www.ijecs.in
International Journal Of Engineering And Computer Science ISSN:2319-7242
Volume 5 Issue 09 September 2016 Page No.17989-17995

Numerical Solution of Fourth Order Boundary Value Problems by Petrov-Galerkin Method with Quartic B-splines as basis functions and Sextic B-Splines as weight functions

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Abstract: ***This paper deals with a finite element method involving Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions to solve a general fourth order boundary value problem with a particular case of boundary conditions. The basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. The weight functions are also redefined into a new set of weight functions which in number match with the number of redefined basis functions. The proposed method was applied to solve several examples of fourth order linear and nonlinear boundary value problems. The obtained numerical results were found to be in good agreement with the exact solutions available in the literature.***

Keywords: **Petrov-Galerkin method, Quartic B-spline, Sextic B-spline, Fourth order boundary value problem, Absolute error.**

1. Introduction

In this paper, we consider a general fourth order boundary value problem

$$a_0(x)y^{(4)}(x)+a_1(x)y'''(x)+a_2(x)y''(x)+a_3(x)y'(x)+a_4(x)y(x)=b(x), c<x<d \quad (1)$$

subject to boundary conditions

$$y(c)=A_0, y(d)=C_0, y'(c)=A_1, y'(d)=C_1 \quad (2)$$

where A_0, A_1, C_0, C_1 are finite real constants and $a_0(x), a_1(x), a_2(x), a_3(x), a_4(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

The fourth order boundary value problems occur in a number of areas of applied mathematics, among which are fluid mechanics, elasticity and quantum mechanics as well as in science and engineering. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [1]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on fourth order boundary value problems by using different methods for numerical solutions. Papamichael and Worsey [2] developed the solution of a special case of linear fourth order boundary value problems by cubic spline method. Agarwal and Chow [3] presented the solution of nonlinear fourth order boundary value problems by the Picard’s iterative method and the quasilinear iterative method. Taiwo and Evans [4] developed perturbed collocation method to solve a general linear fourth order boundary value problem. Wazawz [5] presented modified decomposition method to solve a special case of fourth order boundary value problems. Waleed and Luis [6] developed decomposition method to solve fourth order boundary value problems. Erturk and Momani [7] presented a numerical comparison between di_erential transform methodand the Adomian decomposition method for solving fourth-order boundary value problems. Momani and Noor [8] presented a numerical comparison between the Differential transform method, Adomian decomposition and Homotopy perturbation method for solving a fourth-order boundary value problem. Samuel and Sinkala [9] developed higher order B-spline collocation method to solve fourth order boundary value problems. Syed and Noor [10], Noor and Syed [11] developed Homotopy perturbation method and Variational iteration technique respectively for the solution of fourth order boundary value problems. Ahniyaz et al. [12] developed Sinc-Galerkin method to solve a general linear fourth order boundary value problem. Manoj and Pankaj [13], Ramadan et al. [14], Pankaj et al. [15] and Ghazala and Amin [16] presented the solution of a special case of linear fourth order boundary value problems by spline techniques. Kasi Viswanadham et al. [17], Kasi Viswanadham and Sreenivasulu [20] developed Galerkin methods with quintic Bsplines and cubic B-splines respectively to solve a general fourth order boundary value problem. Rashidinia and Ghasemi [18], Kasi Viswanadham and Showri Raju [19] have developed B-spline collocation method, cubic B-spline collocation method respectively to solve a general fourth order boundary value problem. So far, fourth order boundary value problems have not been solved by using Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions. This motivated us to solve a fourth order boundary value problem by Pertrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions.

In this paper, we try to present a simple finite element method which involves Petrov-Gelerkin approach with quartic B-splines as basis functions and sextic B-splines as weight functions to solve the fourth order boundary value problem of the type (1)-(2). This paper is organized as follows. Section 2, deals with the justification for using Petrov-Galerkin Method.

In section 3, the definition of quartic B-splines and sextic B-splines has been described. In section 4, description of the Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions has been presented and in section 5, solution procedure to find the nodal parameters is presented. In section 6, the proposed method is tested on several linear and nonlinear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [21]. Finally, in the last section, the conclusions are presented.

2. Justification for using Petrov-Galerkin method

In Finite Element Method(FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz method, Galerkin method, Least Squares method, Petrov-Galerkin method and Collocation method etc.

