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Finite Element Solution of Poisson Equation over Polygonal Domains using an Explicit Integration Scheme and a novel Auto Mesh Generation Technique

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Abstract :
This paper presents an explicit finite element integration scheme to compute the stiffness matrices for linear convex quadrilaterals. Finite element formulationals express stiffness matrices as double integrals of the products of global derivatives. These integrals can be shown to depend on triple products of the geometric properties matrix and the matrix of integrals containing the rational functions with polynomial numerators and linear denominator in bivariates as integrands over a 2-square. These integrals are computed explicitely by using symbolic mathematics capabilities of MATLAB. The proposed explicit finite element integration scheme can be applied solve boundary value problems in continuum mechanics over convex polygonal domains.We have also developed an automatic all quadrilateral mesh generation technique for convex polygonal domain which provides the nodal coordinates and element connectivity. We have used the explicit integration scheme and the novel mesh generation technique to solve the Poisson equation with given Dirichlet boundary conditions over convex polygonal domains.
Key words: Explicit Integration, Finite Element Method, Matlab Symbolic Mathematics, All Quadrilateral Mesh Generation Technique,Poisson Equation,Dirichlet Boundary Conditions ,Polygonal Domain

1. Introduction :

In recent years, the finite element method (FEM) has emerged as a powerful tool for the approximate solution of differential equations governing diverse physical phenomena. Today, finite element analysis is an integral and major component in many fields of engineering design and manufacturing. Its use in industry and research is extensive, and indeed without it many practical problem in science, engineering and emerging technologies such as nanotechnology, biotechnology, aerospace, chemical. etc would be incapable of solution [1,2,3]. In FEM, various integrals are to be determined numerically in the evaluation of stiffness matrix, mass matrix, body force vector, etc. The algebraic integration needed to derive explicit finite element relations for second order continuum mechanics problems generally defies our analytic skill and in most cases, it appears to be a prohibitive task. Hence, from a practical point of view, numerical integration scheme is not only necessary but very important as well. Among various numerical integration schemes, Gaussian quadrature, which can evaluate exactly the $(2n-1)^{th}$ order polynomial with n Gaussian integration points, is mostly used in view of the accuracy and efficiency of calculation. However, the integrands of global derivative products in stiffness matrix computations of practical applications are not always simple polynomials but rational expressions which the Gaussian quadrature cannot evaluate exactly

[7-15]. The integration points have to be increased in order improve the integration accuracy but it is also desirable to make these evaluations by using as few Gaussian points as possible, from the point of view of the computational efficiency. Thus it is an important task to strike a proper balance between accuracy and economy in computation. Therefore analytical integration is essential to generate a smaller error as well as to save the computational costs of Gaussian quadrature commonly applied for science, engineering and technical problems. In explicit integration of stiffness matrix, complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric elements, the presence of determinant of the Jacobian matrix (we refer this as Jacobian here after) in the denominator of the element matrix integrands. This problem is considered in the recent work [16] for the linear convex quadrilateral proposes a new discretisation method and use of pre computed universal numeric arrays which do not depend on element size and shape. In this method a linear polygon is discretized into a set of linear triangles and then each of these triangles is further discretised into three linear convex quadrilateral elements by joining the centroid to the mid-point of sides. These quadrilateral elements are then mapped into 2-squares ($-1\leq \xi,\eta \leq 1$) in the natural space (ξ,η) to obtain the same expression of the Jacobian, namely $c(4+\xi+\eta)$ where c is some appropriate constant which depends on the geometric data for the triangle.

Many important problems in engineering,science and applied mathematics are formulated by appropriate differential equations with some boundary conditions imposed on the desired unknown function or the set of functions. There exists a large literature which demonstrates numerical accuracy of the finite element method to deal with such issues [1]. Clough seems to be the first who introduced the fnite elements to standard computational procedures [2]. A further historical development and present{day concepts of fnite element analysis are widely described in references [1, 3]. In this paper the well-known Laplace and Poisson equations will be examined by means of the finite element method applied to an appropriate 'mesh'. The class of physical situations in which wemeet these equations is really broad. Let's recall such problems like heat conduction, seepage through porous media, irrotational flow of ideal fluids, distribution of electrical or magnetic potential, torsion of prismatic shafts, lubrication of pad bearings and others [4]. Therefore, in physics and engineering arises a need of some computational methods that allow to solve accurately such a large variety of physical situations.The considered method completes the above-mentioned task. Particularly, it refers to a standard discrete pattern allowing to find an approximate solution to continuum problem. At the beginning, the continuum domain is discretized by dividing it into a finite number of elements which properties must be determined from an analysis of the physical problem (e. g. as a result of experiments). These studies on particular problem allow to construct so{called the stiffness matrix for each element that, for instance, in elasticity comprising material properties like stress strain relationships [2, 5]. Then the corresponding nodal loads associated with elements must be found. The construction of accurate elements constitutes the subject of a mesh generation recipe proposed by the author within the presented article. In many realistic situations, mesh generation is a time{consuming and error prone process because of various levels of geometrical complexity. Over the years, there were developed both semi automatic and fully automatic mesh generators obtained, respectively, by using the mapping methods or, on the contrary, algorithms based on the Delaunay triangulation method [6], the advancing front method [7] and tree methods [8]. It is worth mentioning that the first attempt to create fully automatic mesh generator capable to produce valid finite element meshes over arbitrary domains has been made by Zienkiewicz and Phillips [9].

In the present paper, we propose a similar discretisation method for linear polygon in Cartesian two space (x,y). This discrtisation is carried in two steps, We first discretise the linear polygon into a set of linear triangles in the Cartesian space (x,y) and these linear triangles are then mapped into a standard triangle in a local space (u,v). We further discretise the standard triangles into three linear quadrilaterals by joining the centroid to the midpoints of triangles in (u,v) space which are finally mapped into 2-square in the local (ξ, η) space. We then establish a derivative product relation between the linear convex quadrilaterals in the Cartesian space, (x,y) which are interior to an arbitrary triangle and the linear quadrilaterals in the local space (u,v) interior to the standard triangle. In this procedure, all computations in the local space (u,v) for product of global derivative integrals are free from geometric properties and hence they are pure numbers. We then propose a numerical scheme to integrate the products of global derivatives. We have shown that the matrix product of global derivative integrals is expressible as matrix triple product comprising of geometric properties matrices and the product of local derivative integrals matrix. We have obtained explicit integration of the product of local derivatives which is now possible by use of symbolic integration commands available in leading mathematical softwares MATLAB, MAPLE, MATHEMATIKA etc. In this paper, we have used the MATLAB symbolic mathematics to compute the integrals of the products of local derivatives in (u, v) space .The proposed explicit integration scheme is shown as a useful technique in the formation of element stiffness matrices for second order boundary problems governed by partial differential equations.

2. POISSON EQUATION

2.1 Statement of the Problem

The Poisson equation

$$-\nabla^2 u = f \qquad \text{...(1)}$$

is the simplest and most famous elliptic partial differential equations.The source (or load) function is given on some two or three dimensional domain$\Omega \subset \mathcal{R}^2$ or $\mathcal{R}^3$. A solution u satisfying (1.1) will also satisfy boundary conditions on the boundary $\partial\Omega$ of Ω ;for example

$$\alpha u + \beta \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega \qquad \text{...(2)}$$

where $\partial u / \partial n$ denotes directional derivative in the direction normal to the boundary $\partial\Omega$ (conveniently pointing outwards) and α and β are constants, although variable coefficients are also possible.The combination of (1.1) and (1.2) together is referred to as boundary value problem. If the constant β in (1.2) is zero,then the boundary condition is known as the Dirichlet type, and the boundary value problem is referred as the Dirichlet problem for the Poisson equation. Alternatively, if the constant α in (1.2) is zero,then we correspondingly have a Neumann boundary value problem. A third possibility is that Dirichlet conditions hold on part of the boundary $\partial\Omega_D$ and Neumann conditions(or indeed mixed conditions where α and β are both nonzero) hold on remainder $\partial\Omega \backslash \partial\Omega_D$. The case $\alpha = 0, \beta = 1$ in (1.2) demands special attention.First, since u=constant satisfies the homogeneous problem with $f = 0, g = 0$,it is clear that a solution to a Neumann problem can only be unique up to an additive constant.Second,integrating (1.1) over Ω using Gauss's theorem gives

$$-\int_{\partial\Omega} \frac{\partial u}{\partial n} = --\int_{\Omega} \nabla^2 u = \int_{\Omega} f \qquad \text{..(3)}$$

thus a necessary condition for the existence of a solution to the Neumann problem is that the source and boundary data satisfy the compatibility condition:

$$\int_{\partial\Omega} g + \int_{\Omega} f = 0 \qquad \text{--(4)}$$

2.2 Weak Formulation of the Poisson Boundary Value Problem

A sufficiently smooth function u satisfying both eqns(1) and (2) is known as classical solution to the Poisson boundary value problem. For a Dirichlet problem, u is a classical solution only if it has continuousecond derivatives in Ω (i.e. u is $C^2(\Omega)$ and is continuous up to the boundary i.e.u is in $C^0(\overline{\Omega})$). In case of nonsmooth domains or discontinuous source functions,the function u satisfying eqns(1) and (2) may not be smooth (or regular) enough to be regarded as classical solution. For problems which arise from , perfectly reasonable mathematical models an alternative description of the boundary value problem is required. Since this alternative description is less restrictive in terms of admissible data it is called weak formulation.

To derive a weak formulation of a Poisson problem,we require that for an appropriate set of test functions v,

$\int_{\Omega} (\nabla^2 u + f)\, v = 0$.

....(5)

This formulation exists provided that the integrals are well defined. If u is a classical solution then it must also satisfy eqn (5). If v is sufficiently smooth however,then the smoothness required of u can be reduced by using the derivative of a product rule and the divergence theorem

$$-\int_{\Omega} v\nabla^2 u = \int_{\Omega} \nabla u \,.\, \nabla v - \int_{\Omega} \nabla.(v\nabla u)$$

$$= \int_{\Omega} \nabla u \,.\, \nabla v - \int_{\partial\Omega} v\frac{\partial \mathrm{u}}{\partial \mathrm{n}},$$

so that

$\int_{\Omega} \nabla u \,.\, \nabla v = \int_{\Omega} vf + \int_{\partial\Omega} v\,\frac{\partial \mathrm{u}}{\partial \mathrm{n}}$

....(6a)

The point here is that the problem posed byeqn(6) may have a solution u called a weak solution,that is not smooth enough to be a classical solution. If a classical solution does exist then eqn(6)is equivalent to eqns (1) and (2) and the weak solution is classical.

The case of Neumann problem $(\alpha = 0\ , \beta = 1)$ in eqn(2) is particularly straight forward. Substituting from eqn(2) into eqn(6)gives us the following formulation: find u defined on Ω such that

$\int_{\Omega} \nabla u \,.\, \nabla v = \int_{\Omega} vf + \int_{\partial\Omega} vg$

....(6b)

for all suitable test functions v .

2.3 Finite Elements for Poisson's Equation with Dirichlet conditions: Implementation and Review Of Theory

2.3.1 Weak Form

Given Poisson Equation:

$-\Delta u(\mathbf{x}) = f(\mathbf{x})$ **for all** $\mathbf{x} \in \boldsymbol{\Omega}$

............................(7a)

$u = g(\boldsymbol{x})$ on $\partial\Omega$

...............................(7b)

We have already obtained in eqn(6) with $(\alpha = 1\ , \beta = 0)$ the weak form of the equation by multiplying both sides by a test function v (i.e a function which is infinitely differentiable and has compact support,integrating over the domain Ω and performing integration by parts or by application of Divergence(GREEN) theorem. The result is

$\int_{\Omega} \nabla u \,.\, \nabla v \, d\boldsymbol{x} = \int_{\Omega} vf \, d\boldsymbol{x}$

..............................(7c)

$u = g(\boldsymbol{x})$ on $\partial\Omega$

...............................(7d)

For all test functions v .

