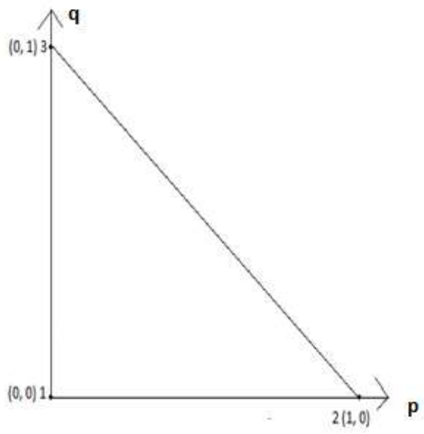

Fig.2a An Arbitrary Linear Triangle in the (x, y) space

Fig.2b A Right Isosceles Triangle in the (p,q) space

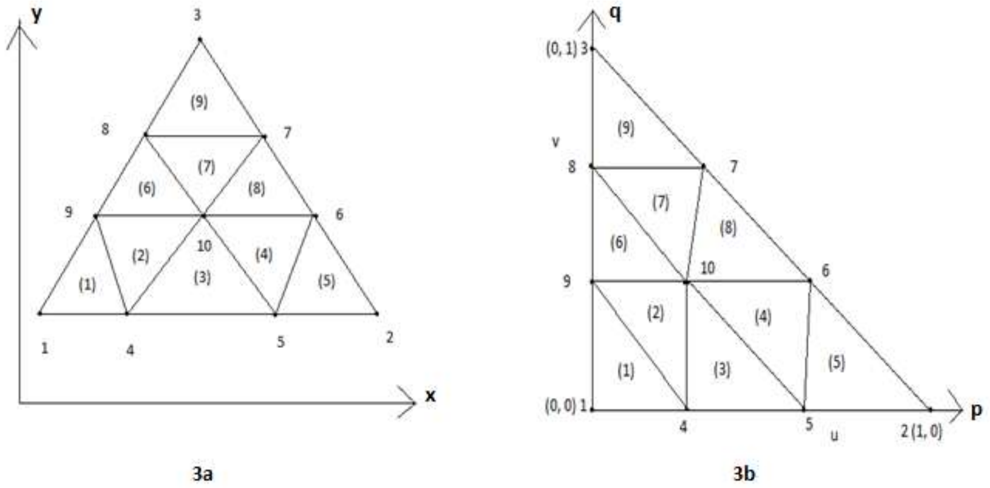

Fig.3a Division of an arbitrary triangle into Nine triangles in Cartesian space (x,y)

Fig.3b Division of a right isosceles triangle into Nine right isosceles triangles in (p,q) space

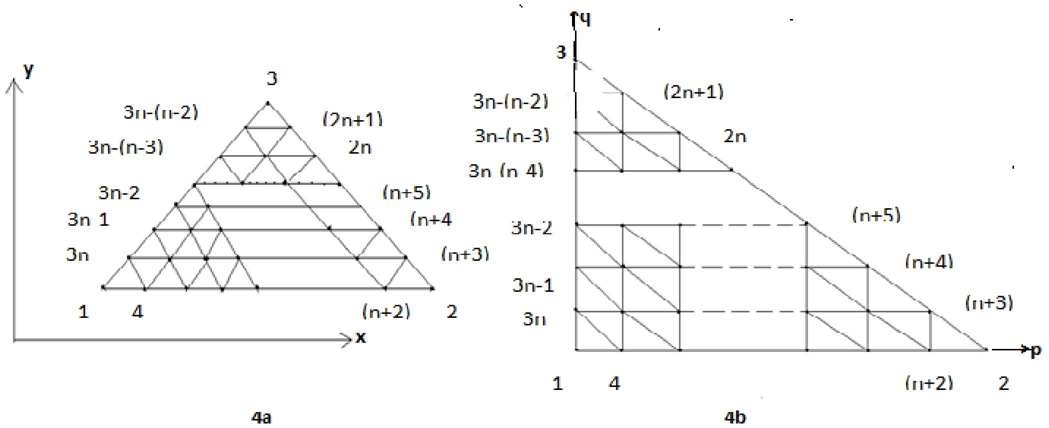

Fig.4a Division of an arbitrary triangle into n^2 triangle in Cartesian space (x, y), where each side is divided into n divisions of equal length

Fig.4b Division of a right isosceles triangle into n^2 right isosceles triangle in (u, v) space, where each side is divided into n divisions of equal length

We have shown the division of an arbitrary triangle in Fig. 4a , Fig. 4b, We divided each side of the triangles (either in Cartesian space or natural space) into n equal parts and draw lines parallel to the sides of the triangles. This creates (n+1) (n+2) nodes. These nodes are numbered from triangle base line l_{12} (letting l_{ij} as the line joining the vertex (x_i, y_i) and (x_j, y_j)) along the line $q = 0$ and upwards up to the line $q = 1$. The nodes 1, 2, 3 are numbered anticlockwise and then nodes 4, 5, ------, (n+2) are along line $q = 0$ and the nodes (n+3), (n+4), ------, 2n, (2n+1) are numbered along the line l_{23} i.e. $p + q = 1$ and then the node (2n+2), (2n+3), -------, 3n are numbered along the line $p = 0$. Then the interior nodes are numbered in increasing order from left to right along the line $q = \frac{1}{n}, \frac{2}{n}, - - -, \frac{n-1}{n}$ bounded on the right by the line $+q = 1$. Thus the entire triangle is covered by (n+1) (n+2)/2 nodes. This is shown in the $\underline{rr}$ matrix of size $(n + 1) \times (n + 1)$, only nonzero entries of this matrix refer to the nodes of the triangles

$$\underline{rr} = \begin{bmatrix} 1, & 4, & 5,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & (n+2) & 2 \\ 3n, & (3n+1),\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, 3n+(n-2), & (n+3) & & 0 \\ 3n-1, 3n+(n-1)\ldots\ldots\ldots & ,3n+(n-2)+(n-3), & (n+4) & 0 & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ 3n-(n-3), & \frac{(n+1)(n+2)}{2}, & 2n & 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 \\ 3n-(n-2), & (2n+1), & 0 & 0\ldots\ldots\ldots\ldots\ldots\ldots & 0 \\ 3 & 0 & 0 & 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 \end{bmatrix}$$

(21)

The nodal coordinates of the standard triangle is given by

coord=

$$\begin{bmatrix} (0,0) & (1/n,0) & (2/n,0)\ \ldots\ldots\ldots\ldots & ((n-2)/n,0) & ((n-1)/n,0) & (1,0) \\ (0,1/n) & (1/n,1/n) & (2/n,1/n)\ \ldots\ldots\ldots\ldots & ((n-2)/n,1/n) & ((n-1)/n,1/n) & \\ (0,2/n) & (1/n,2/n) & (2/n,2/n)\ \ldots\ldots\ldots\ldots & ((n-2)/n,2/n) & & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ . & & & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ (0,(n-2)/n) & (1/n,(n-2)/n) & (2/n\ (n-2)/n) & & & \\ (0,(n-1)/n) & (1/n,(n-1)/n) & & & & \\ (0,1) & & & & & \end{bmatrix}$$

(22)

Thus from the above matrices of eqs(21)-(22),we understand the following:

rr(1,1)=1 and the coordinate for node 1 of the standard triangle is (0,0)

rr(1,2)=4 and the coordinate for node 4 of the standard triangle is $((\frac{1}{n},0)$

rr(1,3)=5 and the coordinate for node 5 of the standard triangle is $((\frac{2}{n},0)$

....

rr(1,n)=(n+2) and the coordinate for node (n+2) of the standard triangle is $((\frac{(n-1)}{n},0)$

rr(1,n +1)=2 and the coordinate for node 2 of the standard triangle is (1,0)

.........etc (23)

We can generate the nodal address matrix and the nodal coordinate and use the in the linear affine transformation given in eq(20) to compute the nodal coordinates of the arbitrary triangle in the Cartesian space.

6 Composite Integration Formulas

In some physical applications,we are required to compute integrals of functions which are expressed in explicit form.In finite element method and boundary element method evaluation of two dimensional integrals over polygonal domains with complicated and explicit functions as integrands is of great importance. This is the subject matter of of several investigations [12 -18] .We now consider the evaluation of the integral

$II_{\Omega_N}(f) = \iint_{\Omega_N} f(x,y)dxdy\,,\;\; where$ Ω_N is a polygonal domain (24) $II_{\Omega_N}(f)$ can be computed as finite sum **o**f linear integrals and this can be expressed as

$$II_{\Omega_N}(f) = \sum_{i=1}^{N} \iint_{\Delta_i} f(x,y)dxdy \quad (25)$$

Where it is assumed that $\Omega_N = \sum_{i=1}^{N} \Delta_i$ *,and* Δ_i *is an arbitrary triangle*

We shall now further propose the composite integration formula for the arbitrary triangle Δ_i.

We can write

$$\iint_{\Delta_i} f(x,y)dxdy = \sum_{j=1}^{n^2} \sum_{e=1}^{4} \iint_{\Delta_{i,j}^{e}} f(x,y)dxdy \quad (26)$$

Where, ith arbitrary triangle *is* Δ_i and ,jth division of Δ_i *is defined as* $\Delta_{i,j}$,

given the vertices of arbitrary triangle $\Delta_{i,j}$, without obtaining any further new mesh, we can express $\Delta_{i,j} = \sum_{e=1}^{4} \Delta_{i,j}^{e}$
Sustituting eq(26) into eq(25),we obtain

$$II_{\Omega_N}(f) = \sum_{i=1}^{N} \sum_{j=1}^{n^2} \sum_{e=1}^{4} \iint_{\Delta_{i,j}^{e}} f(x,y)dxdy \quad (27)$$

Letting $\Delta_{i,j}^{e} = \Delta$, an arbitrary triangle with vertices $((\,x_k\,,\;\, y_k), k = 1,2,3))$ we can express from eqs(13a,13b)

$$\iint_{\Delta_{i,j}{}^{e}} f(x,y)dxdy = \iint_{\Delta} f(x,y)dxdy$$

$$= \mathbf{2}\Delta_{\mathbf{123}} \int_0^1 \int_0^1 f\big(x(u,v), y(u,v)\big) \left|\frac{\partial(\varphi,\psi,)}{\partial(u,v)}\right| du\, dv$$ Where,

$2\Delta_{123} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$=2 X area of the triangle Δ

and the Numerical Integration Schemes of section 4 can be now applied to compute eq(27) as,we have now

$$II_{\Omega_N}(f) = \sum_{i=1}^{N} \sum_{j=1}^{n^2} \sum_{e=1}^{4} \iint_{\Delta} f(x,y)dxdy \quad (28)$$

7. Applications toNumerical Computations

We now consider the integration over polygonal domains.Since the simplest polygons are certainly those having four sides and five sides and they can be assembled as a combination arbitrary triangles,we take up the testing of complicated integrals over these domains which are popularly known as quadrilaterals and pentagons.