In Petrov-Galerkin method, the residual of approximation is made orthogonal to the weight functions. When we use Petrov-Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [22, 23] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to the boundary conditions [24]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed and also the number of weight functions should match with the number of basis functions. Hence in this paper we employed the use of Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions to approximate the solution of fourth order boundary value problem.

3. Definition of quartic B-spline and Sextic B-spline

The quartic B-splines and sextic B-splines are defined in [25]-[27]. The existence of quartic spline interpolate s(x) to a function in a closed interval [c, d] for spaced knots (need not be evenly spaced) of a partition $c = x_0 < x_1 < \ldots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the quartic B-splines. Introduce eight additional knots x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3} and x_{n+4} in such a way that

$x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0$ and $x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}$.

Now the quartic B-splines $B_i(x)'s$ are defined by

$$B_i(x)=\begin{cases}\sum\limits_{r=i-2}^{i+3}\dfrac{(x_r-x)_+^4}{\pi'(x_r)}, & x\in[x_{i-2},\ x_{i+3}]\\ 0, & otherwise\end{cases}$$

where

$$(x_r-x)_+^4=\begin{cases}(x_r-x)^4, & if\ x_r\ge x\\ 0, & if\ x_r\le x\end{cases}$$

and $$\pi(x)=\prod_{r=i-2}^{i+3}(x-x_r)$$

where $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x),\ldots,B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_4(\pi)$ of quartic polynomial splines. Schoenberg [24] has proved that quartic B-splines are the unique nonzero splines of smallest compact support with the knots at

$x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0<x_1<\ldots<x_{n-1}<x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}$.

In a similar analogue sextic B-splines $R_i(x)$'s are defined by

$$R_i(x)=\begin{cases}\sum\limits_{r=i-3}^{i+4}\dfrac{(x_r-x)_+^6}{\pi'(x_r)}, & x\in[x_{i-3},\ x_{i+4}]\\ 0, & otherwise\end{cases}$$

where

$$(x_r-x)_+^6=\begin{cases}(x_r-x)^6, & if\ x_r\ge x\\ 0, & if\ x_r\le x\end{cases}$$

and $$\pi(x)=\prod_{r=i-3}^{i+4}(x-x_r)$$

where $\{R_{-3}(x), R_{-2}(x), R_{-1}(x), R_0(x), R_1(x),\ldots, R_{n-1}(x), R_n(x), R_{n+1}(x), R_{n+2}(x)\}$ forms a basis for the space $S_6(\pi)$ of sextic polynomial splines with the introduction of four more additional knots $x_{-6}, x_{-5}, x_{n+5}, x_{n+6}$ to the already existing knots x_{-4} to x_{n+4}. Schoenberg [27] has proved that sextic B-splines are the unique nonzero splines of smallest compact support with the knots at

$x_{-6}<x_{-5}<x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0<x_1<\ldots$
$<x_{n-1}<x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}<x_{n+6}.$

4. Description of the method

To solve the boundary value problem (1) subject to boundary conditions (2) by the Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions, we define the approximation for $y(x)$ as

$$y(x)=\sum_{j=-2}^{n+1}\alpha_j B_j(x) \tag{3}$$

where $\alpha_j's$ are the nodal parameters to be determined and $B_j(x)'s$ are quartic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quartic B-splines $\{B_{-2}(x), B_{-1}(x), B_0(x),\ldots, B_n(x), B_{n+1}(x)\}$, the basis functions $B_{-2}(x)$ $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$ and $B_{n+1}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. The procedure for redefining of the basis functions is as follows.