2.3.2 Finite Elements

To find an approximation to the solution u , we choose a finite dimensional space V_h and ask that eqn(7a-b) is satisfied only for $v \; in \; V_h$ rather than for all test functions v. Then we look for a function $u_h \in V_h$ which satisfies

$$\int_{\Omega} \nabla u_h \, . \nabla v \, d\boldsymbol{x} = \int_{\Omega} vf \, d\boldsymbol{x} \quad \text{for all} \quad v \in V_h$$
..............................(8)

u_h is called the finite element solution and functions in V_h are called finite elements.
Note that it is also common for the triangles or quadrilaterals in the mesh to be called elements.
If a basis for V_h is $\{\varphi_j\}_{j=1}^{j=N}$ then we can write $u_h = \sum_{\mathbf{j=1}}^{\mathbf{j=N}} \boldsymbol{\alpha_j \varphi_j}$. Substituting this in eqn(8) and choosing v to be a basis function φ_i gives the following set of equations

$$\sum_{j=1}^{N} \boldsymbol{\alpha_j} \int_{\Omega} \nabla\varphi_i \, . \nabla\varphi_j \, d\boldsymbol{x} = \int_{\Omega} f\varphi_i \, d\boldsymbol{x} \quad ,i=1,2,3,...., \quad N$$
.................(9)

This is really a linear system of the form
$\boldsymbol{Ku=f}$
.................(10)

Where, $\boldsymbol{u} = (\alpha_1, \alpha_2, \alpha_3, \dots \dots \dots \alpha_N)^T$ and

$$K_{i,j} = \int_{\Omega} \nabla\varphi_i \, . \nabla\varphi_j \, d\boldsymbol{x}$$
....................(11a)

$$\boldsymbol{f_i} = \int_{\Omega} f\varphi_i \, d\boldsymbol{x}$$
......................(11b)

and $\boldsymbol{K}$ is called stiffness matrix because the linear system looks like Hookes law if $\boldsymbol{f}$ represents forces and $\boldsymbol{u}$ represents displacemewnts.
In general,$\Omega = \sum_{e=1}^{N_e} \Omega^e$, where N_e is the number of elements discritised in the domain Ω. In two dimensions the mesh elements are triangles or quadrilaterals.The choice of finite element spaces are usually piecewise polynomials.

2.3.3 Overview on the implementation of Finite Element Method

Once we have choosen the finite element space (and the element type),then we can implement the finite element method.The implementation is divided into three steps:
1**. Mesh Generation:**how does one perform a triangulation or quadrangulation of the domain Ω ?
2. **Assembling the Stiffness Matrix**:how does one compute the entries in the stiffness matrix in an efficient way?
3**. Solving the linear System:**What kind of methodse suited for solving the linear system?
In this paper,we present new approach to mesh generation [ ] and explicit computations for the entries in the stiffness matrix [ ] which is vital in Assembling the Stiffness Matrix,since we believe that the methods of solving linear system are well researched and standardised.
We shall first take up the derivations regarding the topic on **Assembling the Stiffness Matrix.** The **Mesh Generation** topic will be discussed immediately there after.

2.3.4 Assembling the Stiffness Matrix

In order to assemble the stiffness matrix,we need to compute integrals of the form(see eqn(11) in section 2.3.2)

$$K_{i,j} = \int_{\Omega} \nabla\varphi_i \, . \nabla\varphi_j \, d\boldsymbol{x}$$
....(11a)

The most obvious way to assemble the stiffness matrix is to compute the integrals $K_{i,j}$ for the nodal pairs i and j ; this is a node oriented computation and we need to know the common support of basis functions φ_i *and* φ_j.This means we need to know which elements contain both i and j.The mesh generator provides us with the information regarding the nodes on a particular element so we would need to do some extra processing to find the elements that contain a particular node.This is an issue which is very complicated.Hence,in practice assembling is focussed on elements rather than on nodes.We note that on a particular element,the basis functions have a simple expression and the elements themselves are very simple domains like triangles and quadrilaterals. It is very easy to make a change of variables for integrals over triangles and quadrilaterals to standard triangles and squares. In the element oriented computation,we rewrite or interpret the integral in eqn(11) as

$K_{i,j} = \sum_{\Omega^e\ \varepsilon\Omega_h^e} K_{i,j}^e =$ ....(12a)

where

$K_{i,j}^e = \int_{\Omega^e} \nabla\varphi_i \,.\, \nabla\varphi_j \, d\,\boldsymbol{x}$ ....(12b)

and Ω_h^e is the set of (mesh) elements in Ω contributing to $K_{i,j}$ and $\Omega = \sum_{e=1}^{N_e} \Omega^e$, $\Omega^e$is an element contained in the set Ω_h^e .This says us that we can compute $K_{i,j}$ by computing the integrals over each element Ω^e and then summing up over all elements Ω_h^e .

Notice that the integrals

$K_{i,j}^e = \int_{\Omega^e} \nabla\varphi_i \,.\, \nabla\varphi_j \, d\,\mathbf{x}$ look like the entries $K_{i,j} = \int_{\Omega} \nabla\varphi_i \,.\, \nabla\varphi_j \, d\,\boldsymbol{x} = \sum_{\Omega^e\ \varepsilon\Omega_h^e} \int_{\Omega^e} \nabla\varphi_i \,.\, \nabla\varphi_j \, d\,\boldsymbol{x}$ except the domain of integration is an element Ω^e. So,we only need to save all entries of K^e =[$K_{i,j}^e$] which corresponds to nodes on Ω^e. Then if Ω^e has d nodes , we can think of K^e as a dxd matrix. In view of the above,the procedure for computing the stiffness maerix is done on an element by element basis.

We must also compute the integrals

$\boldsymbol{f_i} = \int_{\Omega} f\varphi_i \, d\boldsymbol{x} = \sum_{\boldsymbol{e=1}}^{\boldsymbol{N_e}} \boldsymbol{f_i^e}$ ....(12c)

where

$\boldsymbol{f_i^e} = \int_{\Omega^e} f\varphi_i d\,\mathbf{x}$ ....(12d)

Now further assume that on an element Ω^{e} , $u_h = \mathbf{u^e} = \sum_{\mathbf{j=1}}^{\mathbf{j=d}} \mathbf{u_j^e} \boldsymbol{\varphi}_\mathbf{j}$

From eqn(9) and eqns(12a-d) it follows that ***Ku=f*** **is equivalent to**

$\sum_{e=1}^{N_e} K^e \,\mathbf{u^e} = \sum_{e=1}^{N_e} \boldsymbol{f^e}$ **..(12e)**

Where

$\mathbf{u^e} = (\mathbf{u_1^e}\,, \mathbf{u_2^e}, \ \mathbf{u_3^e}, \ \ldots\ldots\ldots\ldots \mathbf{u_d^e}\ )^T$, $\boldsymbol{f}^{\mathbf{e}} = (\boldsymbol{f}_{\mathbf{1}}^{\mathbf{e}}\,, \boldsymbol{f}_{\mathbf{2}}^{\mathbf{e}}, \ \boldsymbol{f}_{\mathbf{3}}^{\mathbf{e}}, \ \ldots\ldots\ldots\ldots \boldsymbol{f}_{\mathbf{d}}^{\mathbf{e}}\ )^T$ **...(12f)**

d referes to number of nodes per element, N_e referes to the total number of elements in the domain Ω

2.3.5 Computing the Integrals $\boldsymbol{K_{i,j}^e}$ and $\boldsymbol{f_i^e}$

In order to compute the local/element stiffness matrices,we need to compute the integrals $K_{i,j}^{e} = \int_{\Omega^{e}} \nabla\varphi_{\,i} \,.\, \nabla\varphi_{j}\, d\,\mathbf{x}$ .These integrals are computed by making a change of variables to a reference element. We now outline a brief procedure for element oriented computation

(1)For each element Ω^{e} , compute it's local stiffness matrix K^{e}. This requires computing the integrals $K_{i,j}^{e} = \int_{\Omega^{e}} \nabla\varphi_{\,i} \,.\, \nabla\varphi_{j}\, d\,\mathbf{x}$ which we compute by transforming to a reference element. In two dimensions Ω^{e} is an arbitrary linear triangle and each triangle will be further discritised three convex quadrilaterals Q_{3e-2} , Q_{3e-1} and Q_{3e} Each triangle will be transformed to the corresponding reference elements:the standard triangle(a right isosceles triangle) and further Each triangle will be transformed to the corresponding reference elements:the standard triangle(a right isosceles triangle) and further Each triangle will be transformed to the corresponding reference elements:the standard triangle(a right isosceles triangle) and further each quadrilateral will be transformed into a standard square(1-square or a 2-square).Since in two dimensional space **x**= (x,y) the explicit form of $K_{i,j}^{e} = \int_{\Omega^{e}} \nabla\varphi_{\,i} \,.\, \nabla\varphi_{j}\, d\,\mathbf{x}$ is given by

$K_{i,j}^{e} = \int_{\Omega^{e}} \nabla\varphi_{\,i} \,.\, \nabla\varphi_{j}\, d\mathbf{x}$ $=\int_{\Omega^{e}}\{\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\}\, dxdy = \sum_{e=1}^{N_e}\sum_{n=0}^{2}\int_{Q_E}\{\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\}dxdy$

$=\sum_{e=1}^{N_e}\sum_{n=0}^{2} S_{i,j}^{E}$ ....(12g)

Where $S_{i,j}^{E} = \int_{Q_E}\{\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\}dxdy$ and E=3e+n-2,e=1,2,... N_e andn=0,1,2

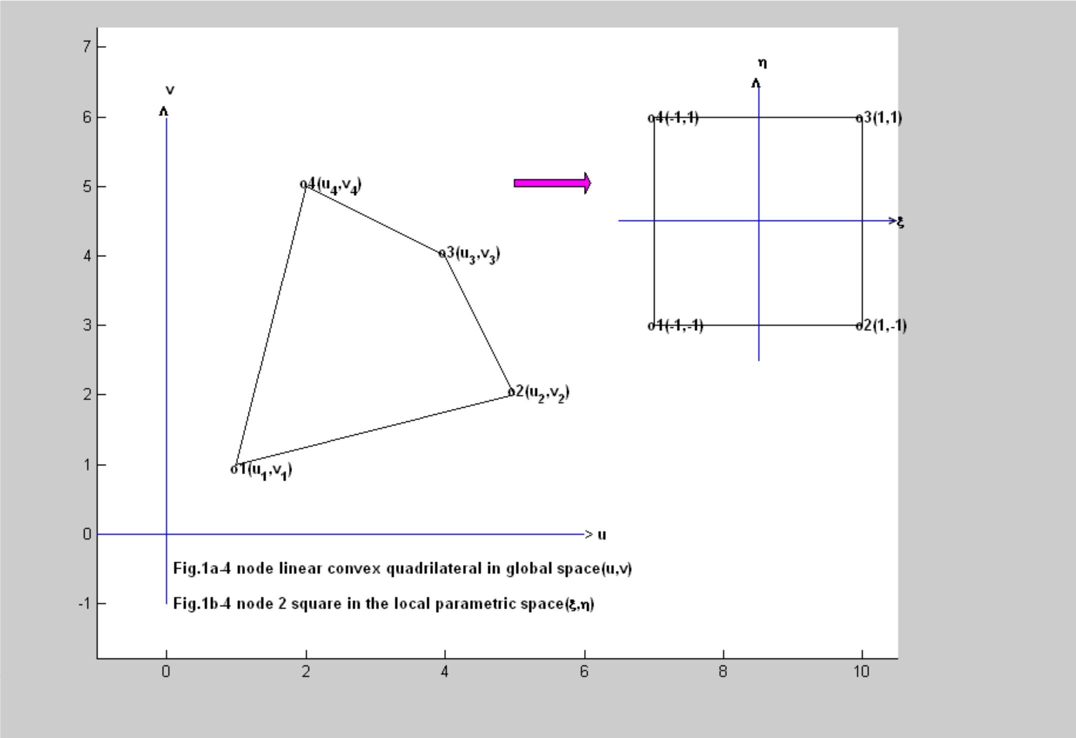
and hence we must be careful about the derivatives when we perform the change of variables. These bring extra factors involving the affine transformations (when Ω^{e} is an arbitrary linear triangle) and bilinear transformations(when Ω^{e} is an arbitrary linear convex quadrilateral)

$\boldsymbol{f_i^e}$ = $\int_{\Omega^{e}} f\varphi_i$ dxdy can be computed in a straight forward manner if f is a simple function otherwise we have to apply numerical integration

(2) For each element Ω^{e}, first compute the local stiffness matrices $S^{E} = [S_{i,j}^{E}]$ and then add contribution of $K^{e} = S^{3e-2} + S^{3e-1} + S^{3e}$, to the global stiffness matrix K. We repeat this procedure for all elements i.e for e=1,2,......, N_e; where N_e is the number of elements Ω^{e} $which$ are discritised in the domain Ω ,in fact we have $\Omega = \sum_{e=1}^{N_e}\Omega^{e}$ $=\sum_{e=1}^{N_e}\sum_{n=0}^{2} Q_E$, E=3e+n-2

2.3.6 Linear Convex Quadrilateral Elements :

Let us first consider an arbitrary four noded linear convex quadrilateral in the global (Cartesian) coordinate system (u, v) as in Fig 1a, which mapped into a 2-square in the local(natural) parametric coordinate (ξ, η) as in Fig 1b.