7.1 Standard 1-Square OR Unitary Square

We consider the integral*s*

$$\int_0^1 \int_0^1 e^{x+y}\, dxdy \quad (29)$$

and the coarse mesh for the standard 1-square or the unitary square is as shown in Fig.5.We then apply the numerical schemes on numerical integration developed in the previous sections and these results are summarised in Table-1.

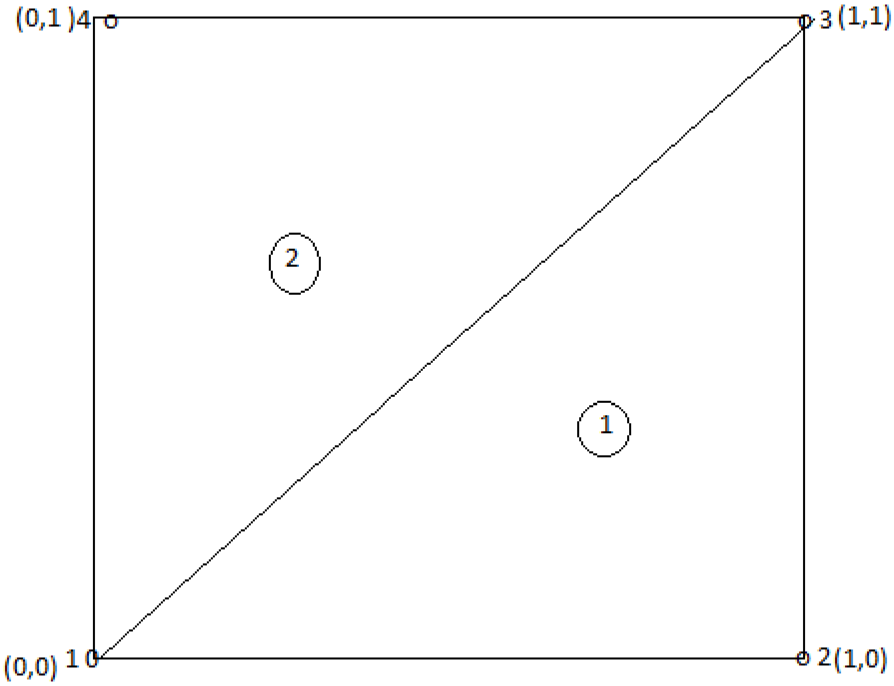

Fig.5 Standard 1-Square OR Unitary Square

7.2 Standard 2-Square

We next consider the integrals

$$\int_0^{\frac{\pi}{4}}\int_0^{\sin(y)}\frac{1}{\sqrt{(1-x^2)}}\,dxdy$$

$$=\int_{-1}^{1}\int_{-1}^{1}\frac{\frac{1}{2}\sin(\frac{\pi}{8}(1+s))\frac{\pi}{8}}{\left\{1-\frac{1}{2}(\sin(\frac{\pi}{8}(1+s))^2(1+t)^2\right\}^{\frac{1}{2}}}\,\text{dsdt} \qquad (30)$$

and the coarse mesh for the standard 2-square is as shown in Fig.6.We then apply the numerical schemes We then apply the numerical schemes on numerical integration developed in the previous sections and these results are are summarised in Table-2.

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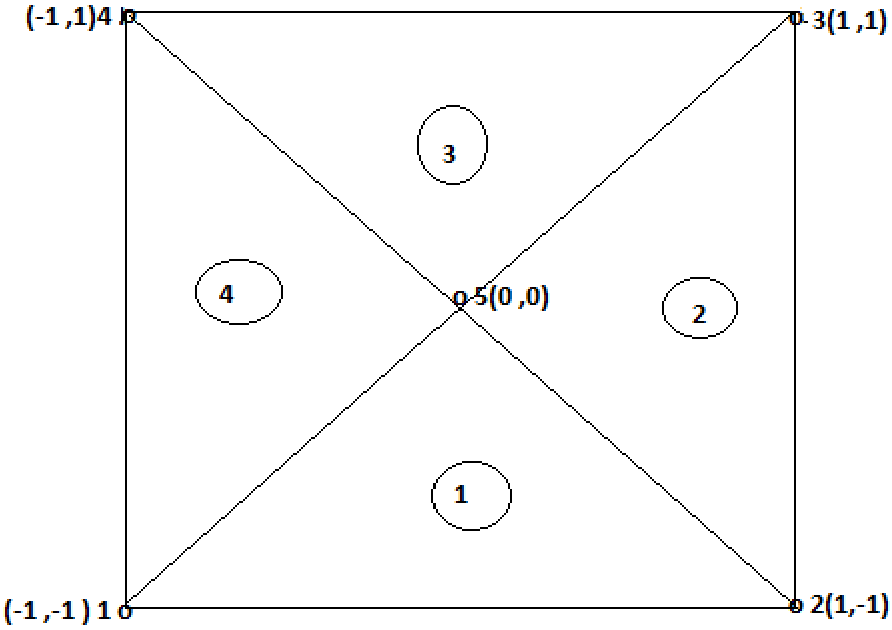

Fig.6 Standard 2-SQUARE

7.3 Pentagon We now consider the pentagonal domain whose vertices are {1(-1,-1), 2(1,-1) ,3(2,0) ,4(0,1) , 5(-2,0).We choose the interior point 6(0,0) and create a coarse mesh of five triangles as shown in **Fig.7**.

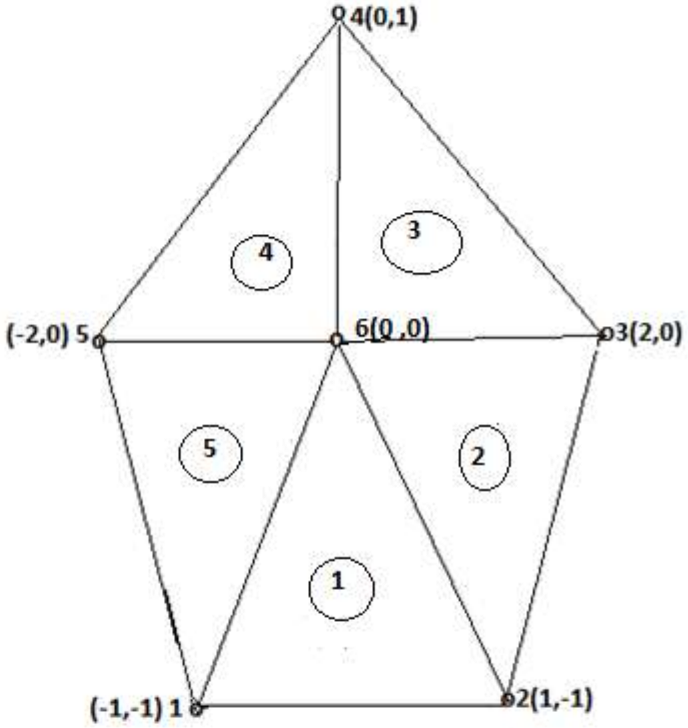

Fig.7 Division of a Pentagon into five triangles

We have demonstrated the computation of some complicated integrals over the Pentagonal domain of Fig.7.These resulted are summarised in Tables,3-11.

8 Conclusions

The purpose of this paper is to develop efficient numerical integration schemes for arbitrary linear polygons in a 2-space which are very useful in finite element method, boundary integration method and mathematical modeling of several phenomena in science and engineering. The present study concentrates on those phenomena which may require integrating arbitrary functions over linear polygons which may be either convex or nonconvex. We can discretise these domains by using either triangles or quadrilaterals. In this paper, we propose to discretise the polygonal domain into a fine mesh of triangles and for the same fine mesh each of these triangles is then divided into four triangle by joining the midpoints of the three sides. The composite integration scheme is developed by discretising the arbitrary triangle into n^2,(n=1,2,3,4,5,……) triangles and then each of these triangles is divided further into four triangles by joining the midpoints of sides. We map each of these arbitrary triangles into a 2-square. Then we apply Gauss Legendre quadrature rules over the 2-square to find sampling points and weight coefficients applicable for the entire polygonal domain. The composite integration scheme is tested on examples of some typical integrals over the 1-square and 2-square domains and the pentagonal domains.The performance of the proposed numerical schemes is very efficient and accurate. The following MATLAB programs are also appended.

(1)coordinates_stdtriangle.m

(2)nodal_address_rtisosceles_triangle.m

(3) glsampleptsweights.m

(4) gausslegendrequadratureNtriangularmesh4compositeintegration.m

The sample output of these programs is given in Tables 1-9 here.We hope that this study will be useful.

APPENDIX

Table-1

$$\int_0^1\int_0^1 e^{x+y}\ dxdy = 2.952492442012559750$$

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),ngpntr=ng*(ng+1)

ng/ntr	8	32	72	128	200
2.	2.952267642088299	2.952492442012561	2.952492442012559	2.952492442012560	2.952492442012558
4.	2.952492441685622	2.952492442012561	2.952492442012559	2.952492442012560	2.952492442012558
6	2.952492442012559	2.952492442012561	2.952492442012559	2.952492442012560	2.952492442012558
8.	2.952492442012559	2.952492442012561	2.952492442012559	2.952492442012560	2.952492442012558
10	2.952492442012561	2.952492442012561	2.952492442012559	2.952492442012560	2.952492442012558

symmetric rule and a mesh of all triangles ref numerical scheme(2)),ngpntr=ng*ng

ng/ntr	8	32	72	128	200
2.	2.952437205621090	2.952492442012559	2.952492442012559	2.952492442012560	2.952492442012558
4.	2.952492442004837	2.952492442012559	2.952492442012559	2.952492442012560	2.952492442012558
6.	2.952492442012560	2.952492442012559	2.952492442012559	2.952492442012560	2.952492442012558
8.	2.952492442012559	2.952492442012559	2.952492442012559	2.952492442012560	2.952492442012558
10	2.952492442012559	2.952492442012559	2.952492442012559	2.952492442012560	2.952492442012558

new symmetric rules and a mesh of all triangles ref numerical scheme(3), ngpntr=ng*ng

ng/ntr	8	32	72	128	200
2	2.952279641419893	2.952492442012560	2.952492442012559	2.952492442012561	2.952492442012558
4	2.952492441719476	2.952492442012560	2.952492442012559	2.952492442012561	2.952492442012558
6	2.952492442012559	2.952492442012560	2.952492442012559	2.952492442012561	2.952492442012558
8	2.952492442012559	2.952492442012560	2.952492442012559	2.952492442012561	2.952492442012558
10	2.952492442012559	2.952492442012560	2.952492442012559	2.952492442012561	2.952492442012558