Using the definition of quartic B-splines and the Dirichlet boundary conditions of (2), we get the approximate solution at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-2}^{1} \alpha_j B_j(x_0) = A_0 \tag{4}$$

$$y(d) = y(x_n) = \sum_{j=n-2}^{n+1} \alpha_j B_j(x_n) = C_0 \tag{5}$$

Eliminating α_{-2} and α_{n+1} from the equations (3), (4) and (5), we get

$$y(x) = w(x) + \sum_{j=-1}^{n} \alpha_j P_j(x) \tag{6}$$

where $$w(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+1}(x_n)} B_{n+1}(x) \tag{7}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1, 0, 1 \\ B_j(x), & j = 2, 3, \ldots, n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+1}(x_n)} B_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{8}$$

The new set of basis functions in the approximation y(x) is $\{P_j(x), j = -1, 0, \ldots, n\}$. Here $w(x)$ takes care of given set of Dirichlet boundary conditions and $P_j(x)$'s vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation is n + 2, where as the number of weight functions is n+6. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows:

Let us write the approximation for v(x) as

$$v(x) = \sum_{j=-3}^{n+2} \beta_j R_j(x) \tag{9}$$

where $R_j(x)'s$ are sextic B-splines and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of the given boundary conditions (2). That means $v(x)$ defined in (9) satisfies the conditions

$$v(c) = 0, v(d) = 0, v'(c) = 0, v'(d) = 0 \tag{10}$$

Applying the boundary conditions (10) to (9), we get the approximate solution at the boundary points as

$$v(c) = v(x_0) = \sum_{j=-3}^{2} \beta_j R_j(x_0) = 0 \tag{11}$$

$$v(d) = v(x_n) = \sum_{j=n-3}^{n+2} \beta_j R_j(x_n) = 0 \tag{12}$$

$$v'(c) = v'(x_0) = \sum_{j=-3}^{n+2} \beta_j R'_j(x_0) = 0 \tag{13}$$

$$v'(d) = v'(x_n) = \sum_{j=n-3}^{n+2} \beta_j R'_j(x_n) = 0 \tag{14}$$

Eliminating β_{-3}, β_{-2}, β_{n+1} and β_{n+2} from the equations (9) and (11) to (14), we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=-1}^{n} \beta_j T_j(x) \tag{15}$$

where

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S'_j(x_0)}{S'_{-2}(x_0)} S_{-2}(x), & j = -1, 0, 1, 2 \\ S_j(x), & j = 3, 4, \ldots, n-4 \\ S_j(x) - \dfrac{S'_j(x_n)}{S'_{n+1}(x_n)} S_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases} \tag{16}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j(x_0)}{R_{-3}(x_0)} R_{-3}(x), & j = -2, -1, 0, 1, 2 \\ R_j(x), & j = 3, 4, \ldots, n-4 \\ R_j(x) - \dfrac{R_j(x_n)}{R_{n+2}(x_n)} R_{n+2}(x), & j = n-3, n-2, n-1, n, n+1 \end{cases} \tag{17}$$

Now the new set of weight functions for the approximation $v(x)$ is $\{T_j(x), j = -1, 0, \ldots, n\}$. Here $T_j(x)$'s and their derivatives vanish on the boundary.

Applying the Petrov-Galerkin method to (1) with the new set of basis functions $\{P_j(x), j = -1, 0, \ldots, n\}$ and with the new set of weight functions $\{T_j(x), j = -1, 0, \ldots, n+1\}$, we get

$$\int_{x_0}^{x_n} \left[a_0(x) y^{(4)}(x) + a_1(x) y'''(x) + a_2(x) y''(x) + a_3(x) y'(x) \right. \\ \left. + a_4(x) y(x)\right] T_i(x)\, dx = \int_{x_0}^{x_n} b(x) T_i(x) dx \quad \text{for } i = -1, 0, \ldots, n. \tag{18}$$

Integrating by parts the first term on the left hand side of (18) and after applying the boundary conditions prescribed in (2), we get

$$\int_{x_0}^{x_n} a_0(x) T_i(x) y^{(4)}(x) dx = \frac{d^2}{dx^2}\left[a_0(x) T_i(x)\right]_{x_n} C_1 \\ - \frac{d^2}{dx^2}\left[a_0(x) T_i(x)\right]_{x_0} A_1 - \int_{x_0}^{x_n} \frac{d^3}{dx^3}\left[a_0(x) T_i(x)\right] y'(x) dx \tag{19}$$