$\binom{u}{v} = \sum_{k=1}^{4} \binom{u_k}{v_k} \; M_k(\xi,\eta)$ ------------------------ (13)

Where (u_k, v_k) , (k=1,2,3,4) are the vertices of the original arbitrary linear convex quadrilateral in (u, v) plane and $M_k(\xi,\eta)$ denote the well known bilinear basis functions [1-3] in the local parametric space (ξ, η) and they are given by

$M_k(\xi,\eta) = \frac{1}{4}\,(1+\xi\xi_k)(1+\eta\eta_k)$, $k = 1,2,3,4$ ----------------------- (14a)

Where { $(\xi_k,\eta_k), k = 1,2,3,4\}$ = {(-1,-1),(1,-1),(1,1),(-1,1)} ----------------------- (14b)

describe two transformations over a linear convex quadrilateral element from the original global space into the local parametric space.

2.3.7 Isoparametric Transformation :

For the isoparametric coordinate transformation over the linear convex quadrilateral element as shown in Fig 1, we select the field variables, say ϕ, ψ , etc governing the physical problem as

$\binom{\phi}{\psi} = \sum_{k=1}^{4} \binom{\phi_k}{\psi_k} N_k{}^{e}(\xi,\eta)$ ------------------------ (15)

Where ϕ_k , ψ_k refer to unknowns at node k and the shape functions $N_k{}^e = M_k$, and M_k are defined as in Eqn.(2a-b)

2.3.8 Subparmetric Transformation :

For the subparametric transformation over the nde – noded element we define the field variables ϕ, ψ (say) governing the physical problem as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=1}^{nde} \begin{pmatrix} \phi_k{}^e \\ \psi_k{}^e \end{pmatrix} N_k{}^e(\xi, \eta) \quad \text{-------------------------- (16)}$$

Where ϕ_k , ψ_k refer to unknowns at node k and nde >4

In our recent paper[ ], the explicit finite element integration scheme is presented by using the isoparametric transformation over the 4 node linear convex quadrilateral element which is applied to torison of square shaft,on considering symmetry mesh generation for 1/8 of the cross section which is a triangle was discritised into an all quadrilateral mesh.In this paper we consider applications to polygonal domains.

2.3.9 Explicit Form of the Jacobian and Global Derivatives :

Jacobian

Let us consider an arbitrary linear convex quadrilateral in the global Cartesian space (u, v) as in Fig 1a , c which is mapped into a 8- node 2- square in the local parametric space (ξ, η) as in Fig 1b, d

From the Eq.(13) and Eq.(14), we have

$$\frac{\partial u}{\partial \xi} = \sum_{k=1}^{4} u_k \frac{\partial M_k}{\partial \xi} = \frac{1}{4}[\,(-u_1 + u_2 + u_3 - u_4) + (u_1 - u_2 + u_3 - u_4)\,\eta\,] \quad \text{------------ (17a)}$$

$$\frac{\partial u}{\partial \eta} = \sum_{k=1}^{4} u_k \frac{\partial M_k}{\partial \eta} = \frac{1}{4}[\,(-u_1 - u_2 + u_3 + u_4) + (u_1 - u_2 + u_3 - u_4)\,\xi\,] \quad \text{------------ (17b)}$$

$$\frac{\partial v}{\partial \xi} = \frac{1}{4}[\,(-v_1 + v_2 + v_3 - v_4) + (v_1 - v_2 + v_3 - v_4)\,\eta\,] \quad \text{------------ (17c)}$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}[\,(-v_1 - v_2 + v_3 + v_4) + (v_1 - v_2 + v_3 - v_4)\,\xi\,] \quad \text{------------ (17d)}$$

Hence the Jacobian, J can be expressed as [1, 2, 3]

$$J = \frac{\partial(u,v)}{\partial(\xi,\eta)} = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \alpha + \beta\xi + \gamma\,\eta \quad \text{-------------- (18a)}$$

Where

$\alpha = \frac{1}{8}[\,(u_4 - u_2)(v_1 - v_3) + (u_3 - u_1)(v_4 - v_2)\,]$

$\beta = \frac{1}{8}[\,(u_4 - u_3)(v_2 - v_1) + (u_1 - u_2)(v_4 - v_3)\,]$

$\gamma = \frac{1}{8}[\,(u_4 - u_1)(v_2 - v_3) + (u_3 - u_2)(v_4 - v_1)\,]$ ---------------- (18b)

Global Derivatives:

If N_i^e denotes the basis functions of node i of any order of the element e, then the chain rule of differentiation from Eq.(1) we can write the global derivative as in [1, 2, 3]

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix} = \frac{1}{J}\begin{bmatrix} \frac{\partial v}{\partial \eta} & -\frac{\partial v}{\partial \xi} \\ -\frac{\partial u}{\partial \eta} & \frac{\partial u}{\partial \xi} \end{bmatrix}\begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix}$$ --- ------------------------ (19)

Where $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \xi}$ and $\frac{\partial v}{\partial \eta}$ are defined as in Eqs.(17a)–(17d) while J is defined in Eq.(18a-b) , ($i, j = 1,2,3, - - - - -$, nde) , nde = the number of nodes per element.

2.3.10 Discretisation of an Arbitrary Triangle :

A linear convex polygon in the physical plane (x, y) can be always discretised into a finite number of linear triangles. However, we would like to study a particular discretization of these triangles further into linear convex quadrilaterals. This is stated in the following Lemma [6].

Lemma 1. Let Δ PQR be an arbitrary triangle with the vertices $P(x_p, y_p)$, $Q(x_q, y_q)$ and $R(x_r, y_r)$ and S, T, U be the midpoints of sides PQ, QR and RP respectively and let Z be its centroid. We can obtain three linear convex quadrilaterals ZTRU, ZUPS and ZSQT from triangle Δ PQR as shown in Fig2. If we map each of these quadrilaterals into 2-squares in which the nodes are oriented in counter clockwise from Z then Jacobian J for each element e is given by

$J = J^e = \frac{1}{48}\Delta\, pqr\,(4 + \xi + \eta), \qquad e = 1,2,3$
--------------- (20)

Where $\Delta\, pqr$ is the area of the triangle $\Delta\, PQR$

$$2\Delta\, pqr = \begin{vmatrix} 1 & x_p & y_p \\ 1 & x_q & y_q \\ 1 & x_r & y_r \end{vmatrix} = [\,(x_p - x_r)(y_q - y_r) - (x_q - x_r)(y_p - y_r)\,]$$ ------------ (21)

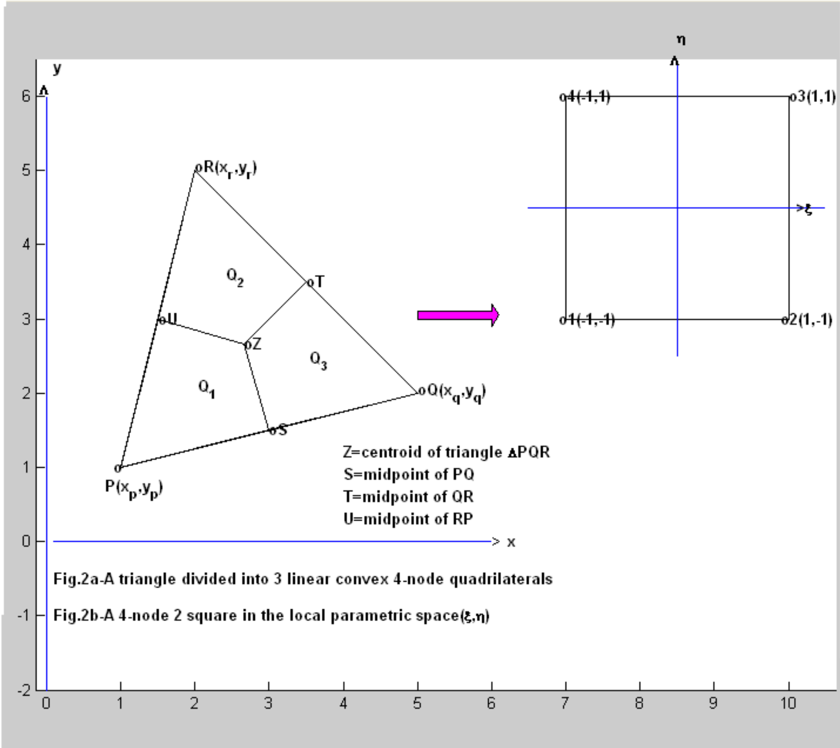


Fig.2a-A triangle divided into 3 linear convex 4-node quadrilaterals

Fig.2b-A 4-node 2 square in the local parametric space(ξ,η)

Proof : Proof is straight forward and it can be elaborated on the lines of proof given in [17].

Lemma 2. Let Δ PQR be an arbitrary triangle with the vertices $P(x_p, y_p)$, $Q(x_q, y_q)$ and $R(x_r, y_r)$, let S, T, U be the midpoints of sides PQ, QR, and RP and let Z be the centroid of Δ PQR, Then we obtain three quadrilaterals Q_1, Q_2, Q_3 spanning the vertices <ZUPS> , <ZSQT> and <ZTRU> . these quadrilaterals can be mapped into the quadrilateral spanning vertices GECF with G(1/3, 1/3), E(0, ½) , C(0, 0) and F(1/2, 0) of the right isosceles triangle Δ ABC with spanning vertices A(1, 0), B(0, 1) and C(0, 0) in the (u, v) space as shown in Fig 3a and Fig 3b

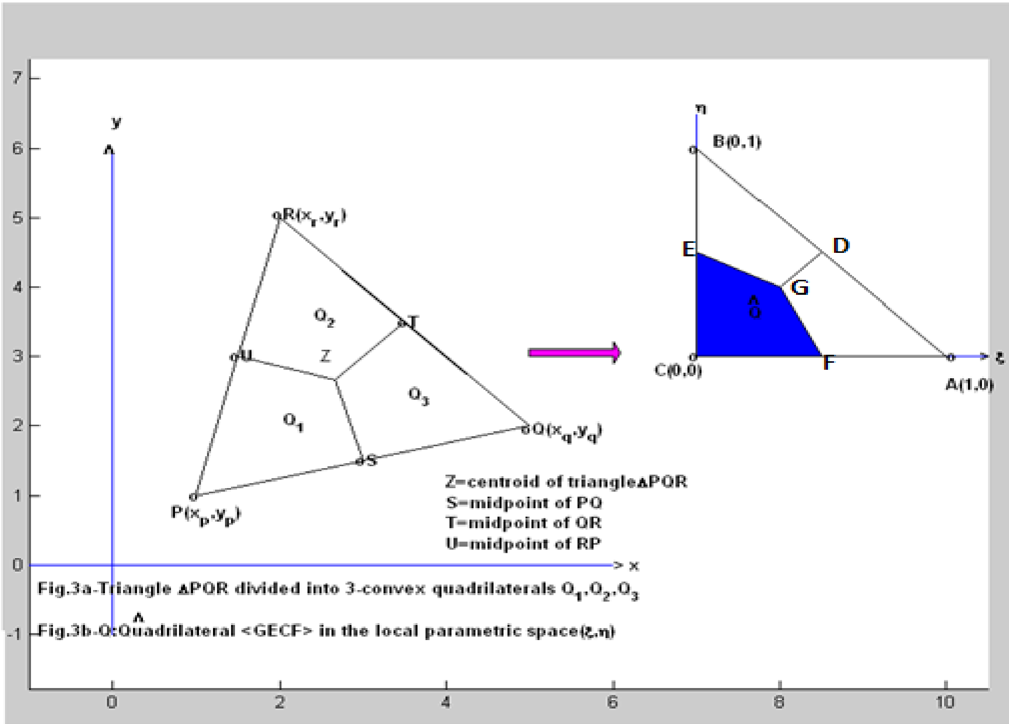


Fig.3a-Triangle ∆PQR divided into 3-convex quadrilaterals Q_1, Q_2, Q_3

Fig.3b-$\hat{Q}$:Quadrilateral <GECF> in the local parametric space(ξ,η)