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4), ngpntr= ng*(ng+1)

ng/ntr	8	32	72	128	200
2.	2.952478263213804	2.952492442012560	2.952492442012558	2.952492442012560	2.952492442012558
4.	2.952492442011263	2.952492442012560	2.952492442012558	2.952492442012560	2.952492442012558
6.	2.952492442012558	2.952492442012560	2.952492442012558	2.952492442012560	2.952492442012558
8.	2.952492442012560	2.952492442012560	2.952492442012558	2.952492442012560	2.952492442012558
10	2.952492442012560	2.952492442012560	2.952492442012558	2.952492442012560	2.952492442012558

Table-2

$$\int_0^{\frac{\pi}{4}}\int_0^{\sin(y)}\frac{1}{\sqrt{(1-x^2)}}\ dxdy = \int_{-1}^{1}\int_{-1}^{1}\frac{\frac{1}{2}\sin(\frac{\pi}{8}(1+s))\frac{\pi}{8}}{\{1-\frac{1}{2}(\sin(\frac{\pi}{8}(1+s))^2(1+t)^2\}^{\frac{1}{2}}}\ \text{dsdt} = 0.30842513753404243$$

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),ngpntr=ng*(ng+1)

ng/ntr	**16**	**64**	**144**	**256**	**400**

ng/ntr	16	64	144	256	400
2	0.308418773541046	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
4.	0.308425135967861	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
6.	0.308425137533608	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
8.	0.308425137534043	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
10	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042

symmetric rule and a mesh of all triangles ref numerical scheme(2), ngpntr=ng*ng

ng/ntr	**16**	**64**	**144**	**256**	**400**
2	0.308456210911457	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
4.	0.308425148832194	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
6.	0.308425137541636	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
8.	0.308425137534048	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042
10	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534042

new symmetric rules and a mesh of all triangles ref numerical scheme(3)), ngpntr=ng*ng

ng/ntr	**16**	**64**	**144**	**256**	**400**
2.	0.308410570454756	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
4.	0.308425133320935	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
6.	0.308425137530986	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
8	0.308425137534040	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
10	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4), ngpntr=ng*(ng+1)

ng/ntr	**16**	**64**	**144**	**256**	**400**
2.	0.308418873626232	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
4.	0.308425135371277	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
6.	0.308425137532562	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
8.	0.308425137534041	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042
10	0.308425137534042	0.308425137534042	0.308425137534042	0.308425137534043	0.308425137534042

Table-3

NUMERICAL VALUES FOR THE INTEGRAL

$$\int\int_P \frac{(x^4+y^3)}{(1+x^2)}dxdy = 1.92403054263265005374705651775$$

Where P=pentagon

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),), ngpntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10.	1.924030542632655	1.924030542632651	1.924030542632648	1.924030542632649	1.924030542632649
20.	1.924030542632650	1.924030542632651	1.924030542632648	1.924030542632649	1.924030542632649
30.	1.924030542632649	1.924030542632651	1.924030542632648	1.924030542632649	1.924030542632649
40.	1.924030542632651	1.924030542632651	1.924030542632648	1.924030542632649	1.924030542632649

symmetric rule and a mesh of all triangles ref numerical scheme(2)), ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10.	1.924030542632634	1.924030542632649	1.924030542632649	1.924030542632650	1.924030542632650
20.	1.924030542632651	1.924030542632649	1.924030542632649	1.924030542632650	1.924030542632650
30.	1.924030542632648	1.924030542632649	1.924030542632649	1.924030542632650	1.924030542632650
40.	1.924030542632646	1.924030542632649	1.924030542632649	1.924030542632650	1.924030542632650

new symmetric rules and a mesh of all triangles ref numerical scheme(3),ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10.	1.924030542632676	1.924030542632650	1.924030542632648	1.924030542632648	1.924030542632649
20.	1.924030542632651	1.924030542632650	1.924030542632648	1.924030542632648	1.924030542632649
30.	1.924030542632651	1.924030542632650	1.924030542632648	1.924030542632648	1.924030542632649
40.	1.924030542632647	1.924030542632650	1.924030542632648	1.924030542632648	1.924030542632649

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4), pntr=ng*(ng+1)

ng/ntr	20	80	180	320	500

10.1.924030542632643	1.924030542632649	1.924030542632648	1.924030542632650	1.924030542632649
20.1.924030542632651	1.924030542632649	1.924030542632648	1.924030542632650	1.924030542632649
30.1.924030542632650	1.924030542632649	1.924030542632648	1.924030542632650	1.924030542632649
40.1.924030542632651	1.924030542632649	1.924030542632648	1.924030542632650	1.924030542632649

Table-4

NUMERICAL VALUES FOR THE INTEGRAL

$\int \int_{P} (1-x)\sin(10xy)\,\boldsymbol{dxdy}$= -0.013103719669957

Where P= pentagon

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),), ngpntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10	-0.013103719673868	-0.013103719669957	-0.013103719669956	-0.013103719669957	-0.013103719669958
20	-0.013103719669957	-0.013103719669957	-0.013103719669956	-0.013103719669957	-0.013103719669958
30	-0.013103719669958	-0.013103719669957	-0.013103719669956	-0.013103719669957	-0.013103719669958
40	-0.013103719669956	-0.013103719669957	-0.013103719669956	-0.013103719669957	-0.013103719669958

symmetric rule and a mesh of all triangles ref numerical scheme(2)), ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10.	-0.013103719630136	-0.013103719669957	-0.013103719669957	-0.013103719669956	-0.013103719669957
20.	-0.013103719669957	-0.013103719669957	-0.013103719669957	-0.013103719669956	-0.013103719669957
30.	-0.013103719669955	-0.013103719669957	-0.013103719669957	-0.013103719669956	-0.013103719669957
40.	-0.013103719669957	-0.013103719669957	-0.013103719669957	-0.013103719669956	-0.013103719669957

new symmetric rules and a mesh of all triangles ref numerical scheme(3),ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10.	-0.013103719669414	-0.013103719669958	-0.013103719669957	-0.013103719669957	-0.013103719669958
20.	-0.013103719669958	-0.013103719669958	-0.013103719669957	-0.013103719669957	-0.013103719669958
30.	-0.013103719669956	-0.013103719669958	-0.013103719669957	-0.013103719669957	-0.013103719669958
40.	-0.013103719669955	-0.013103719669958	-0.013103719669957	-0.013103719669957	-0.013103719669958

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4),pntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10	-0.013103719642995	-0.013103719669957	-0.013103719669957	-0.013103719669957	-0.013103719669957
20	-0.013103719669958	-0.013103719669957	-0.013103719669957	-0.013103719669957	-0.013103719669957
30	-0.013103719669958	-0.013103719669957	-0.013103719669957	-0.013103719669957	-0.013103719669957
40	-0.013103719669957	-0.013103719669957	-0.013103719669957	-0.013103719669957	-0.013103719669957

Table-5

NUMERICAL VALUES FOR THE INTEGRAL

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$\int \int_{\boldsymbol{P}} (0,2x + 0.3y)^{19} \, \boldsymbol{dxdy}$ = -0.00000005569067891985714285714285714285 Where P=pentagon

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1)

ng/ntr	20	80	180	320	500
10	-5.569067891985712e-8	-5.569067891985710e-8	-5.569067891985704e-8	-5.569067891985715e-8	-5.569067891985714e-8
20	-5.569067891985716e-8	-5.569067891985710e-8	-5.569067891985704e-8	-5.569067891985715e-8	-5.569067891985714e-8
30	-5.569067891985716e-8	-5.569067891985710e-8	-5.569067891985704e-8	-5.569067891985715e-8	-5.569067891985714e-8
40	-5.569067891985700e-8	-5.569067891985710e-8	-5.569067891985704e-8	-5.569067891985715e-8	-5.569067891985714e-8

symmetric rule and a mesh of all triangles ref numerical scheme(2)), ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10	-5.569067891985717e-8	-5.569067891985714e-8	-5.569067891985702e-8	-5.569067891985716e-8	-5.569067891985708e-8
20	-5.569067891985718e-8	-5.569067891985714e-8	-5.569067891985702e-8	-5.569067891985716e-8	-5.569067891985708e-8
30	-5.569067891985711e-8	-5.569067891985714e-8	-5.569067891985702e-8	-5.569067891985716e-8	-5.569067891985708e-8
40	-5.569067891985717e-8	-5.569067891985714e-8	-5.569067891985702e-8	-5.569067891985716e-8	-5.569067891985708e-8

new symmetric rules and a mesh of all triangles ref numerical scheme(3),ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10	-5.569067891985718e-8	-5.569067891985716e-8	-5.569067891985710e-8	-5.569067891985718e-8	-5.569067891985716e-8
20	-5.569067891985722e-8	-5.569067891985716e-8	-5.569067891985710e-8	-5.569067891985718e-8	-5.569067891985716e-8
30	-5.569067891985717e-8	-5.569067891985716e-8	-5.569067891985710e-8	-5.569067891985718e-8	-5.569067891985716e-8
40	-5.569067891985703e-8	-5.569067891985716e-8	-5.569067891985710e-8	-5.569067891985718e-8	-5.569067891985716e-8

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4), pntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10	-5.569067891985715e-8	-5.569067891985716e-8	-5.569067891985709e-8	-5.569067891985716e-8	-5.569067891985718e-8
20	-5.569067891985717e-8	-5.569067891985716e-8	-5.569067891985709e-8	-5.569067891985716e-8	-5.569067891985718e-8
30	-5.569067891985712e-8	-5.569067891985716e-8	-5.569067891985709e-8	-5.569067891985716e-8	-5.569067891985718e-8
40	-5.569067891985712e-8	-5.569067891985716e-8	-5.569067891985709e-8	-5.569067891985716e-8	-5.569067891985718e-8

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Table-6

NUMERICAL VALUES FOR THE INTEGRAL

$\int \int_P (0,17x+0.25y)^{25}\,dxdy$**=-0.0000000000069990091267998353953694168811778**

