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Common Fixed Point Theorem Satisfying Implicit Relation On Menger Space.

Dr. M Ramana Reddy

Assistant Professor of mathematics, Sreenidhi Institute of Science and Technology,Hyderabad.

Abstact: The main object of this article is to obtain fixed point theorem in the setting of probabilistic metric space using weak compatibility, semi compatibility and an implicit relation.
Keywords: Probabilistic metric space, Menger space, Weakly compatibility, Semi compatibility and Implicit relations

I.Introduction

The study of these spaces expanded rapidly with the pioneering works of Schweizer and Sklar [1]. In 1972, Sehgal and Bharucha-Reid [2] initiated the study of contraction mappings on probabilistic metric (briefly, PM) spaces. Since then there has been a massive growth of fixed point theorems using certain conditions on the mappings or on the space itself.

In 1986 Jungck [3] introduced the notion of compatible mappings in metric spaces. Mishra [4] extended the notion of compatibility to probabilistic metric spaces. This condition has further been weakened by introducing the notion of weakly-compatible mappings by Jungck and Rhoades [5]. The concept of weakly compatible mappings is most general as each pair of compatible mappings is weakly compatible but the reverse is not true.

Cho et. al. [6] have introduced the notion of semi-compatible maps in a d-topological space.In 1999 Popa [7] proved theorem for weakly compatible non-continuous mapping using implicit relations. Singh and Jain [8] have established some fixed point theorems in Menger space using semi-compatibility of the mappings. In 2008 Altun and Turkoglu [9] proved two common fixed point theorems on complete FM-space with an implicit relation. Popa in [7] used the family Φ of implicit function to find the fixed points of two pairs of semi compatible maps in a d complete topological space, where Φ be the family of real continuous function $\varphi: (R^+)^4 \to R$ satisfying the properties

(a) for every $u \geq 0, v \geq 0$ with $\varphi(u,v,u,v) \geq 0$ or $\varphi(u,v,v,u) \geq 0$ we have $u \geq v$.

(b) $\varphi(u,u,1,1) \geq 0$ implies that $u \geq 1$.

II. Preliminaries

Definition 2.1: A probabilistic metric space (PM space) is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ into the collections of all distribution $F \in R$. For $x, y \in X$ we denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at u in R.The functions $F_{x,y}$ are assumed to satisfy the following conditions:

(i) $F_{x,y}(u) = 1 \;\; \forall\, u > 0$ iff $x = y$.

(ii) $F_{x,y}(0) = 0 \;\; \forall\, x, y$ in X.

(iii) $F_{x,y} = F_{y,x} \;\; \forall\, x, y$ in X.

(iv)If $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1$ then $F_{x,z}(u + v) = 1$

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for all x, y and z in X and $, v > 0$.

Definition 2.2: A commutative, associative and non decreasing mapping $t: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm if and only if

(i) $t(a, 1) = a \quad \forall a \in [0,1]$
(ii) $t(a, 0) = 0$
(iii) $t(c, d) \geq t(a, b)$ for $c \geq a$, $d \geq b$.

Definition 2.3: A Menger space is a triplet(X, F, t) where (X, F) is a PM- space, t is a t-norm and the generalized triangle inequality

$$F_{x,z}(u + v) \geq t(F_{x,z}(u), F_{y,z}(v))$$

holds for all x, y and z in X and $u, v > 0$.

Definition 2.4: Let (X, F, t) be a Menger space. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0, 1)$, then (ε, λ) - neighborhood of x is called $U_x(\varepsilon, \lambda)$, is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X: F_{x,y}(\varepsilon) > (1 - \lambda)\} \; .$$

An (ε, λ) - topology in X is the topology induced by the family $\{U_x(\varepsilon, \lambda): x \in X, \varepsilon > 0 \text{ and } \lambda \in (0, 1)\}$ of neighborhood.

Definition 2.5: A sequence $\{x_n\}$ in (X, F, t) is said to be convergent to a point x in X if for every $\varepsilon > 0$ and $\lambda > 0$,there is an integer $N = N(\varepsilon, \lambda)$ such that

$$x_n \in U_x(\varepsilon, \lambda) \text{ for all } n \geq N$$

or equivalently $F(x_n, x; \varepsilon) > 1 - \lambda$ λ forall $n \geq N$.