Proof : The sum of the quadrilaterals $Q_1+ Q_2+ Q_3 = \Delta$ PQR as shown in Fig 2a & Fig 3a. The linear transformations

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} w + \begin{pmatrix} x_q \\ y_q \end{pmatrix} u + \begin{pmatrix} x_r \\ y_r \end{pmatrix} v$$

--------------------------- (22a)

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \end{pmatrix} w + \begin{pmatrix} x_r \\ y_r \end{pmatrix} u + \begin{pmatrix} x_p \\ y_p \end{pmatrix} v$$

-------------------------- (22b)

$$\begin{pmatrix} x^{(3)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \end{pmatrix} w + \begin{pmatrix} x_p \\ y_p \end{pmatrix} u + \begin{pmatrix} x_q \\ y_q \end{pmatrix} v$$

------------------------ (22c)

with $w = 1 - u - v$

------------------------ (22d)

map the arbitrary triangle Δ PQR into a right isosceles triangle A(1, 0) , B(0, 1) and C(0, 0) in the uv–plane We can now verify that quadrilateral Q_1 spanned by vertices Z($\frac{x_p+x_q+x_r}{3}$, $\frac{y_p+y_q+y_r}{3}$) , U($\frac{x_r+x_p}{2}$, $\frac{y_r+y_q}{2}$) , P(x_p, y_p) ,S($\frac{x_p+x_q}{2}$, $\frac{y_p+y_q}{2}$) in xy- plane is mapped into the quadrilateral spanning the vertices G(1/3, 1/3) , E(0, ½), C(0, 0) and F(1/2, 0) by use of the transformation given in Eq.(22a),

Similarly, we see that the quadrilateral Q_2 spanned by vertices Z,S,Q,T is mapped into the quadrilateral spanned by vertices G(1/3, 1/3), E(0, ½), C(0, 0) and F(½, 0) by use of the transformation of Eq.(22b), Finally the quadrilateral Q_3 in the xy- plane is mapped into the quadrilateral GECF in uv- plane by use of the linear transformation of Eq.(22c),

This completes the proof

We have shown in the present section that an arbitrary triangle can be discretised into three linear convex quadrilaterals. Further, each of these quadrilaterals can be mapped into a unique quadrilateral in uv-plane spanned by vertices (1/3, 1/3), (0, ½), (0, 0) and (½, 0) .

3.0 Computing the Integrals : $K^e_{i,j}$

3.1 Global Derivative Integrals: $\int_\Omega \nabla\varphi_i . \nabla\varphi_j \, d\boldsymbol{x} \quad K^e_{i,j}$

If $N_i^{(e)}$ denotes the basis function for node i of element e , then by chain rule of partial differentiation

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{bmatrix}$$

------------------------- (23)

We note that to transform $Q_e (e = 1,2,3)$ of ΔPQR in Cartesian space (x,y) into $\widehat{Q}$, the quadrilateral spanned by vertices (1/3,1/3), (0,½), (0,0) and (1/2,0) in uv-plane we must use the earlier transformations.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} + \begin{pmatrix} x_q - x_p \\ y_q - y_p \end{pmatrix} u + \begin{pmatrix} x_r - x_p \\ y_r - y_p \end{pmatrix} v \quad \text{for } Q_1 \text{ in } \Delta PQR$$

-------------------------- (24a)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \end{pmatrix} + \begin{pmatrix} x_r - x_q \\ y_r - y_q \end{pmatrix} u + \begin{pmatrix} x_p - x_q \\ y_p - y_q \end{pmatrix} v \quad \text{for } Q_2 \text{ in } \Delta PQR$$

------------------------- (24b)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \end{pmatrix} + \begin{pmatrix} x_p - x_r \\ y_p - y_r \end{pmatrix} u + \begin{pmatrix} x_q - x_r \\ y_q - y_r \end{pmatrix} v \quad \text{for } Q_3 \text{ in } \Delta PQR$$

------------------------ (24c)

and the above transformations viz Eqs.(24a)-(24c) are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x_a - x_c \\ y_a - y_c \end{pmatrix} u + \begin{pmatrix} x_b - x_c \\ y_b - y_c \end{pmatrix} v$$

------------------------ (24d)

which can map an arbitrary triangle ΔABC , A(x_a ,y_a) ,B(x_b ,y_b) , C(x_c , y_c) in xy – plane into a right isosceles triangle in the uv – plane

Hence, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (x_a - x_c) & (x_b - x_c) \\ (y_a - y_c) & (y_b - y_c) \end{pmatrix}^{-1} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$$

------------------------- (25)

This gives

u = $(\propto_a + \beta_a x + \gamma_a y)/(2\,\Delta_{abc})$

v = $(\propto_b + \beta_b x + \gamma_b y)/(2\,\Delta_{abc})$

------------------------ (26)

$\propto_a = (x_b y_c - x_c y_b)$, $\quad \propto_b = (x_c y_a - x_a y_c)$,

$\beta_a = (y_b - y_c)$, $\quad \beta_b = (y_c - y_a)$,

$\gamma_a = (x_c - x_b)$, $\quad \gamma_b = (x_a - x_c)$,

$$\frac{\partial(x,y)}{\partial(u,v)} = 2\Delta_{abc} = \begin{vmatrix} 1 & x_a & y_a \\ 1 & x_b & y_b \\ 1 & x_c & y_c \end{vmatrix} = 2 * \text{area of the triangle } \Delta ABC$$

$$= (\gamma_b \beta_a - \gamma_a \beta_b)$$

------------------- (27)

Hence from Eq.(23) and Eq.(22), we obtain

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\beta_a}{2\Delta_{abc}} & \frac{\beta_b}{2\Delta_{abc}} \\ \frac{\gamma_a}{2\Delta_{abc}} & \frac{\gamma_b}{2\Delta_{abc}} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} \beta_a^* & \beta_b^* \\ \gamma_a^* & \gamma_b^* \end{pmatrix} \begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix}$$

----------------------------- (28)

where $\beta_a^* = \frac{\beta_a}{(2\Delta_{abc})}$, $\quad \beta_b^* = \frac{\beta_b}{(2\Delta_{abc})}$

$\gamma_a^* = \frac{\gamma_a}{(2\Delta_{abc})}$, $\quad \gamma_b^* = \frac{\gamma_b}{(2\Delta_{abc})}$

--------------------------- (29)

Letting,

$$D_{x,y}^{i,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} \quad , \quad P = \begin{pmatrix} \beta_a{}^* & \beta_b{}^* \\ \gamma_a{}^* & \gamma_b{}^* \end{pmatrix} \quad , \quad D_{u,v}^{i,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix}$$

---------------------------- (30)

We obtain from Eq.(24),

$$D_{x,y}^{i,e} = P\, D_{u,v}^{i,e}$$

----------------------------- (31)

So that from Eq.(30) and Eq.(31) , we obtain

$$G_{x,y}^{i,j,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{pmatrix} \left(\frac{\partial N_j^e}{\partial x} \quad \frac{\partial N_j^e}{\partial y} \right) = (D_{x,y}^{i,e})\,(D_{x,y}^{j,e})^T$$

$$= \begin{pmatrix} \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} & \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial y} \\ \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial x} & \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y} \end{pmatrix}$$

-

---------------------------- (32a)

$$G_{u,v}^{i,j,e} = \begin{pmatrix} \frac{\partial N_i^e}{\partial u} \\ \frac{\partial N_i^e}{\partial v} \end{pmatrix} \left(\frac{\partial N_j^e}{\partial u} \quad \frac{\partial N_j^e}{\partial v} \right) = (D_{u,v}^{i,e})\,(D_{u,v}^{j,e})^T$$

$$= \begin{pmatrix} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial u} & \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial v} \\ \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial u} & \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial v} \end{pmatrix}$$

-

---------------------------- (32b)

We have now from Eq.(27) and Eq.(28)

$$G_{x,y}^{i,j,e} = (P\, D_{u,v}^{i,e})\,\left(\left(D_{u,v}^{j,e}\right)^T P^T\right)$$

$$= P\, G_{u,v}^{i,j,e}\, P^T$$

--------------------------- (33)

We now define the submatrices of global derivative integrals in (x,y) and (u,v) space associated with the nodes i and j as ;

$$S^{i,j,e} = \iint_{Q_e} G_{x,y}^{i,j,e} \;\; dx\, dy\,, \qquad (e=1,2,3)$$

--------------------------- (34)

$$K^{i,j,e} = \iint_{\widehat{Q}} G_{u,v}^{i,j,e} \;\; du\, dv$$

--------------------------- (35)

where, we have already defined the quadrilaterals Q_e (e=1,2,3) in (x,y) space and $\widehat{Q}$ in (u,v) space in Fig 3a-b. From Eqs.(28)-(33) , we obtain the following relations connecting the submatrices $S^{i,j,e}$ and $K^{i,j,e}$ We now obtain the submatrices $S^{i,j,e}$ and $K^{i,j,e}$ in an explicit form from Eqs.(32a)- (32b) as

$$S^{i,j,e} = \iint_{Q_e} G_{x,y}^{i,j,e} \;\; dx\, dy = \begin{pmatrix} \iint_{Q_e} \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x}\, dxdy & \iint_{Q_e} \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial y}\, dxdy \\ \iint_{Q_e} \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial x}\, dxdy & \iint_{Q_e} \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}\, dxdy \end{pmatrix}$$

$$= \begin{pmatrix} S_{2i-1\,,2j-1}^{e} & S_{2i-1\,,2j}^{e} \\ S_{2i\,,2j-1}^{e} & S_{2i\,,2j}^{e} \end{pmatrix} \quad \text{(say)}$$

------------------(36)

and in similar manner

$$K^{i,j,e} = \iint_{\widehat{Q}} G_{u,v}^{i,j,e} \;\; du\, dv = \begin{pmatrix} \iint_{\widehat{Q}} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial u}\, dudv & \iint_{\widehat{Q}} \frac{\partial N_i^e}{\partial u}\frac{\partial N_j^e}{\partial v}\, dudv \\ \iint_{\widehat{Q}} \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial u}\, dudv & \iint_{\widehat{Q}} \frac{\partial N_i^e}{\partial v}\frac{\partial N_j^e}{\partial v}\, dudv \end{pmatrix}$$

$$= \begin{pmatrix} K^{e}_{2i-1,2j-1} & K^{e}_{2i-1,2j} \\ K^{e}_{2i,2j-1} & K^{e}_{2i,2j} \end{pmatrix} \quad \text{(say)}$$

------------------(37)

We now obtain from the above Eq.(23)-(33)

$$S^{i,j,e} = \iint_{Q_e} G^{i,j,e}_{x,y} \; dx\,dy = \iint_{\hat{Q}} (P\, G^{i,j,e}_{u,v} P^T)\, \frac{\partial(x,y)}{\partial(u,v)} \; du\,dv$$
$$= 2\Delta_{abc} \iint_{\hat{Q}} (P\, G^{i,j,e}_{u,v} P^T)\; du\,dv$$
$$= 2\Delta_{abc}\, P\,(\iint_{\hat{Q}} G^{i,j,e}_{u,v}\, du\,dv)\; P^T$$
$$= 2\Delta_{abc}\; P\,(K^{i,j,e})\; P^T$$

-----------------------(38)

We can thus obtain the global derivative integrals in the physical space or Cartesian space (x,y) by using the matrix triple product established in eqn.(33).