Where P=pentagon

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),), ngpntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10	-6.999009126799026e-12	-6.999009126799839e-12	-6.999009126799825e-12	-6.999009126799838e-12	-6.999009126799839e-12
20	-6.999009126799838e-12	-6.999009126799839e-12	-6.999009126799825e-12	-6.999009126799838e-12	-6.999009126799839e-12
30	-6.999009126799839e-12	-6.999009126799839e-12	-6.999009126799825e-12	-6.999009126799838e-12	-6.999009126799839e-12
40	-6.999009126799848e-12	-6.999009126799839e-12	-6.999009126799825e-12	-6.999009126799838e-12	-6.999009126799839e-12

symmetric rule and a mesh of all triangles ref numerical scheme(2)), ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10	-6.999009126783056e-12	-6.999009126799843e-12	-6.999009126799826e-12	-6.999009126799845e-12	-6.999009126799856e-12
20	-6.999009126799845e-12	-6.999009126799843e-12	-6.999009126799826e-12	-6.999009126799845e-12	-6.999009126799856e-12
30	-6.999009126799846e-12	-6.999009126799843e-12	-6.999009126799826e-12	-6.999009126799845e-12	-6.999009126799856e-12
40	-6.999009126799851e-12	-6.999009126799843e-12	-6.999009126799826e-12	-6.999009126799845e-12	-6.999009126799856e-12

new symmetric rules and a mesh of all triangles ref numerical scheme(3),ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10	-6.999009126800596e-12	-6.999009126799841e-12	-6.999009126799833e-12	-6.999009126799846e-12	-6.999009126799838e-12
20	-6.999009126799841e-12	-6.999009126799841e-12	-6.999009126799833e-12	-6.999009126799846e-012	-6.999009126799838e-12
30	-6.999009126799842e-12	-6.999009126799841e-12	-6.999009126799833e-12	-6.999009126799846e-12	-6.999009126799838e-12
40	-6.999009126799838e-12	-6.999009126799841e-12	-6.999009126799833e-12	-6.999009126799846e-12	-6.999009126799838e-12

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4), pntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10	-6.999009126788399e-12	-6.999009126799848e-12	-6.999009126799827e-12	-6.999009126799832e-12	-6.999009126799834e-12
20	-6.999009126799843e-12	-6.999009126799848e-12	-6.999009126799827e-12	-6.999009126799832e-12	-6.999009126799834e-12
30	-6.999009126799842e-12	-6.999009126799848e-12	-6.999009126799827e-12	-6.999009126799832e-12	-6.999009126799834e-12
40	-6.999009126799833e-12	-6.999009126799848e-12	-6.999009126799827e-12	-6.999009126799832e-12	-6.999009126799834e-12

Table-7

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NUMERICAL VALUES FOR THE INTEGRAL

$\int \int_P cos(30(x+y))\, dxdy =$ -0.0074361772990243732198584488994964

Where P=pentagon

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),), ngpntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10	-1.752711593146233e-2	-7.436177299024268e-3	-7.436177299024170e-3	-7.436177299024323e-3	-7.436177299024713e-3
20	-7.436177449259440e-3	-7.436177299024268e-3	-7.436177299024170e-3	-7.436177299024323e-3	-7.436177299024713e-3
30	-7.436177299024501e-3	-7.436177299024268e-3	-7.436177299024170e-3	-7.436177299024323e-3	-7.436177299024713e-3
40	-7.436177299024327e-3	-7.436177299024268e-3	-7.436177299024170e-3	-7.436177299024323e-3	-7.436177299024713e-3

symmetric rule and a mesh of all triangles ref numerical scheme(2)), ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10	-5.072720900308940e-2	-7.436177299024448e-3	-7.436177299024491e-3	-7.436177299024358e-3	-7.436177299024621e-3
20	-7.436179524196942e-3	-7.436177299024448e-3	-7.436177299024491e-3	-7.436177299024358e-3	-7.436177299024621e-3
30	-7.436177299024598e-3	-7.436177299024448e-3	-7.436177299024491e-3	-7.436177299024358e-3	-7.436177299024621e-3
40	-7.436177299024516e-3	-7.436177299024448e-3	-7.436177299024491e-3	-7.436177299024358e-3	-7.436177299024621e-3

new symmetric rules and a mesh of all triangles ref numerical scheme(3), ngpntr=ng*ng

ng/ntr	20	80	180	320	500
10	-6.186384659565267e-3	-7.436177299024377e-3	-7.436177299024252e-3	-7.436177299024377e-3	-7.436177299024292e-3
20	-7.436177299019797e-3	-7.436177299024377e-3	-7.436177299024252e-3	-7.436177299024377e-3	-7.436177299024292e-3
30	-7.436177299024342e-3	-7.436177299024377e-3	-7.436177299024252e-3	-7.436177299024377e-3	-7.436177299024292e-3
40	-7.436177299024326e-3	-7.436177299024377e-3	-7.436177299024252e-3	-7.436177299024377e-3	-7.436177299024292e-3

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4), pntr=ng*(ng+1)

ng/ntr	20	80	180	320	500
10	-4.027489894207207e-2	-7.436177299024219e-3	-7.436177299023990e-3	-7.436177299024158e-3	-7.436177299024509e-3
20	-7.436179524132867e-3	-7.436177299024219e-3	-7.436177299023990e-3	-7.436177299024158e-3	-7.436177299024509e-3
30	-7.436177299024296e-3	-7.436177299024219e-3	-7.436177299023990e-3	-7.436177299024158e-3	-7.436177299024509e-3
40	-7.436177299024185e-3	-7.436177299024219e-3	-7.436177299023990e-3	-7.436177299024158e-3	-7.436177299024509e-3

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Table-8

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NUMERICAL VALUES FOR THE INTEGRAL

$\int \int_P \sqrt{((x-0.5)^2+(y-0.5)^2)}\, dxdy$=**5.8095345948989937115376443744064**

Where P=pentagon

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),), ngpntr=ng*(ng+1)

ng/ntr	7 20	9 80	1280	1620	2000
10	5.809534593192251	5.809534585261450	5.809534594898721	5.809534590364454	5.809534594898848
20	5.809534594863271	5.809534585261450	5.809534594898721	5.809534590364454	5.809534594898848
30	5.809534594895498	5.809534585261450	5.809534594898721	5.809534590364454	5.809534594898848
40	5.809534594898334	5.809534585261450	5.809534594898721	5.809534590364454	5.809534594898848

symmetric rule and a mesh of all triangles ref numerical scheme(2)), ngpntr=ng*ng

ng/ntr	7 20	9 80	1280	1620	2000
10	5.809534593467319	5.809534595592756	5.809534594898802	5.809534595225411	5.809534594898893
20	5.809534594872111	5.809534595592756	5.809534594898802	5.809534595225411	5.809534594898893
30	5.809534594896493	5.809534595592756	5.809534594898802	5.809534595225411	5.809534594898893
40	5.809534594898536	5.809534595592756	5.809534594898802	5.809534595225411	5.809534594898893

new symmetric rules and a mesh of all triangles ref numerical scheme(3),ngpntr=ng*ng

ng/ntr	720	9 80	1280	1620	2000
10	5.809534593244261	5.809534599323652	5.809534594898790	5.809534596980826	5.809534594898881
20	5.809534594869717	5.809534599323652	5.809534594898790	5.809534596980826	5.809534594898881
30	5.809534594896302	5.809534599323652	5.809534594898790	5.809534596980826	5.809534594898881
40	5.809534594898502	5.809534599323652	5.809534594898790	5.809534596980826	5.809534594898881

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4),pntr=ng*(ng+1)

ng/ntr	720	9 80	1280	1620	2000
10	5.809534593593015	5.809534594037160	5.809534594898811	5.809534594493496	5.809534594898893
20	5.809534594873428	5.809534594037160	5.809534594898811	5.809534594493496	5.809534594898893
30	5.809534594896579	5.809534594037160	5.809534594898811	5.809534594493496	5.809534594898893
40	5.809534594898546	5.809534594037160	5.809534594898811	5.809534594493496	5.809534594898893

Table-9

NUMERICAL VALUES FOR THE INTEGRAL

$\int \int_P e^{-((x-\frac{1}{2})^2+(y-\frac{1}{2})^2)} dxdy$=1.7291752918422164454108867422149

Where P=pentagon

ng=order o f gauss legendre rule,ntr=number of triangles, ngpntr= number of gauss points per triangle

asymmetric rule and a mesh of all triangles ref numerical scheme(1),), ngpntr=ng*(ng+1)

ng/ntr	7 20	9 80	1280	1620	2000
10	1.729175291842217	1.729175291842216	1.729175291842215	1.729175291842215	1.729175291842215
20	1.729175291842217	1.729175291842216	1.72917529184221	1.729175291842215	1.729175291842215
30	1.729175291842217	1.729175291842216	1.729175291842215	1.729175291842215	1.729175291842215
40	1.729175291842210	1.729175291842216	1.729175291842215	1.729175291842215	1.729175291842215

symmetric rule and a mesh of all triangles ref numerical scheme(2)), ngpntr=ng*ng

ng/ntr	7 20	9 80	1280	1620	2000
10	1.729175291842217	1.729175291842217	1.729175291842214	1.729175291842215	1.729175291842216
20	1.729175291842216	1.729175291842217	1.729175291842214	1.729175291842215	1.729175291842216
30	1.729175291842216	1.729175291842217	1.729175291842214	1.729175291842215	1.729175291842216
40	1.729175291842216	1.729175291842217	1.729175291842214	1.729175291842215	1.729175291842216

new symmetric rules and a mesh of all triangles ref numerical scheme(3),ngpntr=ng*ng

ng/ntr	720	9 80	1280	1620	2000
10	1.729175291842217	1.729175291842216	1.729175291842216	1.729175291842215	1.729175291842215
20	1.729175291842217	1.729175291842216	1.729175291842216	1.729175291842215	1.729175291842215
30	1.729175291842216	1.729175291842216	1.729175291842216	1.729175291842215	1.729175291842215
40	1.729175291842216	1.729175291842216	1.729175291842216	1.729175291842215	1.729175291842215

symmetric gauss legendre rules and a mesh of all triangles ref numerical scheme(4), pntr=ng*(ng+1)

ng/ntr	720	9 80	1280	1620	2000
10	1.729175291842217	1.729175291842216	1.729175291842215	1.729175291842215	1.729175291842215
20	1.729175291842217	1.729175291842216	1.729175291842215	1.729175291842215	1.729175291842215
30	1.729175291842217	1.729175291842216	1.729175291842215	1.729175291842215	1.729175291842215
40	1.729175291842215	1.729175291842216	1.729175291842215	1.729175291842215	1.729175291842215