Definition 2.6: A sequence $\{x_n\}$ in (X, F, t) is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$,there exists an integer $N = N(\varepsilon, \lambda)$ such that

$$F(x_n, x_m; \varepsilon) > 1 - \lambda \quad \lambda \text{ for all } n, m \geq N.$$

Definition 2.7: A Menger space (X, F, t) with the continuous t -norm is said to be complete if every cauchy sequence in X converges to a point in X.

Definition 2.8: Let (X, F, t) be a Menger space, two mappings $f, g: X \rightarrow X$ are said to be weakly compatible if they commute at coincidence point.

III. Main Results

Theorem.3.1 Let (X, F, t) be a complete Menger space, where t is continuous and $t(p, p) \geq p$ for all $p \in [0, 1]$. Let A, B, S, T, P and Q are self mappings from X into itself such that

(I) $P(X) \subseteq AB(X)$ and $Q(X) \subseteq ST(X)$

(II) $AB = BA, ST = TS, PT = TP, QB = BQ$

(III) The pair (P, ST) is semi compatible and (Q, AB) is weak compatible.

(IV) Either P or ST is continuous;

For some $\varphi \in \Phi$, there exist $k \in (0,1)$ such that $\forall\, x, y \in X$ and $p > 0$

(V)$\varphi\{t(F_{Px,Qy}(kp)), t(F_{STx,ABy}(p)), t(F_{Px,STx}(p)), t(F_{Qy,ABy}(kp))\} \geq 0$

(VI)$\varphi\{t(F_{Px,Qy}(kp)), t(F_{STx,ABy}(p)), t(F_{Px,STx}(kp)), t(F_{Qy,ABy}(p))\} \geq 0$

Then A, B, S, T, P and Q have unique fixed point in X.

Proof: Let x_0 be any arbitrary point of X,

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as $P(X) \subseteq AB(X)$ and $Q(X) \subseteq ST(X)$
there is x_1, x_2 in X such that

$$Px_0 = ABx_1, Qx_1 = STx_2 .$$

Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n+1} = Px_{2n} = ABx_{2n+1} \text{ and } y_{2n+2} = Qx_{2n+1} = STx_{2n+2},$$

for n= 0, 1, 2……….
Now by (V)

$$\varphi\begin{Bmatrix} t(F_{Px_{2n},Qx_{2n+1}}(kp)), t(F_{STx_{2n},ABx_{2n+1}}(p)), \\ t(F_{Px_{2n},STx_{2n}}(p)), t(F_{Qx_{2n+1},ABx_{2n+1}}(kp)) \end{Bmatrix} \geq 0$$

$$\varphi\begin{Bmatrix} t(F_{y_{2n+1},y_{2n+2}}(kp)), t(F_{y_{2n},y_{2n+1}}(p)), \\ t(F_{y_{2n+1},y_{2n}}(p)), t(F_{y_{2n+2},y_{2n+1}}(kp)) \end{Bmatrix} \geq 0$$

By (a)

$$t(F_{y_{2n+2},y_{2n+1}}(kp)) \geq t(F_{y_{2n+1},y_{2n}}(p))$$

$$F_{y_{2n+2},y_{2n+1}}(kp) \geq F_{y_{2n+1},y_{2n}}(p)$$

Substituting again $x = x_1 = x_{2n+1}$ and $y = x_{2n+1}$ in (VI), We have

$$\varphi\begin{Bmatrix} t(F_{y_{2n+3},y_{2n+2}}(kp)), t(F_{y_{2n+1},y_{2n+2}}(p)), t(F_{y_{2n+3},y_{2n+2}}(kp)), \\ t(F_{y_{2n+1},y_{2n+2}}(p)) \end{Bmatrix} \geq 0$$

By (a) $\quad F_{y_{2n+3},y_{2n+2}}(kp) = F_{y_{2n+2},y_{2n+1}}(p)$
Hence $\{y_n\}$ is Cauchy sequence in X.
Therefore $\{y_n\}$ converges to u in X, and its sequences $\{Px_{2n}\}, \{ABx_{2n+1}\}$, $\{Qx_{2n+1}\}$, $\{Sx_{2n+1}\}$ also converges to u.
Case 1: If ST is continuous. We have $STPx_{2n} \to STu$ and $(ST)^2x_{2n} \to STu$.
So, semi compatibility of the pair (P, ST) gives

$$PSTx_{2n+1} \to STu \text{ as } n \to \infty.$$

Step (i):Substituting $x = STx_{2n}$, $y = x_{2n+1}$ in (V)
We obtain that

$$\varphi\begin{Bmatrix} t(F_{PSTx_{2n},Qx_{2n+1}}(kp)), t(F_{STSTx_{2n},ABx_{2n+1}}(p)), t(F_{PSTx_{2n},STSTx_{2n}}(p)), \\ t(F_{Qx_{2n+1},ABx_{2n+1}}(kp)) \end{Bmatrix} \geq 0$$