From Eq.(35) and noting the fact that $\hat{Q}$ is the quadrilateral in (u, v) space spanned by the vertices (1/3, 1/3), (0, 1/2), (0, 0)and (1/2, 0) we obtain

$$K^{i,j,e} = \iint_{\hat{Q}} G^{i,j,e}_{u,v} \; du\,dv$$
$$= \int_{-1}^{1}\int_{-1}^{1} G^{i,j,e}_{u,v}\, \frac{\partial(u,v)}{\partial(\xi,\eta)} \; d\xi\, d\eta$$

------------------------(39)

We now refer to section 5 of this paper, in this section we have derived the necessary relations to integrate the integrals of Eq.(39), As in Eq.(22a-d) , we use the transformation

$$u(\xi,\ \eta) = \tfrac{1}{3} N_1(\xi,\ \eta) + \tfrac{1}{2} N_4(\xi,\ \eta)$$
$$v(\xi,\ \eta) = \tfrac{1}{3} N_1(\xi,\ \eta) + \tfrac{1}{2} N_2(\xi,\ \eta)$$

-----------------------(40)

to map the quadrilateral $\hat{Q}$ to the 2-square $-1\leq \xi, \eta \leq 1$ Using Eq.(40) in Eq.(39), we obtain

$$K^{i,j,e} = \iint_{\hat{Q}} G^{i,j,e}_{u,v} \;\left(\frac{4+\xi+\eta}{96}\right) d\xi d\eta$$

----------------------- (41)

The submatrices for the quadrilateral Q_e is expressed from Eq.(38) as

$$S^{i,j,e} = (2\Delta_{abc})\; P\,(K^{i,j,e})\; P^T$$

---------------------- (42)

In eqn.(38) , $2\Delta_{abc} = 2$ x area of the triangle spanning vertices A(x_a ,y_a) ,B(x_b ,y_b) , C(x_c, y_c) which is scalar. The matrices P, P^T depend purely on the nodel coordinates (x_a ,y_a) ,(x_b ,y_b) , (x_c, y_c) the matrix $K^{i,j,e}$ can be explicity computed by the relations obtained in section 2 and 3 . We find that $K^{i,j,e}$ is a (2X2) matrix of integrals whose integrands are rational functions with polynomial numerator and the linear denominator $(4+\xi+\eta)$. Hence these integrals can be explicity computed. The explicit values of these integrals are expressible in terms of logarithmic constants. We have used symbolic mathematics software of MATLAB to compute the explicit values and their conversion to any number of digits can be obtained by using variable precision arithmetic (vpa) command. The matrix K^e as noted in Eq.(33) is of order ($2xn_{de}$) x ($2xn_{de}$) . We have computed K^e for the four noded isoparametric quadrilateral element. This is listed **in Table 1,1 and Table 1.2** ,

We may note that In order to compute the local/element stiffness matrices for the Poisson Boundary Value problem, we need to compute the integrals Eqns(12a-b)

$$K^{e}_{i,j} = \int_{\Omega^e} \nabla\varphi_i \,.\, \nabla\varphi_j \; d\mathbf{x} = \int_{\Omega^e} \{\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\}\, dxdy\ ,$$

,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,(43a)

from the above derivations, we can rewrite $K_{i,j}^{e}$ in the notations of this sections by taking $\varphi_{\,i} = \mathrm{N}_{\,i}$ and $\varphi_{\,j} = N_{\,j}$ *and* $\Omega^{e} = Q_{e}$ so that

$K_{i,j}^{e} = \int_{Q_e} \nabla \mathrm{N}_{\,i} \,.\, \nabla N_j \, d\mathbf{x} = \int_{Q_e} \{ \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \} \, dxdy = \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} + \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}}$
....................................(43b)

3.2 Computation of $K_{i,j}^{e}$

The explicit integration scheme explained above compute four derivative product integrals as given in eqn(36) and they are necessary to compute the stiffness matrix entries of plane stress/plane strain problems in elasticity and sevral other applications in continuum mechanics.But this computation requires matrix triple product as given in eqn (42).Since,we only need the sum of two of these integrals viz : $\mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} + \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}}$ .We now present an efficient method to compute this sum by using matrix product.

Let $F_{p.q}^{i,j} = \frac{\partial N_i}{\partial p}\frac{\partial N_j}{\partial q}$, $I_{p.q}^{i,j} = \int_{Q_e} F_{p.q}^{i,j} \, dpdq$, then we have from eqns(36-37) :

$$\mathrm{S}^{\mathrm{i,j,e}} = \iint_{\mathrm{Q_e}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{x\,,y}} \;\; \mathrm{dx\,dy} = \begin{pmatrix} \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{x}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{x}} \,\mathrm{dxdy} & \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{x}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{y}} \,\mathrm{dxdy} \\ \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{y}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{x}} \,\mathrm{dxdy} & \iint_{\mathrm{Q_e}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{y}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{y}} \,\mathrm{dxdy} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} & \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}} \\ \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}-1} & \mathrm{S}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}} \end{pmatrix} \text{ (say)}$$

$$= \begin{pmatrix} I_{x,x}^{i,j} & I_{x,y}^{i,j} \\ I_{y,x}^{i,j} & I_{y,y}^{i,j} \end{pmatrix}$$
....(44a)

$$\mathrm{K}^{\mathrm{i,j,e}} = \iint_{\widehat{\mathrm{Q}}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{u\,,v}} \;\; \mathrm{du\,dv} = \begin{pmatrix} \iint_{\widehat{\mathrm{Q}}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{u}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{u}} \,\mathrm{dudv} & \iint_{\widehat{\mathrm{Q}}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{u}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{v}} \,\mathrm{dudv} \\ \iint_{\widehat{\mathrm{Q}}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{v}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{u}} \,\mathrm{dudv} & \iint_{\widehat{\mathrm{Q}}} \frac{\partial \mathrm{N_i^e}}{\partial \mathrm{v}}\frac{\partial \mathrm{N_j^e}}{\partial \mathrm{v}} \,\mathrm{dudv} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}-1} & \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}-1\,,2\mathrm{j}} \\ \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}-1} & \mathrm{K}^{\mathrm{e}}_{2\mathrm{i}\,,2\mathrm{j}} \end{pmatrix} \text{ (say)}$$

$$= \begin{pmatrix} I_{u,u}^{i,j} & I_{u,v}^{i,j} \\ I_{v,u}^{i,j} & I_{v,v}^{i,j} \end{pmatrix}$$
....(44b)

Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$, $P^{T} = \begin{pmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{pmatrix}$
....(45)

From eqns(44a-b) and (45)

$\mathrm{S}^{\mathrm{i,j,e}} = \iint_{\mathrm{Q_e}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{x\,,y}} \;\; \mathrm{dx\,dy} = 2\Delta_{\mathrm{abc}} \, \mathrm{P} \, (\iint_{\widehat{\mathrm{Q}}} \mathrm{G}^{\mathrm{i,j,e}}_{\mathrm{u\,,v}} \mathrm{du\,dv}) \;\; \mathrm{P}^{\mathrm{T}}$

$$=2\Delta_{abc}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{pmatrix}\begin{pmatrix} I_{u,u}^{i,j} & I_{u,v}^{i,j} \\ I_{v,u}^{i,j} & I_{v,v}^{i,j}\end{pmatrix}\begin{pmatrix} P_{11} & P_{21} \\ P_{12} & P_{22}\end{pmatrix}$$

$$= 2\Delta_{abc}\begin{pmatrix} \{ P_{11}(P_{11}I_{u.u}^{i,j} + P_{12}I_{u,v}^{i,j}) + P_{12}(P_{11}I_{v,u}^{i,j} + P_{12}I_{v,v}^{i,j}) \} & \{ P_{11}(P_{21}I_{u,u}^{i,j} + P_{22}I_{u.v}^{i,j}) + P_{12}(P_{21}I_{v,u}^{i,j} + P_{22}I_{v.v}^{i,j})\} \\ \{ P_{21}(P_{11}I_{u,u}^{i,j} + P_{12}I_{u,v}^{i,j}) + P_{22}(P_{11}I_{u.u}^{i,j} + P_{12}I_{v,v}^{i,j}) \} & \{ P_{21}(P_{21}I_{u,u}^{i,j} + P_{22}I_{u,v}^{i,j}) + P_{22}(P_{21}I_{v,u}^{i,j} + P_{22}I_{v,v}^{i,j})\} \end{pmatrix}$$

....(46)

From eqn(43b) ,eqn(44a) and eqn(46) , we find

trace $(S^{i,j,e})$= trace$(\iint_{Q_e} G_{x,y}^{i,j,e})$=$(S_{2i-1,2j-1}^{e} + S_{2i,2j}^{e})$= $K_{i,j}^{e} = \int_{Q_e} \nabla N_i \,.\nabla N_j \, d\mathbf{x}$ =$\int_{Q_e}\{\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\}\,dxdy$

$$=(P_{11}^2 + P_{21}^2)I_{u,u}^{i,j}+(P_{11}\,P_{12} + P_{21}\,P_{22})\,(I_{u,v}^{i,j} + I_{v,u}^{i,j}) +=(P_{12}^2 + P_{22}^2)I_{v,v}^{i,j}$$
..........(47)

W e can obtain the above integraql $\int_{Q_e}\{\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\}\,dxdy$ by use of matrix operations which doesnot need the computation matrix triple product.This procedure is presented below

From eqn (44b) and eqn(45),let us do the following:

$$(P^T\,P)\,.*\begin{pmatrix} I_{u,u}^{i,j} & I_{u,v}^{i,j} \\ I_{v,u}^{i,j} & I_{v,v}^{i,j}\end{pmatrix} = \begin{bmatrix} (P_{11}^2 + P_{21}^2) & (\,P_{11}\,P_{12} + P_{21}\,P_{22}) \\ (\,P_{11}\,P_{12} + P_{22}\,P_{21}) & (P_{12}^2 + P_{22}^2)\end{bmatrix}$$ ....(48)

We observe from eqn(48) that sum of all the entries gives us the value of the integral i.e

$$\int_{Q_e}\{\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\}\,dxdy = sum(sum\left(\;(P^T\,P).*\begin{pmatrix} I_{u,u}^{i,j} & I_{u,v}^{i,j} \\ I_{v,u}^{i,j} & I_{v,v}^{i,j}\end{pmatrix}\;\right))$$
............................(49)

Where,sum is aq Matlab function.We note that S=sum(**X**) gives the sum of the elements of vector X. If X is a matrix then S is a row vector with the sum over each column.It is clear that sum(sum(**X**)) gives the sum of all the entries in a matrix **X .**

3.2 Computing of Force Vector Integrals $\int_{\Omega^e} f\varphi_i\ \mathrm{dxdy}$

We shall now propose numerical integration for the complicated integrands in the force vector integrals over the domain Ω^e which is an arbitrary linear triangle and $\phi(x,y) = f\varphi_i$ .We also refer to the section 2 for the theory necessary to derive the composite numerical integration formla

We shall now establish a composite integration formula for an arbitrary linear triangular region Δ PQR shown in Fig 2a or Fig 3a. We have for an arbitrary smooth function ϕ(x, y)

II $_{\Delta PQR}$ = $\iint_{\Delta PQR}\phi(x,y)\,dxdy = \sum_{e=1}^{3}\iint_{Q_e}\phi(x,y)\,dxdy$
--------------------- (50)

$= \iint_{\widehat{Q}} \sum_{e=1}^{3}[\phi\,(x^{(e)}(u,v), y^{(e)}(u,v))\frac{\partial\left(x^{(e)}(u,v),y^{(e)}(u,v)\right)}{\partial(u,v)}]\,dudv$

$= (2\,\Delta_{pqr})\ \iint_{\widehat{Q}} \{\sum_{e=1}^{3}[\phi\,(x^{(e)}(u,v),\ y^{(e)}(u,v))\,]\}\,dudv$

-------------------- (51)

Where $(x^{(e)}(u,v),\ y^{(e)}(u,v)), e = 1,2,3)$ are the transformations of Eqs.(8)–(10) and $\widehat{Q}$ is the quadrilateral in uv- plane spanned by vertices G(1/3 ,1/3) , E(0, ½) , C(0, 0) and F(1/2,0) , and Δ_{pqr} is the area of triangle Δ PQR, Now using the transformations defined in Eqs.(1)–(2) we obtain

$II_{\Delta PQR} = (2\,\Delta_{pqr})\ \iint_{\widehat{Q}} \{\sum_{e=1}^{3}[\phi\,(x^{(e)}(u,v),\ y^{(e)}(u,v))\frac{\partial(u,v)}{\partial(\xi,\eta)}\}\,d\xi\,d\eta$

--------------- (52)

In Eq.(14) we have used the transformation

$u(\xi,\ \eta) = \frac{1}{3}N_1(\xi,\ \eta) + \frac{1}{2}N_4(\xi,\ \eta)$

$v(\xi,\ \eta) = \frac{1}{3}N_1(\xi,\ \eta) + \frac{1}{2}N_2(\xi,\ \eta)$

---------------------- (53)

to map the quadrilateral $\widehat{Q}$ into a 2 – square in $\xi\eta$ – plane.