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COMPUTER PROGRAMS

(1)PROGRAM: coordinates_stdtriangle

```
function[ui,vi,wi]=coordinates_stdtriangle(n)
%divides the standard triangle into n^2 right isoscles triangles
% each of side length 1/n
syms ui vi wi table
%corner nodes
ui=sym([0;1;0]);
vi=sym([0;0;1]);
wi=sym([1;0;0]);
%nodes along v=0
if (n-1)>0
k1=3;
for i1=1:n-1
    k1=k1+1;
    ui(k1,1)=sym(i1/n);
    vi(k1,1)=sym(0);
    wi(k1,1)=sym(1-ui(k1,1));
end
%nodes along w=0
k2=k1;
for i2=1:n-1
    k2=k2+1;
    ui(k2,1)=sym((n-i2)/n);
    vi(k2,1)=sym(1-ui(k2,1));
    wi(k2,1)=0;
end
%nodes along u=0
k3=k2;
for i3=1:n-1
    k3=k3+1;
    wi(k3,1)=sym(i3/n);
     vi(k3,1)=sym(1-wi(k3,1));
     ui(k3,1)=sym(0);
 end
end
 if (n-2)>0
 k4=k3;
```

```
   for i4=1:(n-2)
       for j4=1:(n-1)-i4
           k4=k4+1;
           ui(k4,1)=sym(j4/n);
           vi(k4,1)=sym(i4/n);
           wi(k4,1)=sym(1-ui(k4,1)-vi(k4,1));
       end
   end

 end
```

(2) PROGRAM: nodal_address_rtisosceles_triangle.m

```
function[mst_tri]=nodal_address_rtisosceles_triangle(n)
syms mst_tri x
%disp('left edge')
nni=1;
for i=0:(n-2)
    nni=nni+(n-i)+1;
  %disp([nni 3*n-i])
end
%disp('right edge')
nni=n+1;
for i=0:(n-2)
    nni=nni+n-i;
    %disp([nni n+3+i])
end
%disp('interior nodes')
nni=1;jj=0;
for i=0:n-3
    nni=nni+(n-i)+1;
    for j=1:n-2-i
        jj=jj+1;
        nnj=nni+j;
        %disp([nnj 3*n+jj])
    end
end
%disp('triangle  nodal vertices')
elm(1,1)=1;elm(n+1,1)=2;elm((n+1)*(n+2)/2)=3;
%disp('triangle base nodes')
kk=3;
for k=2:n
    kk=kk+1;
    elm(k,1)=kk;
end
%disp('left edge nodes')
nni=1;
for i=0:(n-2)
    nni=nni+(n-i)+1;
    elm(nni,1)=3*n-i;
end
%disp('right edge nodes')
nni=n+1;
for i=0:(n-2)
    nni=nni+n-i;
  elm(nni,1)= n+3+i;
end
%disp('interior nodes')
nni=1;jj=0;
for i=0:n-3
    nni=nni+(n-i)+1;
    for j=1:n-2-i
```

```
        jj=jj+1;
        nnj=nni+j;
        elm(nnj,1)= 3*n+jj;
    end
end
jj=0;kk=0;
for j=0:n-1
    jj=j+1;
    for k=1:(n+1)-j
        kk=kk+1;
        row_nodes(jj,k)=elm(kk,1);
    end
end
row_nodes(n+1,1)=3;
for jj=(n+1):-1:1
%disp(row_nodes(jj,:))
end
kk=0;
for i=1:n
    for k=1:(n+1)-i
        kk=kk+1;
        mst_tri(kk,1)=row_nodes(i,k);
        mst_tri(kk,2)=row_nodes(i,k+1);
        mst_tri(kk,3)=row_nodes(i+1,k);
        %mst_tri(kk,4)=x;
    end
       for k=1:(n)-i
        kk=kk+1;
        mst_tri(kk,1)=row_nodes(i+1,k+1);
        mst_tri(kk,2)=row_nodes(i+1,k);
        mst_tri(kk,3)=row_nodes(i,k+1);
        %mst_tri(kk,4)=x;
    end


end%for i
```

(3) PROGRAM: glsampleptsweights.m

```
function [s,wwww]=glsampleptsweights(n)
% n must be in multiples of 5,i.e.5,10,15,20,25,30,35,40
switch n
   case 5
   table=[                          0,  .5688888888888888888888888888889
          .9061798459386639927976268782994 0,  .2369268850561890875142640407 2021
```

```
function [s,wwww]=glsampleptsweights(n)
% n must be in multiples of 5,i.e.5,10,15,20,25,30,35,40
switch n
   case 5
   table=[                          0,  .56888888888888888888888888888889
          .90617984593866399279762687829940,  .23692688505618908751426404072021
         -.90617984593866399279762687829940,  .23692688505618908751426404072021
          .53846931010568309103631442070020,  .47862867049936646804129151483563
         -.53846931010568309103631442070020,  .47862867049936646804129151483563];
      s=table(:,1);wwww=table(:,2);
  case 10
 table=[ -.14887433898163121088482600112972,   .29552422471475287017389299465132
          .14887433898163121088482600112972,   .29552422471475287017389299465132
         -.43339539412924719079926594316579,  .26926671930999635509122692156937
           .43339539412924719079926594316579, .26926671930999635509122692156937
          -.67940956829902440623432736511485, .21908636251598204399553493422951
           .67940956829902440623432736511485, .21908636251598204399553493422951
          -.86506336668898451073209668842350, .14945134915058059314577633965488
           .86506336668898451073209668842350, .14945134915058059314577633965488
```

```
            .973906528517171720077964012084452,.66671344308688137593568809896211e-1
            .973906528517171720077964012084452,.66671344308688137593568809896211e-1];
            s=table(:,1);wwww=table(:,2);
case 15
table=[                                    0,     .2025782419255612728806201999675 2
       -.2011940939974345223006283033946 0,   .1984314853271115764561183264438 5
        .2011940939974345223006283033946 0,   .1984314853271115764561183264438 5
       -.3941513470775633698972073709810 5,   .1861610000155622110268005618658 2
        .3941513470775633698972073709810 5,   .1861610000155622110268005618658 2
       -.5709721726085388475372267372539 0,   .1662692058169939335532008604812 9
        .5709721726085388475372267372539 0,   .1662692058169939335532008604812 9
       -.7244177313601700474161860546139 5,   .1395706779261543144478047945708 1
        .7244177313601700474161860546139 5,   .1395706779261543144478047945708 1
       -.8482065834104272162006483207742 0,   .1071592204671719350118695467336 8
        .8482065834104272162006483207742 0,   .1071592204671719350118695467336 8
       -.9372733924007059043077589477102 0, .7036604748810812470926741696942 1e-1
        .9372733924007059043077589477102 0, .7036604748810812470926741696942 1e-1
       -.9879925180204854284895657185866 0, .3075324199611726835462839306665 6e-1
        .9879925180204854284895657185866 0, .3075324199611726835462839306665 6e-1];
     s=table(:,1);wwww=table(:,2);
     case 20
 table=[ -.7652652113349733375464040939884 0e-1, .1527533871307258506980843319551 1
        .7652652113349733375464040939884 0e-1,    .1527533871307258506980843319551 1
         -.2277858511416450780804961953685 7,    .1491729864726037467878287370018 3
          .2277858511416450780804961953685 7,    .1491729864726037467878287370018 3
         -.3737060887154195606725481770249 3,    .1420961093183820513292983250617 9
          .3737060887154195606725481770249 3,    .1420961093183820513292983250617 9
         -.5108670019508270980043640509552 5,    .1316886384491766268984944997469 2
          .5108670019508270980043640509552 5,    .1316886384491766268984944997469 2
         -.6360536807265150254528366962263 0,    .1181945319615184173123773777456 0
          .6360536807265150254528366962263 0,    .1181945319615184173123773777456 0
         -.7463319064601507926143050703556 5,    .1019301198172404350367501359101 2
          .7463319064601507926143050703556 5,    .1019301198172404350367501359101 2
         -.8391169718222188233945290617015 0,  .8327674157670474872475814934451 0e-1
          .8391169718222188233945290617015 0,  .8327674157670474872475814934451 0e-1
         -.9122344282513259058677524412033 0,  .6267204833410906356950653237720 6e-1
          .9122344282513259058677524412033 0,  .6267204833410906356950653237720 6e-1
         -.9639719272779137912676661311973 0,  .4060142980038694133103995650622 8e-1
          .9639719272779137912676661311973 0,  .4060142980038694133103995650622 8e-1
         -.9931285991850949247861223884713 0,  .1761400713915211831186197612275 1e-1
          .9931285991850949247861223884713 0,  .1761400713915211831186197612275 1e-1];
         s=table(:,1);wwww=table(:,2);
       case 25
table=[                                    0,     .1231760537267154562717493130652 5
     -.1228646926107103963873598188080 4,   .1222424429903100467183943058252 5
      .1228646926107103963873598188080 4,   .1222424429903100467183943058252 5
     -.2438668837209884320451903627974 5,   .1194557635357847771429602424740 0
      .2438668837209884320451903627974 5,   .1194557635357847771429602424740 0
     -.3611723058093878377358217301276 3,   .1148582591457116530649520138798 7
      .3611723058093878377358217301276 3,   .1148582591457116530649520138798 7
     -.4730027314457149605221821150091 9,   .1085196244742636575809292550594 1
      .4730027314457149605221821150091 9,   .1085196244742636575809292550594 1
     -.5776629302412229677236898416126 5,   .1005359490670506483385689204289 2
```

```