Substituting (19) in (18) and using the approximation for y(x) given in (6), and after rearranging the terms for resulting equations, we get system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{20}$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \{a_1(x) P_j'''(x) + a_2(x) P_j''(x) \\ + \left[-\frac{d^3}{dx^3}[a_0(x) T_i(x)] + a_3(x) \right] P_j'(x) + a_4(x) P_j(x)\} dx$$

$$\text{for } i=-1, 0, \ldots, n;\ j=-1, 0, \ldots, n. \tag{21}$$

$\mathbf{B} = [b_i]$;

$$b_i = \int_{x_0}^{x_n} \{b(x)T_i(x) - a_1(x)T_i(x)w'''(x)$$
$$-a_2(x)T_i(x)w''(x) - \{-\frac{d^3}{dx^3}[a_0(x)T_i(x)]$$
$$+a_3(x)T_i(x)\}w'(x) - a_4(x)T_i(x)w(x)\}dx$$
$$-\frac{d^2}{dx^2}[a_0(x)T_i(x)]_{x_n} C_1 + \frac{d^2}{dx^2}[a_0(x)T_i(x)]_{x_0} A_1$$

for i=-1, 0, ..., n. (22)

and

$$\alpha = [\alpha_{-1}\, \alpha_0 \ldots \alpha_n]^T.$$

5. Solution procedure to find the nodal parameters

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x) dx$ and $r_j(x)$ are the quartic B-spline basis functions or their derivatives. $v_i(x)$ are the sextic B-spline weight functions or their derivatives. It may be noted that $I_m = 0$ if $(x_{i-3}, x_{i+4}) \cap (x_{j-2}, x_{j+3}) \cap (x_m, x_{m+1}) = \varnothing$.

To evaluate each I_m, we employed 6-point Gauss-Legendre quadrature formula. Thus the stiffness matrix $\mathbf{A}$ is a eleven diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package. We have used the FORTRAN-90 program to solve the boundary value problems (1) - (2) by the proposed method.

6. Numerical results

To demonstrate the applicability of the proposed method for solving the fourth order boundary value problems of the type (1) and (2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

$$y^{(4)} + 4y = 1, \qquad -1 \le x \le 1 \tag{23}$$

subject to

$$y(-1) = y(1) = 0,$$
$$y'(-1) = \frac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)},\ y'(1) = \frac{\sin 2 - \sinh 2}{4(\cosh 2 + \cos 2)}.$$

The exact solution for the above problem is

$$y = .25\left[1 - 2\frac{\sinh 1 \sin 1 \sinh x \sin x + \cosh 1 \cos 1 \cosh x \cos x}{(\cos 2 + \cosh 2)}\right].$$

The proposed method is tested on this problem where the domain [-1, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is 1.169741×10^{-6}.

Table 1: Numerical results for Example 1

x	Absolute error by the proposed method
-0.8	2.495944E-07
-0.6	6.258488E-07
-0.4	8.270144E-07
-0.2	4.991889E-07
0	4.470348E-07
0.2	7.450581E-08
0.4	2.160668E-07
0.6	1.169741E-06
0.8	7.785857E-07

Example 2: Consider the linear boundary value problem

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x, \qquad 0 < x < 1 \tag{24}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = 1,\ y'(1) = -e.$

The exact solution for the above problem is $y = x(1-x)e^x$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is 6.943941×10^{-6}.

Table 2: Numerical results for Example 2

x	Absolute error by the proposed method
0.1	3.501773E-07
0.2	1.356006E-06
0.3	2.980232E-06
0.4	3.933907E-06
0.5	2.115965E-06
0.6	2.771616E-06
0.7	3.576279E-07
0.8	5.424023E-06
0.9	6.943941E-06

Example 3: Consider the linear boundary value problem

$$y^{(4)} - y'' - y = e^x(x-3), \qquad 0 < x < 1 \tag{25}$$

subject to $y(0) = 1,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = -e.$

The exact solution for the above problem is $y = e^x(1-x)$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3. The maximum absolute error obtained by the proposed method is 6.437302×10^{-6}.