Now letting $n \to \infty$ and by the continuity of the t-norm, we have

$$\varphi\{t(F_{STu,u}(kp)), t(F_{STu,u}(p)), t(F_{STu,STu}(p)), t(F_{u,u}(kp))\} \geq 0$$

$$\varphi\{t(F_{STu,u}(kp)), t(F_{STu,u}(p)),1,1\} \geq 0$$

As φ is non decreasing in the first argument,
We have

$$\varphi\{F_{STu,u}(p), F_{STu,u}(p), 1,1\} \geq 0$$

Using (b) , we get

$$F_{STu,u}(p) \geq 1 , \forall\, p > 0$$

Which gives $F_{STu,u}(p) = 1$
That is, $STu = u$.
Step (ii) Substituting $x = u$ and $y = x_{2n+1}$ in (V)
We obtain that

$$\varphi\begin{Bmatrix} t(F_{Pu,Qx_{2n+1}}(kp)), t\left(F_{STu,ABx_{2n+1}}(p)\right), t(F_{Pu,STu}(p)), \\ t(F_{Qx_{2n+1},ABx_{2n+1}}(kp)) \end{Bmatrix} \geq 0$$

Taking the $\lim_{n\to\infty}$ and as $STu = u$ and $Qx_{2n+1}, ABx_{2n+1} \to u$

We get $\quad \varphi\{F_{Pu,u}(kp), 1, F_{Pu,u}(p), 1\} \geq 0$

Now as φ is non decreasing in the first argument,

We have

$$\varphi\{F_{Pu,u}(p), 1, F_{Pu,u}(p), 1\} \geq 0$$

Using (a) , we get

$$F_{Pu,u}(p) \geq 1 , \forall\, p > 0$$

Which gives $\quad F_{Pu,u}(p) = 1$

That is, $Pu = u = ST$.

Step (iii) By (I) $P(X) \subseteq AB(X)$, then $\exists\, w \in X$ such that

$$Pu = u = STu = ABw.$$

Substituting $x = x_{2n}$ and $y = w$ in (V),

We obtain that

$$\varphi\begin{Bmatrix} t(F_{Px_{2n},Qw}(kp)), t(F_{STx_{2n},ABw}(p)), \\ t(F_{Px_{2n},STx_{2n}}(p)), t(F_{Qw,ABw}(kp)) \end{Bmatrix} \geq 0$$

Taking the $\lim_{n\to\infty}$ and as $Px_{2n}, STx_{2n} \to u$, we get

$$\varphi\{F_{u,Qw}(kp), 1,1, F_{Qw,u}(kp)\} \geq 0$$

Using (a) , we get

$$F_{u,Qw}(kp) \geq 1 , \forall\, p > 0$$

Which gives $\quad F_{u,Qw}(p) = 1$

That is, $Qw = u$. Therefore $Qw = ABw = u$

Since (Q, AB) is weak compatible, we get $ABQw = QABw$,

which implies $Qu = ABu$.

Step(iv): Now substituting $x = u$ and $y = u$ in (V)

and as $Pu = u = STu$ and $Qu = ABu$,

We get that

$$\varphi\{t(F_{Pu,Qu}(kp)), t(F_{STu,ABu}(p)), t(F_{Pu,STu}(p)), t(F_{Qu,ABu}(kp))\} \geq 0$$

$$\varphi\{t(F_{Pu,Qu}(kp)), t(F_{STu,ABu}(p)),1,1\} \geq 0$$

Now as φ is non decreasing in the first argument,

We have

$$\varphi\{t(F_{Pu,Qu}(p)), t(F_{Pu,Qu}(p)),1,1\} \geq 0.$$

Using (Fu),we get

$$F_{Pu,Qu}(p) \geq 1 , \forall\, p > 0$$

Which gives $F_{Pu,Qu}(p) = 1$. That is $Pu = Qu$.

Thus $u = Pu = STu = Qu = ABu$.

Case 2. If P is continuous, we have $PSTx_{2n} \to Pu$.

Also the pair (P, ST) is semi compatible, Therefore $PSTx_{2n} \to STu$.