.57766293024122296772368984161265,   .10053594906705064833856892042892
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.67356636847346836448512063324760, .91028261982963653556683392020088e-1
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.75925926303735763057728286520435, .80140700335001021310472938554589e-1
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-.89499199787827536885104200678280, .54904695975835194184887346675705e-1
.89499199787827536885104200678280, .54904695975835194184887346675705e-1
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.94297457122897433941401116965845, .40939156701306314339986626129424e-1
-.97666392145951751149831538647960, .26354986615032138346226658418155e-1
.97666392145951751149831538647960, .26354986615032138346226658418155e-1
-.99555696979049809790878494689390, .11393798501026288416680150938575e-1
.99555696979049809790878494689390, .11393798501026288416680150938575e-1];
s=table(:,1);wwww=table(:,2);
        case 30
table=[ -.51471842555317695833025213166720e-1, .10285265289355882523892690167866
.51471842555317695833025213166720e-1,  .10285265289355882523892690167866
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.15386991360858354696379467274326,    .10176238974840548965415889420028
-.25463692616788984643980512981781,  .99593420586795252438990588447572e-1
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.53662414814201989926416979331110,  .86899787201082967042466189352565e-1
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.69785049479331579693229238802665,  .73755974737705195438293294379393e-1
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.88256053579205268154311646253025,  .48402672830594045795742219106848e-1
-.92620004742927432587932427708045,  .38799192569627043899699995479347e-1
.92620004742927432587932427708045,  .38799192569627043899699995479347e-1
-.96002186496830751221687102558180,  .28784707883323365123125596530187e-1
.96002186496830751221687102558180,  .28784707883323365123125596530187e-1
-.98366812327974720997003258160565,  .18466468311090956430751785735094e-1
.98366812327974720997003258160565,  .18466468311090956430751785735094e-1
-.99689348407464954027163005091870,  .79681924961666044453614922773122e-2
.99689348407464954027163005091870, .79681924961666044453614922773122e-2];
s=table(:,1);wwww=table(:,2);
case 35
table=[                                    0,  .88486794907104275070900994063430e-1
-.88371343275659263600929433497550e-1,  .88140530430275447464350036781980e-1
.88371343275659263600929433497550e-1,  .88140530430275447464350036781980e-1
-.17605106116598956997430365644506,  .87104446997183518919209365035061e-1
```

```
    .17605106116598956997430365644506,  .87104446997183518919209365035061e-1
   -.26235294120929605797089520045558,  .85386653392099110204039344049953e-1
    .26235294120929605797089520045558,  .85386653392099110204039344049953e-1
   -.34660155443081394587697983493024,  .83000593728856573777966309984 93e-1
    .34660155443081394587697983493024,  .83000593728856573777966309984 93e-1
   -.42813754151781425418762061300147,  .79964942242324248864588331504507e-1
    .42813754151781425418762061300147,  .79964942242324248864588331504507e-1
   -.50632277324148861502429755583735,  .76303457155442040114742587402338e-1
    .50632277324148861502429755583735,  .76303457155442040114742587402338e-1
   -.58054534474976450993450200818970,  .72044794772560051990764092540863e-1
    .58054534474976450993450200818970,  .72044794772560051990764092540863e-1
   -.65022436466589038867579280898455,  .67222285269086892138021711709746e-1
    .65022436466589038867579280898455,  .67222285269086892138021711709746e-1
   -.71481450155662878326440863122445,  .61873671966080178001707977433193e-1
    .71481450155662878326440863122445,  .61873671966080178001707977433193e-1
   -.77381025228691255526742300920990,  .56040816212370118719175968189372e-1
    .77381025228691255526742300920990,  .56040816212370118719175968189372e-1
   -.82674989909222540683405061274855,  .49769370401353521049632989390025e-1
    .82674989909222540683405061274855,  .49769370401353521049632989390025e-1
   -.87321912502522233152328234914140,  .43108422326170211198090105655970e-1
    .87321912502522233152328234914140,  .43108422326170211198090105655970e-1
   -.91285426135931761446493706355575,  .36110115863463374181090136897543e-1
    .91285426135931761446493706355575,  .36110115863463374181090136897543e-1
   -.94534514820782732953872598553000,  .28829260108894248977968174223735e-1
    .94534514820782732953872598553000,  .28829260108894248977968174223735e-1
   -.97043761603922983321507048258475,  .21322979911483577132043584500603e-1
    .97043761603922983321507048258475,  .21322979911483577132043584500603e-1
   -.98793576444385149803511708918550,  .13650828348361489865321570092698e-1
    .98793576444385149803511708918550,  .13650828348361489865321570092698e-1
   -.99770656909960029726016313931210,  .58834334204430839474263690978547e-2
    .99770656909960029726016313931210,  .58834334204430839474263690978547e-2];
 s=table(:,1);wwww=table(:,2);
case 40
 table=[ -.38772417506050821933193444024624e-1,  .77505947978424796396831052966037e-1
  .38772417506050821933193444024624e-1,  .77505947978424796396831052966037e-1
   -.11608407067525520848345128440802,  .77039818164247950810825852852202e-1
    .11608407067525520848345128440802,  .77039818164247950810825852852202e-1
   -.19269758070137109971551685206515,  .76110361900626227772361120968616e-1
    .19269758070137109971551685206515,  .76110361900626227772361120968616e-1
   -.26815218500725368114118434480860,  .74723169057968249867078378271926e-1
    .26815218500725368114118434480860,  .74723169057968249867078378271926e-1
   -.34199409082575847300749248117920,  .72886582395804045079686716752540e-1
    .34199409082575847300749248117920,  .72886582395804045079686716752540e-1
   -.41377920437160500152487974580371,  .70611647391286766151028945995058e-1
    .41377920437160500152487974580371,  .70611647391286766151028945995058e-1
   -.48307580168617871290856657424482,  .67912045815233890799062536542532e-1
    .48307580168617871290856657424482,  .67912045815233890799062536542532e-1
   -.54946712509512820207593130552950,  .64804013456601025644096819401890e-1
    .54946712509512820207593130552950,  .64804013456601025644096819401890e-1
   -.61255388966798023795261245023070,  .61306242492928927407006960905878e-1
    .61255388966798023795261245023070,  .61306242492928927407006960905878e-1
   -.67195668461417954837935451496150,  .57439769099391540348879646916123e-1
    .67195668461417954837935451496150,  .57439769099391540348879646916123e-1
```

```
-.72731825518992710328099645175495, .53227846983936814145029242551666e-1
 .72731825518992710328099645175495, .53227846983936814145029242551666e-1
-.77830565142651938769497154550650, .48695807635072232721976504885192e-1
 .77830565142651938769497154550650, .48695807635072232721976504885192e-1
-.82461223083331166319632023066610, .43870908185673263581207213110889e-1
 .82461223083331166319632023066610, .43870908185673263581207213110889e-1
-.86595950321225950382078180835460, .38782167974472010160626968740686e-1
 .86595950321225950382078180835460, .38782167974472010160626968740686e-1
-.90209880696887429672825333086850, .33460195282547841077564588554743e-1
 .90209880696887429672825333086850, .33460195282547841077564588554743e-1
-.93281280827867653336085216684520, .27937006980023395834739366432647e-1
 .93281280827867653336085216684520, .27937006980023395834739366432647e-1
-.95791681921379165580454099945275, .22245849194166952990288087942050e-1
 .95791681921379165580454099945275, .22245849194166952990288087942050e-1
-.97725994998377426266337028371290, .16421058381907885512705412731658e-1
 .97725994998377426266337028371290, .16421058381907885512705412731658e-1
-.99072623869945700645305435222135, .10498284531152811265876265047737e-1
 .99072623869945700645305435222135, .10498284531152811265876265047737e-1
-.99823770971055920034962270242060, .45212770985331899801771377849908e-2
 .99823770971055920034962270242060, .45212770985331899801771377849908e-2];
  s=table(:,1);wwww=table(:,2);
end
```