We can now obtain from Eqs.(14)–(15)

$II_{\Delta PQR} = (2\,\Delta_{pqr})\int_{-1}^{1}\int_{-1}^{1}\,[\sum_{e=1}^{3}\left(\frac{4+\xi+\eta}{96}\right)\phi(x^{(e)}(u,v),\ y^{(e)}(u,v))\,]d\xi\,d\eta$

------------- (54)

We can evaluate Eq.(16) either analytically or numerically depending on the form of the integrand.

Using Numerical Integration ;

$$II_{\Delta PQR}=2\Delta_{pqr}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{W_i^{(N)}W_j^{(N)}(4+\xi_i^{(N)}+\eta_j^{(N)})}{96}\right)\sum_{e=1}^{3}\phi(x^{(e)}\left(u_{i,j}^{(N)},v_{i,j}^{(N)}\right),y^{(e)}\left(u_{i,j}^{(N)},v_{i,j}^{(N)}\right))$$

---------------------- (55)

Where,

$u_{i,j}^{(N)} = u(\xi_i^{(N)},\ \eta_j^{(N)})$ and $v_{i,j}^{(N)} = v(\xi_i^{(N)},\ \eta_j^{(N)})$

---------------------- (56)

and $(W_i^{(N)},\ \xi_i^{(N)})$, $(W_j^{(N)},\ \xi_j^{(N)})$ are the weight coefficients and sampling points of N^{th} order Gauss Legendre Quadrature rules.

The above composite rule is applied to numerical Integration over polygonal domains using convex quadrangulation and Gauss Legendre Quadrature Rules[27].

The above method will help in integrating $\int_{\Omega^e} f\varphi_i$ dxdy,when the intgrand $f\varphi_i$ is complicated

4.0 A NEW APPROACH TO MESH GENERATION

The first step in implimenting finite element method isto generate a mesh.In a recent work the author and his co-workers have proposed a new approach to mesh generation which can discretise a convex polygon into an all quadrilateral mesh.This will be presented next.This new approach to mesh generation meets the necessary requirements of regularity on the shape of elements.There are two types of them which usually suffice in finite element computations.The first is called shape regularity. It says that the ratio of the diameter of the element to the radius of the inner circle must be less than some constant. For triangles,the diameter of the triangle is related to the smallest circle which contain the triangle.The inner circle refers to the largest circle which fits inside the triangle. Shape regularity focuses on the shape of individual triangles and doesnot refer to how the shapes of different elements relate to each other. So some elements can be large wshile others might be very small. There is a second type of requirement on the shape of elements.This requirement says that ratio of the maximum diameter of elements to the radius of the inner circle of an element must be less than some constant .If a mesh satisfies this requirement,it is called quasiuniform.This requirement is more important when we perform refinements.We must note that a mesh generation gives us the nodes on a particular element as well as the coordinates of the nodes.We now give an account of this novel mesh generation technique with an aim to use it further in the solution of Poisson problem. Stated in eqn(7a-b).

In our recent paper[ ], the explicit finite element integration scheme is presented by using the isoparametric transformation over the 4 node linear convex quadrilateral element which is applied to torison of square shaft, on considering symmetry of the problem domain, mesh generation for 1/8 of the cross section which is a triangle was discritised into an all quadrilateral mesh. **In this paper we consider applications to polygonal domains**.

4.1 An automatic indirect quadrilateral mesh generator

A wide range of problems in applied science and engineering can be simulated by partial derivative equations(PDE).In the last few decade,one of the most relevant techniques to solve is the Finite Element Method(FEM).It is well known that a good quality mesh is required in order to obtain an accurate solution.Hence the construction of a mesh is on e of the most important steps.

In the next few sections , we present a novel mesh generation scheme of all quadrilateral elements for convex polygonal domains. This scheme converts the elements in background triangular mesh into quadrilaterals through the operation of splitting. We first decompose the convex polygon into simple subregions in the shape of triangles. These simple subregions are then triangulated to generate a fine mesh of triangles. We propose then an automatic triangular to quadrilateral conversion scheme in which each isolated triangle is split into three quadrilaterals according to the usual scheme, adding three vertices in the middle of edges and a vertex at the barrycentre of the triangular element. Further, to preserve the mesh conformity a similar procedure is also applied to every triangle of the domain and this fully discretizes the given convex polygonal domain into all quadrilaterals, thus propogating uniform refinement. In section 4.2, we present a scheme to discretize the arbitrary and standard triangles into a fine mesh of six node triangular elements. In section 4.3, we explain the procedure to split these triangles into quadrilaterals. In section 4.4,we have presented a method of piecing together of all triangular subregions and eventually creating a all quadrilateral mesh for the given convex polygonal domain. In section 4.5,we present several examples to illustrate the simplicity and efficiency of the proposed mesh generation method for standard and arbitrary triangles,rectangles and convex polygonal domains.

4.2 Division of an Arbitrary Triangle

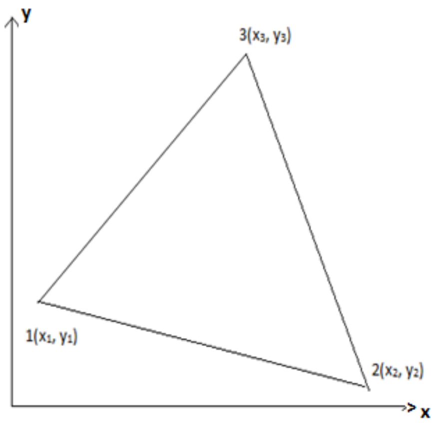
We can map an arbitrary triangle with vertices $((x_i, y_i),\ i = 1, 2, 3)$ into a right isosceles triangle in the (u, v) space as shown in Fig. 4a, b. The necessary transformation is given by the equations.

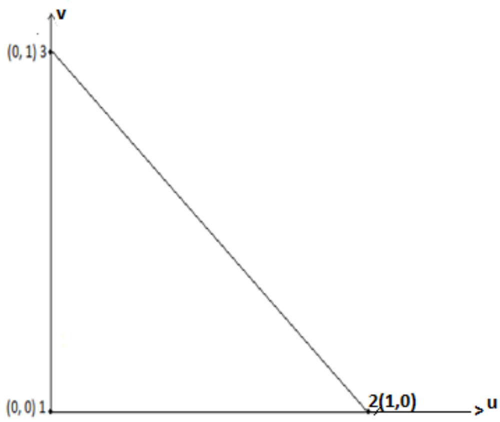
$$x = x_1 + (x_2 - x_1)u + (x_3 - x_1)v$$

$$y = y_1 + (y_2 - y_1)u + (y_3 - y_1)v$$

(57)

The mapping of eqn.(1) describes a unique relation between the coordinate systems. This is illustrated by using the area coordinates and division of each side into three equal parts in Fig. 5a Fig. 5b. It is clear that all the coordinates of this division can be determined by knowing the coordinates $((x_i, y_i),\ i = 1, 2, 3)$ of the vertices for the arbitrary triangle. In general , it is well known that by making 'n' equal divisions on all sides and the concept of area coordinates, we can divide an arbitrary triangle into n^2 smaller triangles having the same area which equals Δ/n^2 where Δ is the area of a linear arbitrary triangle with vertices $((x_i, y_i),\ i = 1, 2, 3)$ in the Cartesian space.



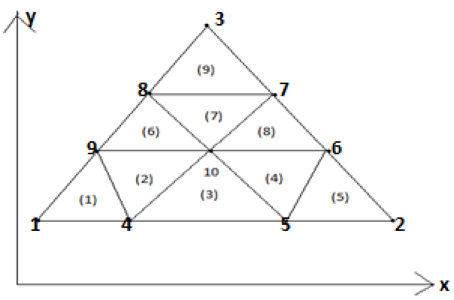


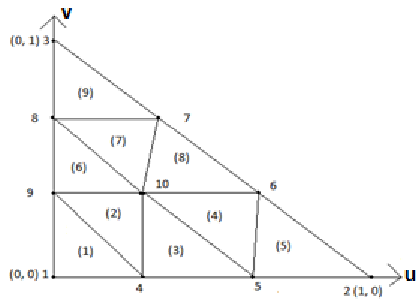
4.a

4 b

Fig. 4a An Arbitrary Linear Triangle in the (x, y) space

Fig. 4b A Right Isosceles Triangle in the (u, v) space





5a

5b

Fig. 5a Division of an arbitrary triangle into Nine triangles in Cartesian space

Fig. 5b Division of a right isosceles triangle into Nine right isosceles triangles in (u, v) space

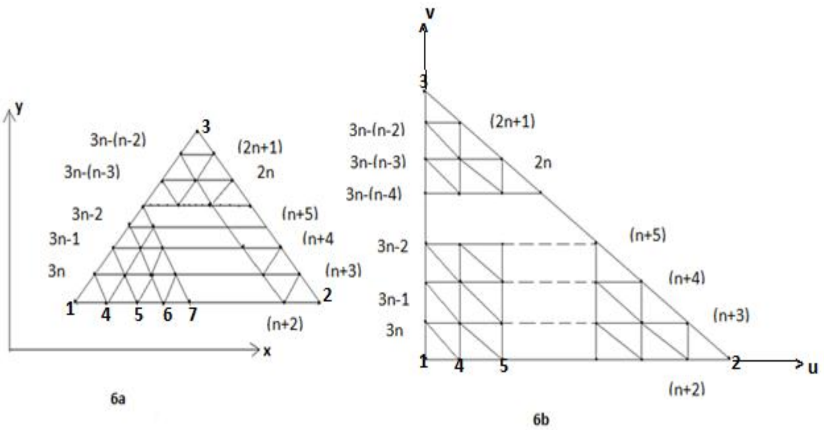


Fig.6a Division of an arbitrary triangle into n^2 triangle in Cartesian space (x, y), where each side is divided into n divisions of equal length

Fig.6b Division of a right isosceles triangle into n^2 right isosceles triangle in (u, v) space, where each side is divided into n divisions of equal length

We have shown the division of an arbitrary triangle in Fig. 6a , Fig. 6b, We divided each side of the triangles (either in Cartesian space or natural space) into n equal parts and draw lines parallel to the sides of the triangles. This creates (n+1) (n+2) nodes. These nodes are numbered from triangle base line l_{12} (letting l_{ij} as the line joining the vertex (x_i, y_i) and (x_j, y_j)) along the line $v = 0$ and upwards up to the line $v = 1$. The nodes 1, 2, 3 are numbered anticlockwise and then nodes 4, 5, ------, (n+2) are along line $v = 0$ and the nodes (n+3), (n+4), ------, 2n, (2n+1) are numbered along the line l_{23} i.e. $u + v = 1$ and then the node (2n+2), (2n+3), -------, 3n are numbered along the line $u = 0$. Then the interior nodes are numbered in increasing order from left to right along the line $v = \frac{1}{n}, \frac{2}{n}, ---, \frac{n-1}{n}$ bounded on the right by the line $+v = 1$. Thus the entire triangle is covered by (n+1) (n+2)/2 nodes. This is shown in the $\underline{rr}$ matrix of size $(n + 1) \times (n + 1)$, only nonzero entries of this matrix refer to the nodes of the triangles