By the uniqueness of the limit $Pu = STu$.

Step (v): Substituting $x = u$ and $y = x_{2n+1}$ in 6.2.1 (V) , we get

$$\varphi\begin{Bmatrix} t(F_{Pu,Qx_{2n+1}}(kp)), t(F_{STu,ABx_{2n+1}}(p)), t(F_{Pu,STu}(p)), \\ t(F_{Qx_{2n+1},ABx_{2n+1}}(kp)) \end{Bmatrix} \geq 0$$

Taking the limit $n \to \infty$ and as $Qx_{2n+1}, ABx_{2n+1} \to u$, We get

$$\varphi\{F_{Pu,u}(kp)),1,F_{Pu,u}(p),1\} \geq 0.$$

Now as φ is non- decreasing in the first argument,

We have

$$\varphi\{F_{Pu,u}(p)),1,F_{Pu,u}(p),1\} \geq 0.$$

Using (Fu),we get

$$F_{Pu,u}(p) \geq 1, \forall p > 0$$

Which gives $u = Pu$.

The rest of the proof follows from step (iii) onwards of the case 1.

Uniqueness of common fixed point:

Let v be another common fixed point of A, B, S, T, P and Q. Then vPv = PvQv = STvABv = STv = vABv.

Now putting xx = uu and yyv y = v in (V), we get

$$\varphi\begin{Bmatrix} t(F_{Pu,Qv}(kp)), t(F_{STu,ABv}(p)), \\ t(F_{Pu,STu}(p)), t(F_{Qv,ABv}(kp)) \end{Bmatrix} \geq 0$$

$$\varphi\begin{Bmatrix} t(F_{u,v}(kp)), t(F_{u,v}(p)), \\ t(F_{u,u}(p)), t(F_{v,v}(kp)) \end{Bmatrix} \geq 0$$

$$\varphi\{t(FF_{u,v}(kp)), t(F_{u,v}(p)),1,1\} \geq 0$$

Now as φ is non decreasing in the first argument,

We have

$$\varphi\varphi\{FF_{u,v}(pkp), FF_{u,v}(pp), 1,1\} \geq 0$$

By using (a) , we have $(FF_{u,v}(pp) \geq 1$, for all $pp > 0$

Which gives uu = vv.

If we take BB = TT = II (the identity map on XIX) in Theorem 3.1, we have the following:

Corollary 3.2: Let (X, F, t) be a complete Menger space, where t is continuous and $t(p,p) \geq p$ for all $p \in [0,1]$. Let A, B, S, T, P and Q are self mappings from X into itself such that

(I) $P(X) \subseteq A(X)$ and $Q(X) \subseteq S(X)$

(II) The pair (P, S) is semi compatible and (Q, A) is weak compatible

(III) Either P or S is continuous;

For some $\varphi \in \Phi$, there exist $k \in (0,1)$ such that $\forall x, y \in X$ and $p > 0$

(IV) $\varphi\{t(F_{Px,Qy}(kp)), t(F_{Sx,Ay}(p)), t(F_{Px,Sx}(p)), t(F_{Qy,Ay}(kp))\} \geq 0$

(V) $\varphi\{t(F_{Px,Qy}(kp)), t(F_{Sx,Ay}(p)), t(F_{Px,Sx}(kp)), t(F_{Qy,Ay}(p))\} \geq 0$

Then A, S, P and Q have unique fixed point in X.

If we take P = Q in Theorem3.1, we have the following:

Corollary 3.3: Let (X, F, t) be a complete Menger space, where t is continuous and $t(p,p) \geq p$ for all $p \in [0,1]$. Let A, B, S, T, P and Q are self mappings from X into itself such that

(I) $P(X) \subseteq AB(X) \cap ST(X)$

(II) The pair (P, ST) is semi compatible and (P, AB) is weak compatible

(III) Either P or ST is continuous;

For some $\varphi \in \Phi$, there exist $k \in (0,1)$ such that $\forall\, x, y \in X$ and $p > 0$

$$(IV)\varphi\{t(F_{Px,Py}(kp)), t(F_{STx,ABy}(p)), t(F_{Px,STx}(p)), t(F_{Py,ABy}(kp))\} \geq 0$$

$$(V)\varphi\{t(F_{Px,Py}(kp)), t(F_{STx,ABy}(p)), t(F_{Px,STx}(kp)), t(F_{Py,ABy}(p))\} \geq 0$$

Then A, B, S, T and P have unique fixed point in X.

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