(4)PROGRAM: gausslegendrequadratureNtriangularmesh4compositeintegration.m

```
function[]=gausslegendrequadratureNtriangularmesh4compositeintegration(
ga,gc,gz,m,mesh,mdiv,ndiv,type)
%gausslegendrequadratureNtriangularmesh4compositeintegration(
%10,10,40,100,32,1,5,type)
%type=1,2,3,4
%ga=first GAUSS RULE,gz=last Gauss Rule
%m=function code
%mesh= coarse outline of polygonal mesh
%div=number of divisions made in a standard triangle=1,2,3,4,........
%
%gauss legendre quadrature rules choosen:5,10,15,20,25,30,35,40
%-------------------------------------------------------------------------------------
---------------

syms Ui Vi Wi
numtr0=1;
switch mesh
case 1%convex polygon with sides%for functions at m=16,17,18,19,20,21
mst_tri=[7 1 2;...%1
         7 2 3;...%2
         7 3 4;...%3
         7 4 5;...%4
         7 5 6;...%5
         7 6 1];%6
gcoord=[0.1 0.0;...%1
        0.7 0.2;...%2
        1.0 0.5;...%3
        .75 .85;...%4
        0.5 1.0;...%5
        0.0 0.25;...%6
        0.5 0.5];%7
[mst_elm,dimension]=size(mst_tri);
case 2%nonconvex polygon with 9 sides
mst_tri=[2 9 1;...%1-3
         4 2 3;...%4-6
```

```
         5 2 4;...%7-9
         5 9 2;...%10-12
         5 7 9;...%13-15
         6 7 5;...%16-18
         7 8 9];%19-21
gcoord=[0.25 0.00;...%1
        0.75 0.50;...%2
        0.75 0.00;...%3
        1.00 0.50;...%4
        0.75 0.75;...%5
        0.75 0.85;...%6
        0.50 1.00;...%7
        0.00 0.75;...%8
        0.25 0.50];%9
[mst_elm,dimension]=size(mst_tri);
case 3%convex poygon:new mesh
mst_tri=[3 7 8;...%1-3
         4 7 3;...%4-6
         5 7 4;...%7-9
         9 7 5;...%10-12
         9 6 7;...%13-15
         1 7 6;...%16-18
         2 7 1;...%19-21
         8 7 2];%22-24
gcoord=[0.1 0.0;...%1
        0.7 0.2;...%2
        1.0 0.5;...%3
        .75 .85;...%4
        0.5 1.0;...%5
        0.0 .25;...%6
        0.5 0.5;...%7
        .75 .25;...%8
        0.3 0.7];%9
[mst_elm,dimension]=size(mst_tri);
case 4%standard triangle functions at cases 34-38
    mst_tri=[1 2 3];
    gcoord=[0.0 0.0;...%1
            1.0 0.0;...%2
            0.0 1.0];%3
    [mst_elm,dimension]=size(mst_tri);
case 5%quadrilateral function at case 35
        mst_tri=[1 2 5;...%1-3
                 2 3 5;...%4-6
                 3 4 5;...%7-9
                 4 1 5];%10-12
        gcoord=[-1 2;...%1
                 2 1;...%2
                 3 3;...%3
                 1 4;...%4
               5/4 5/2];%5
       [mst_elm,dimension]=size(mst_tri);
 case 6%standard square-function at case 40 and case 50
    mst_tri=[1 2 5;...%1-3
             2 3 5;...%4-6
             3 4 5;...%7-9
             4 1 5];%10-12
   gcoord=[-1 -1;...%1
            1 -1;...%2
            1  1;...%3
           -1  1;...%4
            0  0];%5
 [mst_elm,dimension]=size(mst_tri);
 numtr0=4;
```

```
case 7%%divison of a standard triangle into 2^2 right isosceles tiangles
    [Mst_tri]=nodal_address_rtisosceles_triangle(2);
     mst_tri=double(vpa(Mst_tri));
    [Ui,Vi,Wi]=coordinates_stdtriangle(2);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
nitro=16;
case 8%divison of a standard triangle into 3^2 right isosceles tiangles
    [Mst_tri]=nodal_address_rtisosceles_triangle(3);
    mst_tri=double(vpa(Mst_tri));
    [Ui,Vi,Wi]=coordinates_stdtriangle(3);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
nitro=36;
case 9%divison of a standard triangle into 4^2 right isosceles tiangles
     [Mst_tri]=nodal_address_rtisosceles_triangle(4);
     mst_tri=double(vpa(Mst_tri));
    [Ui,Vi,Wi]=coordinates_stdtriangle(4);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
nitro=64;
case 10%divison of a standard triangle into 5^2 right isosceles tiangles
      [Mst_tri]=nodal_address_rtisosceles_triangle(5);
     mst_tri=double(vpa(Mst_tri));
    [Ui,Vi,Wi]=coordinates_stdtriangle(5);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=25;
case 11%divison of a standard triangle into 6^2 right isosceles tiangles
 [Mst_tri]=nodal_address_rtisosceles_triangle(6);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(6);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=36;
case 12%divison of a standard triangle into 7^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(7);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(7);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
```

```
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=49;
case 13%divison of a standard triangle into 8^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(8);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(8);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=64;
case 14%divison of a standard triangle into 9^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(9);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(9);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=81;
case 15%divison of a standard triangle into 10^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(10);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(10);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=100;
case 16%divison of a standard triangle into 11^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(11);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(11);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=121;

case 17%divison of a standard triangle into 12^2 right isosceles tiangles
 [Mst_tri]=nodal_address_rtisosceles_triangle(12);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(12);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
case 18%divison of a standard triangle into 13^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(13);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(13);
ui=double(vpa(Ui));
```

```
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=169;
case 19%divison of a standard triangle into 14^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(14);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(14);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=196;
case 20%divison of a standard triangle into 15^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(15);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(15);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=225;
case 21%divison of a standard triangle into 16^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(16);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(16);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=256;
case 22%1/17%divison of a standard triangle into 17^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(17);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(17);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=17^2;
case 23%1/18%divison of a standard triangle into 18^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(18);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(18);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=18^2;
case 24%1/19%divison of a standard triangle into 19^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(19);
```

```
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(19);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
[mst_elm,dimension]=size(mst_tri);
numtr0=19^2;
case 25%1/20%divison of a standard triangle into 20^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(20);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(20);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=400;
case 26%1/21%divison of a standard triangle into 21^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(21);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(21);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=21^2;
case 27%1/22%divison of a standard triangle into 22^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(22);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(22);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=22^2;
case 28%1/23%divison of a standard triangle into 23^2 right isosceles tiangles
[Mst_tri]=nodal_address_rtisosceles_triangle(23);
 mst_tri=double(vpa(Mst_tri));
[Ui,Vi,Wi]=coordinates_stdtriangle(23);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
wi=double(vpa(Wi));
gcoord(:,1)=double(vpa(Ui));
gcoord(:,2)=double(vpa(Vi));
[mst_elm,dimension]=size(mst_tri);
numtr0=23^2
 case 29%equilateral triangle
    mst_tri=[1 2 3];
    gcoord=[0.0 0.0;...%1
           sqrt(3) 1.0;...%2
              0.0  2.0];%3
    [mst_elm,dimension]=size(mst_tri);
    numtr0=1;
 case 30%isosceles triangle
      mst_tri=[1 2 3];
```

```
    gcoord=[-4.0 1.0;...%1
            -2.5 -3.0;...%2
            -1.0 1.0];%3
    [mst_elm,dimension]=size(mst_tri);
    numtr0=1;
    case 31%scalene triangle
     mst_tri=[1 2 3];
    gcoord=[-3.0 -2.0;...%1
             5.0 -1.0;...%2
            -2.0 1.0];...%3
    [mst_elm,dimension]=size(mst_tri);
    numtr0=1;
    case 32%pentagon%functions at m=100,101,102,103,104,105,107,108,109,110
      mst_tri=[6 1 2;...%1
               6 2 3;...%2
               6 3 4;...%3
               6 4 5;...%4
               6 5 1];%5
      gcoord=[-1.0 -1.0;...%1
               1.0 -1.0;...%2
               2.0  0.0;...%3
               0.0  1.0;...%4
              -2.0  0.0;...%5
               0.0  0.0];%6
      numtr0=5;

    [mst_elm,dimension]=size(mst_tri);
    case 33%unitary square or standard 1-square
        mst_tri=[1 2 3;...%1
                 3 4 1];
       gcoord=[0.0 0.0;...%1
               1.0 0.0;...%2
               1.0 1.0;...%3
               0.0 1.0];%4
      numtr0=2;

    [mst_elm,dimension]=size(mst_tri);




end%switch mesh
%[nel,nnel]=size(nodes);
[nnode,dimension]=size(gcoord);
%if necessary include gauss legendre quadrature
i=0;
for abc=mdiv:ndiv
    i=i+1;
numtri(1,i)=4*numtr0*(abc)^2;
end
disp('number of triangles=')
disp([numtri])
%
for div=mdiv:ndiv
[Mst_tri]=nodal_address_rtisosceles_triangle(div);
 mst=double(vpa(Mst_tri));
%compute element cartesian/global coordinates
for L=1:mst_elm
if div==1
    for M=1:3
      LM=mst_tri(L,M);
      xx(L,M)=gcoord(LM,1);
```

```
      yy(L,M)=gcoord(LM,2);
     end
else
   for M=1:3
      LM=mst_tri(L,M);
      xx(L,M)=gcoord(LM,1);
      yy(L,M)=gcoord(LM,2);
    end
[Ui,Vi,Wi]=coordinates_stdtriangle(div);
ui=double(vpa(Ui));
vi=double(vpa(Vi));
%wi=double(vpa(Wi));
xxL1=xx(L,1);
xxL2=xx(L,2)-xxL1;
xxL3=xx(L,3)-xxL1;
yyL1=yy(L,1);
yyL2=yy(L,2)-yyL1;
yyL3=yy(L,3)-yyL1;

%cartesian/global coordinates
 for ii=4:(div+1)*(div+2)/2
  uii1=ui(ii,1);vii1=vi(ii,1);
  xx(L,ii)=xxL1+xxL2*uii1+xxL3*vii1;
  yy(L,ii)=yyL1+yyL2*uii1+yyL3*vii1;
 end
end%if div
end%for L
%-----------------------------------------------------------------------------