Table 3: Numerical results for Example 3

x	Absolute error by the proposed method
0.1	3.576279E-07
0.2	1.788139E-06
0.3	6.198883E-06
0.4	6.079674E-06
0.5	3.576279E-06
0.6	1.430511E-06
0.7	2.861023E-06
0.8	2.384186E-06
0.9	6.437302E-06

Example 4: Consider the nonlinear boundary value problem

$$y^{(4)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \quad 0 < x < 1 \qquad (26)$$

subject to $y(0)=0,\ y(1)=1,\ y'(0)=0,\ y'(1)=1.$

The exact solution for the above problem is $y = x^5 - 2x^4 + 2x^2$

The nonlinear boundary value problem (26) is converted into a sequence of linear boundary value problems generated by quasilinearization technique[21] as

$$y^{(4)}_{(n+1)} - [2y_{(n)}]y_{(n+1)} = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 - [y_{(n)}]^2,$$

$$\text{n=0, 1, 2, …} \qquad (27)$$

subject to $y_{(n+1)}(0)=0, y_{(n+1)}(1)=1,\ y'_{(n+1)}(0)=0, y'_{(n+1)}(1)=1.$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (27). The obtained numerical results for this problem are presented in Table 4. The maximum absolute error obtained by the proposed method is 4.410744×10^{-6}.

Table 4: Numerical results for Example 4

x	Absolute error by the proposed method
0.1	3.725290E-09
0.2	4.544854E-07
0.3	1.594424E-06
0.4	1.698732E-06
0.5	1.490116E-06
0.6	5.960464E-08
0.7	3.039837E-06
0.8	7.152557E-07
0.9	4.410744E-06

Example 5: Consider the nonlinear boundary value problem

$$y^{(4)} = \sin x + \sin^2 x - \left[y''\right]^2, \qquad 0 < x < 1 \qquad (28)$$

subject to $y(0)=0$, $y(1)=\sin 1$, $y'(0)=1$, $y'(1)=\cos 1$.

The exact solution for the above problem is $y = \sin x$.

The nonlinear boundary value problem (28) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [21] as

$$y^{(4)}_{(n+1)} + [2y''_{(n)}]y''_{(n+1)} = \sin x + \sin^2 x + [y''_{(n)}]^2, \quad n = 0,1,2,\ldots \qquad (29)$$

subject to $y_{(n+1)}(0)=0$, $y_{(n+1)}(1)=\sin 1$,

$y'_{(n+1)}(0) = 1$, $y'_{(n+1)}(1) = \cos 1.$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (29). The obtained numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is 4.112720×10^{-6}.

Table 5: Numerical results for Example 5

x	Absolute error by the proposed method
0.1	4.470348E-07
0.2	1.475215E-06
0.3	3.546476E-06
0.4	4.112720E-06
0.5	3.576279E-06
0.6	1.907349E-06
0.7	2.861023E-06
0.8	8.940697E-07
0.9	2.741814E-06

Example 6: Consider the nonlinear boundary value problem

$$y^{(4)} - 6e^{-4y} = -12(1+x)^{-4}, \quad 0 < x < 1 \qquad (30)$$

subject to

$y(0)=0,\ y(1)=\ln 2,\ y'(0)=1,\ y'(1)=0.5.$

The exact solution for the above problem is $y=ln(1+x)$.

The nonlinear boundary value problem (30) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [21] as

$$y^{(4)}_{(n+1)} + [24e^{-4y_{(n)}}]y_{(n+1)} = -12(1+x)^{-4} + e^{-4y_{(n)}}[6 + 24y_{(n)}],$$

$$\text{n=0, 1, 2...} \qquad (31)$$

subject to

$y_{(n+1)}(0)=0, y_{(n+1)}(1)=\ln 2, y'_{(n+1)}(0)=1, y'_{(n+1)} = 0.5.$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (31). The obtained numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is 3.357418×10^{-7}.

Table 6: Numerical results for Example 6

x	Absolute error by the proposed method
0.1	1.963344E-07
0.2	3.357418E-07
0.3	1.462176E-07
0.4	3.250316E-07
0.5	1.974404E-07
0.6	2.635643E-07
0.7	7.078052E-08
0.8	2.873130E-07
0.9	2.421439E-07

7. Conclusions

In this paper, we have employed a Petrov-Galerkin method with quartic B-splines as basisfunctions and sextic B-splines as weight functions to solve fourth order boundary value problems with special case of boundary conditions. The quartic B-spline basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet boundary conditions are prescribed. The sextic B-splines are redefined into a new set of weight functions which in number match the number of redefined set of basis functions. The proposed method has been tested on three linear and three nonlinear fourth order boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The strength of the proposed method lies in its easy applicability, accurate and efficient to solve fourth order boundary value problems.

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DOI: 10.18535/ijecs/v5i9.34

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