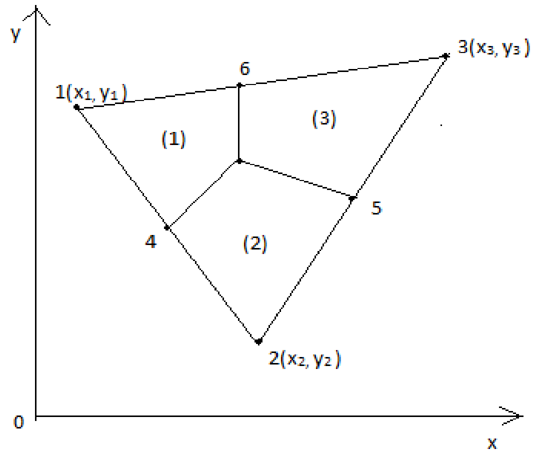
$$\underline{rr} = \begin{bmatrix} 1, & 4, & 5, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & (n+2) & 2 \\ 3n, & (3n+1), \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & 3n+(n-2), & (n+3) & 0 \\ 3n-1, & 3n+(n-1)\dots\dots\dots\dots, & 3n+(n-2)+(n-3), & (n+4) \quad 0 & 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & & & & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & & & & \\ 3n-(n-3), & \frac{(n+1)(n+2)}{2}, & 2n \quad 0\dots\dots\dots\dots\dots\dots\dots\dots\dots & & 0 \\ 3n-(n-2), & (2n+1), & 0 \quad 0\dots\dots\dots\dots\dots\dots. & & 0 \\ 3 & 0 & 0 \quad 0\dots\dots\dots\dots\dots\dots\dots\dots\dots & & 0 \end{bmatrix}$$

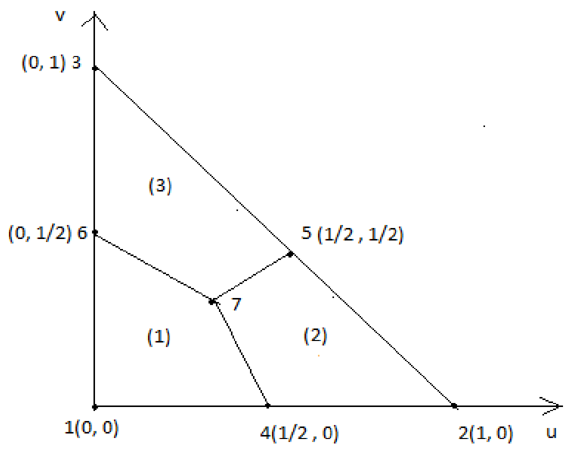
……………………………………………(58)

4.3. Quadrangulation of an Arbitrary Triangle

We now consider the quadrangulation of an arbitrary triangle. We first divide the arbitrary triangle into a number of equal size six node triangles. Let us define l_{ij} as the line joining the points (x_i, y_i) and (x_j, y_j) in the Cartesian space (x, y). Then the arbitrary triangle with vertices at $((x_i, y_i), i = 1,2,3)$ is bounded by three lines l_{12}, l_{23}, and l_{31} . By dividing the sides l_{12}, l_{23}, l_{31} into $n = 2m$ divisions (m, an integer) creates m^2 six node triangular divisions. Then by joining the centroid of these six node triangles to the midpoints of their sides, we obtain three quadrilaterals for each of these triangle. We have illustrated this process for the two and four divisions of l_{12}, l_{23}, and l_{31} sides of the arbitrary and standard triangles in Figs. 4 and 5

Two Divisions of Each side of an Arbitrary Triangle



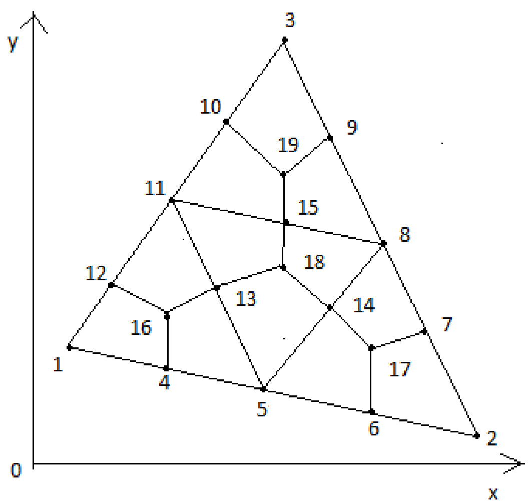


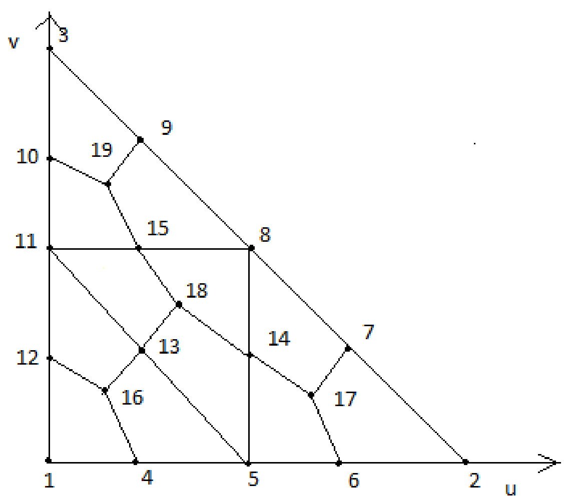
7(a) 7(b)

Fig 7(a). Division of an arbitrary triangle into three quadrilaterals

Fig 7(b). Division of a standard triangle into three quadrilaterals

Four Divisions of Each side of an Arbitrary Triangle





8a 8b

Fig 8a. Division of an arbitrary triangle into 4 six node triangles

Fig 8b. Division of a standard triangle into 4 right isosceles triangle

In general, we note that to divide an arbitrary triangle into equal size six node triangle, we must divide each side of the triangle into an even number of divisions and locate points in the interior of triangle at equal spacing. We also do similar divisions and locations of interior points for the standard triangle. Thus n (even) divisions creates ($n/2)^2$ six node triangles in both the spaces. If the entries of the sub matrix $\underline{rr}\,(i\,;\, i+2,\ j\,;\, j+2)$ are nonzero then two six node triangles can be formed. If $\underline{rr}\,(i+1,\ j+2) = \underline{rr}\ (i+2,\ j+1;\ j+2) = 0$ then one six node triangle can be formed. If the sub matrices $\underline{rr}\,(i\,;\, i+2,\ j\,;\, j+2)$ is a (3×3) zero matrix , we cannot form the six node triangles. We now explain the creation of the six node triangles using the $\underline{rr}$ matrix_of eqn.( ). We can form six node triangles by using node points of three consecutive rows and columns of $\underline{rr}$ matrix. This procedure is depicted in Fig. 9 for three consecutive rows $i\,, i+1,\ i+2$ and three consecutive columns $j\,,\ j+1,\ j+2$ of the $\underline{rr}$ sub matrix

Formation of six node triangle using sub matrix $\underline{rr}$

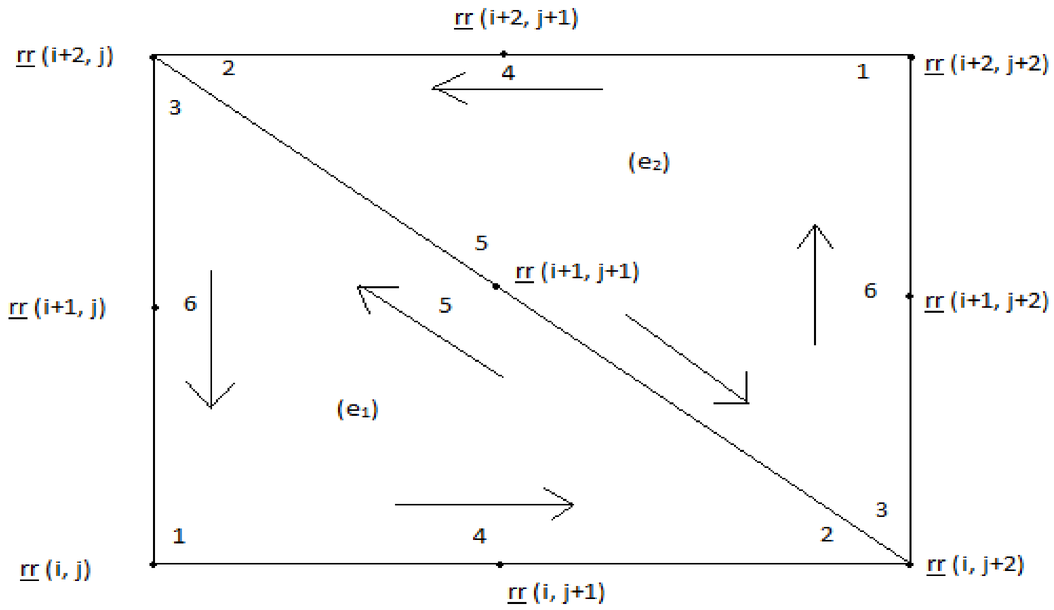


Fig. 9 Six node triangle formation for non zero sub matrix $\underline{rr}$

If the sub matrix ($(\underline{rr}\ (\,k, l), k = i, i+1, i+2), l = j,\ j+1,\ j+2)$ is nonzero, then we can construct two six node triangles. The element nodal connectivity is then given by

(e_1) $<\underline{rr}\ \ (i,\ j), \underline{rr}\ (i, i+2), \underline{rr}\ (i+2,\ j), \underline{rr}\ (i, j+1), \underline{rr}\ (i+1, j+1), \underline{rr}\ (i+1, j) >$

(e_2) $<\underline{rr}\ \ (i+2, j+2), \underline{rr}\ (i+2, j), \underline{rr}\ (i, j+2), \underline{rr}\ (i+2, j+1), \underline{rr}\ (i+1, j+1), \underline{rr}\ \ i+1, j+2) >$ ....(59)

If the elements of sub matrix ($(\underline{rr}\ (\ k, l), k = i, i+1, i+2),\ l = j,\ j+1,\ j+2)$ are nonzero, then as standard earlier, we can construct two six node triangles. We can create three quadrilaterals in each of these six node triangles. The nodal connectivity for the 3 quadrilaterals created in (e_1) are given as

$Q_{3n_1-2} < c_1, \underline{rr}\ (i+1,\ j),\ \underline{rr}\,(i,\ j),\ \underline{rr}\,(i,j+1) >$

$Q_{3n_1-1} < c_1, \underline{rr}\ (i,\ j+1),\ \underline{rr}\,(i,\ j+2),\ \underline{rr}\,(i+1,j+1) >$

$Q_{3n_1} < c_1, \underline{rr}\ (i+1,\ j+1),\ \underline{rr}\,(i+2,\ j),\ \underline{rr}\,(i+1,j) >$ ....(60)

and the nodal connectivity for the 3 quadrilaterals created in (e_2) are given as

$Q_{3n_2-2} < c_2, \underline{rr}\ (i+1,\ j+2),\ \underline{rr}\,(i+2,\ j+2),\ \underline{rr}\,(i+2,\ j+1) >$

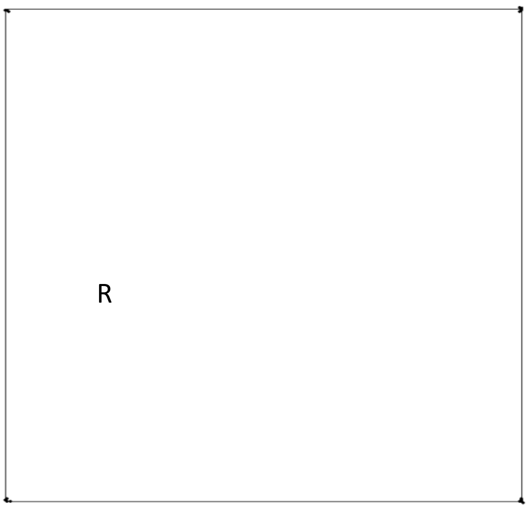
$Q_{3n_2-1} < c_2, \underline{rr}\ (i+2,\ j+1),\ \underline{rr}\,(i+2,\ j),\ \underline{rr}\,(i+1,\ j+1) >$