%---------------------------------------------------------------------------------------
 format long e
ggg=0;
for ng=ga:gc:gz
 ggg=ggg+1;
 %***********generate or refer gauss quadrature rule once only  for a given domain
 %************************
 if div== mdiv
 switch type
     case 1
 nn=ng*(ng+1);
 [ss,ww]=glsampleptsweights(ng+1);
 [sss,www]=glsampleptsweights(ng);

 k=0;
for ia=1:ng+1
    si=ss(ia);wi=ww(ia);
    for ja=1:ng
        k=k+1;
        sj=sss(ja);wj=www(ja);
        psj=(1+sj)/2;qsj=(1-sj)/2;psi=(1+si)/2;qsi=(1-si)/2;
        wij=wi*wj/(4);
        uij(k,1)=qsi;
        vij(k,1)=psi*psj;

        %wij(k,1)=1-uij(k,1)-vij(k,1);
         wt(k,1)=psi*wij;
   end
end

     case 2

      nn=ng^2;
```

```
[ss,ww]=glsampleptsweights(ng);

kk=0;
for ic=1:ng
    si=ss(ic);
    wi=ww(ic);
    aa=(1+si);
  for jc=1:ng
     kk=kk+1;
    sj=ss(jc);wj=ww(jc);
    bb=(1+sj);cc=(1-sj);
uij(kk,1)=aa*bb/4;
vij(kk,1)=aa*cc/4;
 wt(kk,1)=wi*wj*aa/8;
  end
end

 case 3
   nn=ng^2;

[ss,ww]=glsampleptsweights(ng);
kk=0;
for id=1:(ng-1)
    si=ss(id);wi=ww(id);
    for jd=id+1:ng
        kk=kk+1;
        sj=ss(jd);wj=ww(jd);
    aa=(1-((1+sj)/4))*((1+si)/2);
    bb=(1-((1+si)/4))*((1+sj)/2);
    cc=wi*wj*(1-(2+si+sj)/4)/4;
    uij(2*kk-1,1)=aa;uij(2*kk,1)=bb;
    vij(2*kk-1,1)=bb;vij(2*kk,1)=aa;
    wt(2*kk-1,1)=cc; wt(2*kk,1)=cc;
   % phii(kk,1)=aa;xhii(kk,1)=bb;
    %wttt(kk,1)=cc;

    end

end
k=ng^2-ng;
for ie=1:ng
 si=ss(ie);wi=ww(ie);
k=k+1;
%kk=kk+1;
uij(k,1)=(1-((1+si)/4))*((1+si)/2);
vij(k,1)=(1-((1+si)/4))*((1+si)/2);
wt(k,1)=wi*wi*(1-(2+si+si)/4)/4;

end
case 4

     nn=ng*(ng+1);
%*******************************************************************************

 [sss,www]=glsampleptsweights(ng+1);
    [ss,ww]=glsampleptsweights(ng);
kk=0;
for ik=1:ng+1
    si=sss(ik);
    wi=www(ik);
    aa=(1+si);
```

```
  for jk=1:ng
     kk=kk+1;
    sj=ss(jk);wj=ww(jk);
    bb=(1+sj);cc=(1-sj);
uij(kk,1)=aa*bb/4;
vij(kk,1)=aa*cc/4;
 wt(kk,1)=wi*wj*aa/8;
  end
end

 end%switch type
 end%if div==mdiv
%
    no(ggg,1)=ng;
    integralvalue(ggg,div)=0;
for iel=1:mst_elm
for eldiv=1:div*div
funxy=0;
n1=mst(eldiv,1);
n2=mst(eldiv,2);
n3=mst(eldiv,3);
x1=xx(iel,n1);
x2=xx(iel,n2);
x3=xx(iel,n3);
y1=yy(iel,n1);
y2=yy(iel,n2);
y3=yy(iel,n3);
xa=(x1+x2)/2;ya=(y1+y2)/2;
xb=(x2+x3)/2;yb=(y2+y3)/2;
xc=(x1+x3)/2;yc=(y1+y3)/2;
deta1=(xa-x1)*(yc-y1)-(xc-x1)*(ya-y1);
deta2=(x2-xa)*(yb-ya)-(xb-xa)*(y2-ya);
deta3=(xc-xb)*(ya-yb)-(xa-xb)*(yc-yb);
deta4=(xb-xc)*(y3-yc)-(x3-xc)*(yb-yc);
delabc=(x2-x1)*(y3-y1)-(x3-x1)*(y2-y1);
for kkkk=1:nn
UIJ=uij(kkkk,1);
VIJ=vij(kkkk,1);
WIJ=1-UIJ-VIJ;
wc=wt(kkkk,1);
%
wc1=wc*deta1;
wc2=wc*deta2;
wc3=wc*deta3;
wc4=wc*deta4;
xx1=x1*WIJ+xa*UIJ+xc*VIJ;yy1=y1*WIJ+ya*UIJ+yc*VIJ;
xx2=xa*WIJ+x2*UIJ+xb*VIJ;yy2=ya*WIJ+y2*UIJ+yb*VIJ;
xx3=xb*WIJ+xc*UIJ+xa*VIJ;yy3=yb*WIJ+yc*UIJ+ya*VIJ;
xx4=xc*WIJ+xb*UIJ+x3*VIJ;yy4=yc*WIJ+yb*UIJ+y3*VIJ;
funxy=funxy+(wc1*fnxy(m,xx1,yy1)+wc2*fnxy(m,xx2,yy2)+wc3*fnxy(m,xx3,yy3)+wc4*fnxy(m,xx4
,yy4));
end%kkkk
fff=funxy;
integralvalue(ggg,div)=integralvalue(ggg,div)+fff;
end%eldiv
end%iel
%disp([ ggg div integralvalue(ggg,div)])
end%ng
end%div


switch m
  case 16%polygonal domain
```

```
      disp('fn=(x+y)^19');

   case 17 %polygonal domain
      disp('fn=cos(30*(x+y))');
   case 18%polygonal domain
      disp('fn=sqrt((x-1/2)^2+(y-1/2)^2)');
   case 19%polygonal domain
      disp('fn=exp(-((x-1/2)^2+(y-1/2)^2))');
   case 20%polygonal domain
     disp(' fn=exp(-100*((x-1/2)^2+(y-1/2)^2))');
   case 21%polygonal domain
      disp('f1=0.75*exp(-0.25*(9*x-2)^2-0.25*(9*y-2)^2)');
       disp('f2=0.75*exp((-1/49)*(9*x+1)^2-0.1*(9*y+1))');
       disp('f3=0.5*exp(-0.25*(9*x-7)^2-0.25*(9*y-3)^2)');
         disp('f4=-0.2*exp(-(9*y-4)^2-(9*y-7)^2)');
         disp('fn=f1+f2+f3+f4');

    case 34%EX-1 standard triangle
      disp('fn=sqrt(x+y)');
   case 35%EX-2 standard triangle
    disp('fn=1/sqrt(x+y)');
case 36%EX-3 standard triangle
  disp('fn=1/sqrt(x^2+(1-y)^2)');
case 37%EX-4 standard triangle
  disp('fn=pi^2/4*sin(((pi*(x-y+1)))/2)');
case 38%EX-5 standard triangle
  disp('EX-5 standard triangle')
 disp(' fn=exp(abs(x-y))');
 case 39%EX-6 arbitrary quadrilateral
 disp('    fn=exp(-100*((x-1/2)^2+(y-1/2)^2))');

case 40
 disp('fn=sqrt((x-1/2)^2+(y-1/2)^2)');

case 41%polygonal domain
 disp('fn=abs(x^2+y^2-1/4)');
case 42%polygonal domain
 disp('fn=sqrt(abs(3-4*x-3*y))');
case 43%polygonal domain
disp('fm=double((x-0.6))');
disp('(if fm<=0')
   disp(' fm=0');
disp('end')
disp('fn=exp(-((5-10*x)^2)/2)+0.75*exp(-((5-10*y)^2)/2)+0.75*(exp(-((5-10*x)^2)/2-((5-
10*y)^2)/2))+((x+y)^3)*fm');
case 44%polygonal domain
 disp('fm=double((x+y-1))');
disp('if fm<=0')
    disp('fm=0');
disp('end')
%
disp(' fn=((1/9*sqrt(64-81*((x-.5)^2+(y-.5)^2)))-.5)*fm');
case 50%EX-7 standard 2_square
    disp('fx=(1/2)*sin(pi*(1+x)/8)');
    disp('f1=fx*pi/8');
    disp('f2=sqrt(1-(fx*(1+y))^2)');
    disp('fn=f1/f2');
  %   disp(' ');
  case 100%pentagon
     % disp(' ');
      disp('fn=(x^4+y^3)/(1+x^2) ');
```

```
  case 101
      disp(' fn=(1-x)*sin(10*x*y); ');

  case 102
      disp('fn=(.2*x+.3*y)^19; ');

  case 103
     disp('fn=(.17*x+.25*y)^25; ');


   case 104
      disp('fn=(x+y)^19/10^10; ');

   case 105
      disp('fn=(x-y)^20/10^5; ');

   case 106

      disp('  fn=cos(30*(x+y));');

  case 107
      disp('fn=sqrt((x-0.5)^2+(y-0.5)^2);  ');

  case 108
      disp('fn=exp(-((x-1/2)^2+(y-1/2)^2));  ');


   case 109
       disp('fn=exp(-100*((x-1/2)^2+(y-1/2)^2)); ');




  case 200% a standard triangle@4
      disp('fn=x*sqrt(1-y) ');

    case 201% a  standard triangle
        disp('fn=sqrt(x+y) ');

    case 202% a equilateral triangle@29&standard square @33
        disp('fn=exp(x+y) ');
    case 203% a  isosceles triangle@30
        disp('fn=(2*x+y)^3 ');
      case 204% a scalene triangle@31
      disp('fn=x^2*(y+1)^3');

    case 205%quadrilateral shafiqul hossain
         disp('fn=(sqrt(x+y))*(1+x+y)^2');
    case 206% standard triangle@4
        disp('fn=exp(x)*sqrt(1-y) ');
    case 207%standard square
    disp('fn=36*x*y*(1-x)*(1-y)');
otherwise
    disp('something wrong')
end
format long
table(:,1)=no;
switch ndiv
 case 1
table(:,2)=integralvalue(:,1);
```

```
case 2
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
case 3
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
case 4
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
case 5
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
case 6
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
case 7
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
case 8
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
case 9
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
case 10
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
```

```
case 11
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
case 12
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 case 13
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
case 14
  table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
case 15
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
```

```
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
case 16
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
case 17
  table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
 table(:,18)=integralvalue(:,17);
case 18
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
 table(:,18)=integralvalue(:,17);
 table(:,19)=integralvalue(:,18);

case 19
```

```
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
 table(:,18)=integralvalue(:,17);
 table(:,19)=integralvalue(:,18);
 table(:,20)=integralvalue(:,19);

case 20
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
 table(:,18)=integralvalue(:,17);
 table(:,19)=integralvalue(:,18);
 table(:,20)=integralvalue(:,19);
 table(:,21)=integralvalue(:,20);
 case 21
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
 table(:,18)=integralvalue(:,17);
 table(:,19)=integralvalue(:,18);
 table(:,20)=integralvalue(:,19);
 table(:,21)=integralvalue(:,20);
 table(:,22)=integralvalue(:,21);
```

```
 case 22
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
 table(:,18)=integralvalue(:,17);
 table(:,19)=integralvalue(:,18);
 table(:,20)=integralvalue(:,19);
 table(:,21)=integralvalue(:,20);
 table(:,22)=integralvalue(:,21);
 table(:,23)=integralvalue(:,22);
 case 23
 table(:,2)=integralvalue(:,1);
 table(:,3)=integralvalue(:,2);
 table(:,4)=integralvalue(:,3);
 table(:,5)=integralvalue(:,4);
 table(:,6)=integralvalue(:,5);
 table(:,7)=integralvalue(:,6);
 table(:,8)=integralvalue(:,7);
 table(:,9)=integralvalue(:,8);
 table(:,10)=integralvalue(:,9);
 table(:,11)=integralvalue(:,10);
 table(:,12)=integralvalue(:,11);
 table(:,13)=integralvalue(:,12);
 table(:,14)=integralvalue(:,13);
 table(:,15)=integralvalue(:,14);
 table(:,16)=integralvalue(:,15);
 table(:,17)=integralvalue(:,16);
 table(:,18)=integralvalue(:,17);
 table(:,19)=integralvalue(:,18);
 table(:,20)=integralvalue(:,19);
 table(:,21)=integralvalue(:,20);
 table(:,22)=integralvalue(:,21);
 table(:,23)=integralvalue(:,22);
 table(:,24)=integralvalue(:,23);

end%switch ndiv
switch  type
  case 1
        disp('asymmetric rule  and a mesh of all triangles ref numerical scheme(1)')
        disp('  ng*(ng+1) quadrature points over each triangle,ng=ORDER OF Gauss 
Legendre Rule')
 case 2
        disp('symmetric rule and a mesh of all triangles ref numerical scheme(2)  ')
        disp(' ng*ng quadrature points over each triangle,ng=ORDER OF G L RULE')
 case 3
    disp(' new symmetric rules and a mesh of all triangles ref numerical scheme(3)')
    disp('  ng*ng quadrature points over each triangle ,ng=ORDER OF Gauss Legendre 
Rule')
 case 4
```

```
     disp(' symmetric gauss legendre rules and a mesh of all triangles ref numerical
scheme(4)')
     disp(' ng*(ng+1) quadrature points over each triangle,ng=ORDER OF Gauss Legendre
Rule')
end
%disp(['sampling points and weights of gauss legendre quadraturerule=',num2str(ng)])
%disp([uij(1:nn,1) vij(1:nn,1) wt(1:nn,1)])
format long
disp([table])
%disp([table]')
```

(5) PROGRAM ON SOME FUNCTION DEFINITIONS

```
function[fn]=fnxy(n,x,y)
switch n

case 50

fx=(1/2)*sin(pi*(1+x)/8);
f1=fx*pi/8;
f2=sqrt(1-(fx*(1+y))^2);
fn=f1/f2;
case 100%pentagon
        fn=(x^4+y^3)/(1+x^2);
 case 101
     fn=(1-x)*sin(10*x*y);
 case 102
     fn=(.2*x+.3*y)^19;
  case 103

    fn=(.17*x+.25*y)^25;

  case 104
        fn=(x+y)^19/10^10;
 case 105
        fn=(x-y)^20/10^5;
 case 106
    fn=cos(30*(x+y));
  case 107
       fn=sqrt((x-0.5)^2+(y-0.5)^2);
  case 108
       fn=exp(-((x-1/2)^2+(y-1/2)^2));

   case 109
       fn=exp(-100*((x-1/2)^2+(y-1/2)^2));

otherwise
    disp('something wrong')
end
```