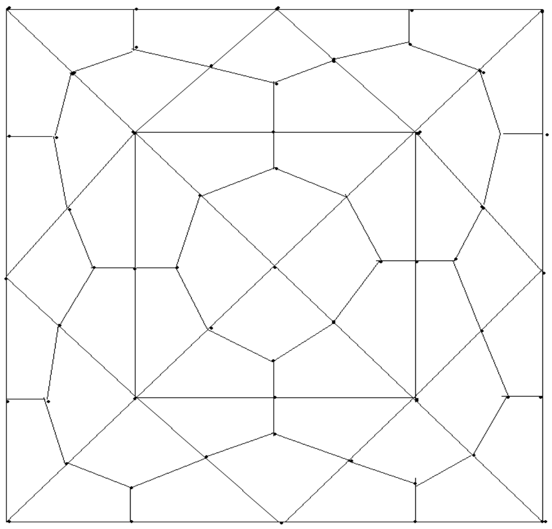
$Q_{3n_1} < c_2, \underline{rr}\ (i+1,\ j+1),\ \underline{rr}\,(i,\ j+2),\ \underline{rr}\,(i+1,j+2) >$ -------------------- (61)

4.4 Quadrangulation of the Polygonal Domain

We can generate polygonal meshes by piecing together triangular with straight sides. Subsection (called LOOPs). The user specifies the shape of these LooPs by designating six coordinates of each LOOP

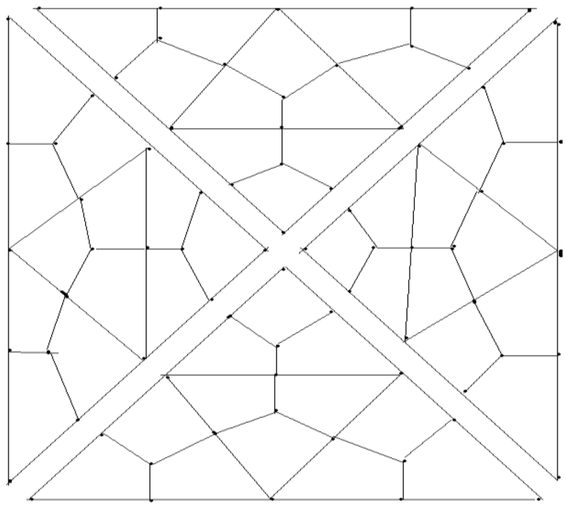
As an example, consider the geometry shown in Fig. 8(a). This is a a square region which is simply chosen for illustration. We divide this region into four LOOPs as shown in Fig.8(d). These LOOPs 1,2,3 and 4 are triangles each with three sides. After the LOOPs are defined, the number of elements for each LOOP is selected to produce the mesh shown in Fig. 8(c).The complete mesh is shown in Fig.8(b)



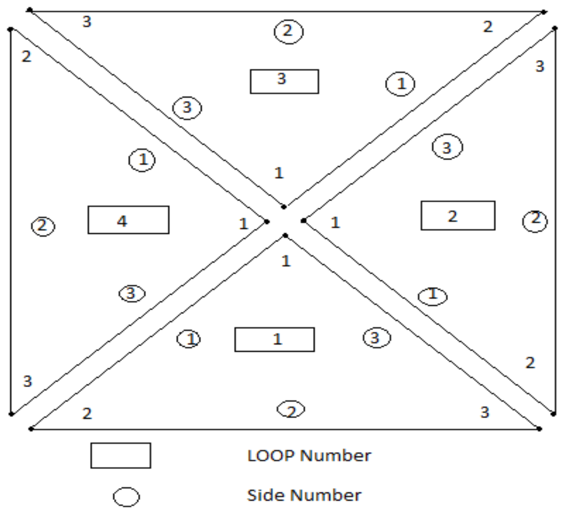


10a 10b

(i)Fig.10a: Region R to be analyzed (ii) Fig.10b: Example of completed mesh



10c



10d

(iii)Fig.10c:Exploded view showing four loops (iv)Fig.10d:Example of a loop and side numbering scheme

How to define the LOOP geometry, specify the number of elements and piece together the LOOPs will now be explained

Joining LOOPs : A complete mesh is formed by piecing together LOOPs. This piecing is done sequentially thus, the first LOOP formed is the foundation LOOP, with subsequent LOOPs joined either to it or to other LOOPs that have already been defined. As each LOOP is defined, the user must specify for each of the three sides of the current LOOP.

In the present mesh generation code, we aim to create a convex polygon. This requires a simple procedure. We join side 3 0f LOOP 1 to side 1 of LOOP 2, side 3 of LOOP 2 will joined to side 1 of LOOP 3, side 3 of LOOP 3 will be joined to side 1 of LOOP 4. Finally side 3 of LOOP 4 will be joined to side 1 of LOOP 1.

When joining two LOOPs, it is essential that the two sides to be joined have the same number of divisions. Thus the number of divisions remains the same for all the LOOPs. We note that the sides of LOOP (i) and side of LOOP $(i+1)$ share the same node numbers. But we have to reverse the sequencing of node numbers of side 3 and assign them as node numbers for side 1 of LOOP $(i+1)$. This will be required for allowing the anticlockwise numbering for element connectivity

4.5. Application Programs

4.5.1 Mesh Generation Over an Arbitrary Triangle

In applications to boundary value problems due to symmetry considerations, we may have to discretize an arbitrary triangle. Our purpose is to have a code which automatically generates convex quadrangulations of the domain by assuming the input as coordinates of the vertices. We use the theory and procedure developed in section 2 and section 3 of this paper for this purpose. The following MATLAB codes are written for this purpose.

(1) polygonal_domain_coordinates. m
(2) nodaladdresses_special_convex_quadrilaterals. m
(3) generate_area_coordinate_over_standard_triangle. m
(4) quadrilateral_mesh4arbitrarytriangle_q4. m

4.5.2 Mesh Generation over a Convex Polygonal Domain

In several physical applications in science and engineering, the boundary value problem require meshes generated over convex polygons. Again our aim is to have a code which automatically generates a mesh of convex quadrilaterals for the complex domains such as those in [21,22]. We use the theory and procedure developed in sections 2, 3 and 4 for this purpose. The following MATLAB codes are written for this purpose.

(1) quadrilateral_mesh4MOINEX_q4. m
(2) nodaladdresses_special_convex_quadrilaterals_trial. m
(3) polygonal_domain_coordinates. m
(4) generate_area_coordinate_over_standard_triangle. m
(5) quadrilateral_mesh4convexpolygonsixside_q4. m

5.0 Application Examples

Let us use the explicit integration scheme and the auto mesh generation techniques which are developed in the previous sections to solve the Poisson Equation with Dirichlet boundary value problem:

$-\Delta u = f,\ \boldsymbol{x\epsilon\Omega} \subset \boldsymbol{\mathcal{R}^2}$ ....(1)

$\boldsymbol{u = g},$ $\boldsymbol{x\epsilon\partial\Omega}$ ....(2)

Where $\boldsymbol{\Omega}$ is a polygonal domain and Δ is the standard standard Laplace operator

5.1 POISSON EQUATION OVER A TRIANGULAR DOMAIN

In a recent paper[26] a new approach to automatic generation of all quadrilateral mesh for finite analysis is proposed and it was applied to discretise the 1/8-th of the square cross section a triangular region into an all quadrilateral mesh. We have demonstrated the proposed explicit integration scheme to solve the St. Venant Torsion problem for a square cross section. Monotonic convergence from below is observed with known analytical solutions for the Prandtl stress function and the torisonal constant which are expessed in terms of infinite series

The following MATLAB PROGRAMS were used for this purpose:

(1) D2LaplaceEquationQ4Ex3automeshgen.m
(2)coordinate_rtisoscelestriangle00_h0_hh.m
(3)nodaladdresses4special_convex_quadrilaterals.m
(4)quadrilateralmesh_square_cross_section_q4.m
We are not listing these programs since they are already presented in [28]

5.2 POISSON EQUATION OVER A LINEAR CONVEX POLYGONAL DOMAIN

Example 1

$$-\Delta \boldsymbol{u} = \boldsymbol{2\pi^2 sin(\pi x) sin(\pi y)}\ ,\ (\boldsymbol{x},\boldsymbol{y})\boldsymbol{\epsilon\Omega} \subset \boldsymbol{\mathcal{R}^2}$$

$\boldsymbol{u(x,0)=0, on\ \ y=0, 0\leq x\leq 1}$

$\boldsymbol{u(x,1)=0, on\ \ y=1, 0\leq x\leq 1}$,

$\boldsymbol{u(1,y)=0, on\ \ x=1, 0\leq y\leq 1/2}$,

$\boldsymbol{u(x,y)=sin(\pi x)sin(\pi y)}$ **, on the line x= 1 - 0.5t ,y=0..5+ 0.5t ,0$\leq t\leq 1$**
..(62)

Where Δ is a standard Laplace operator and Ω is a pentagonal domain joining the vertices {(0,0),(1,0),(1,0.5),(0.5,1),(0,1)}

The exact solution of the above boundary value problem is $u(x,y)=sin(\pi x)sin(\pi y)$.

Example 2

$-\Delta u = 2\pi^2 sin(\pi x)sin(\pi y)$, $(x,y)\epsilon\Omega\subset\mathcal{R}^2$

u =0 , on the boundary $\partial\Omega$
..(63)

Where Δ is a standard Laplace operator and Ω is a square domain $[0,1]^2$.

We have written the following codes to solve the Poisson Equations with Dirichlet Boundary Conditions over linear convex polygonal domains

(1)D2LaplaceEquationQ4MoinExautomeshgen.m
(2) polygonal_domain_coordinates.m
(3) nodaladdresses_special_convex_quadrilaterals_trial
(4) quadrilateral_mesh4MOINEX_q4.m
(5) D2PoissonEquationQ4Ex01_02_MeshgridContour

Conclusions:

This paper proposes the explicit integration scheme for a unique linear convex quadrilateral which can be obtained from an arbitrary linear triangle by joining the centroid to the midpoints of sides of the triangle. The explicit integration scheme proposed for these unique linear convex quadrilaterals is derived by using the standard transformations in two steps. We first map an arbitrary triangle into a standard right isosceles triangle by using a affine linear transformation from global (x, y) space into a local space (u, v). We discritise this standard right isosceles triangle in (u, v) space into three unique linear convex quadrilaterals. We have shown that any unique linear convex quadrilateral in (x, y) space can be mapped into one of the unique quadrilaterals in (u, v) space. We can always map these linear convex quadrilaterals into a 2-aquare in (ξ,η) space by an approximate transformation. Using these two mapping, we have established an integral derivative product relation between the linear convex quadrilaterals in the (x, y) space interior to the arbitrary triangle and the linear convex quadrilaterals of the local space (u, v) interior to the standard right isosceles triangle. Further , we have shown that the product of global derivative integrals $S^{i,j,e}$ in (x, y) space can be expressed as a matrix triple product P $(K^{i,j,e})$ P^T X (2 * area of the arbitrary triangle in (x, y) space) in which P is a geometric properties matrix and $K^{i,j,e}$ is the product of global derivative integrals in (u, v) space.We have shown that the explicit integration of the product of local derivative integrals in (u, v) space over the unique quadrilateral spanning vertices (1/3 ,1/3), (0, ½), (0, 0) , (1/2, 0) is now possible by use of symbolic processing capabilities in MATLAB which are based on Maple – V software package. The proposed explicit integration scheme is a useful technique for boundary value problems governed by either a single equation or a system of second order partial differential equations. The physical applications of such problems are numerous in science, and engineering, the examples of single equations are the well known Laplace and Poisson equations with suitable boundary conditions and the examples of the system of equations are plane stress, plane stress and axisymmetric stress analysis etc in linear elasticity. We have demonstrated the proposed explicit integration scheme to solve the Poisson Boundary Value Problem for a pentagonal and square domains. Monotonic convergence from below is observed with known analytical solutions for the governing unknown function of Poisson Boundary Value Problem.We have shown the solutions in Tables which list both the FEM and exact solutions.The graphical solutions of contour level curves are also displayed. We conclude that efficient scheme on explicit integration of stiffness matrix and a novel automesh generation technique developed in this paper will be useful for the solution of many physical problems governed by second order partial differential equations.

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TABLES

Table-1.1

Values of integrals of the product of global derivatives over the quadrilateral {(xk,yk),k=1,2,3,4}={(1/3,1/3),(0,1/2),(0,0),(1/2,0)}, in the interior of the standard triangle (see eqn (37))
IntJdnidnjuvrs=[Ke (2*i-1,2*j-1) Ke (2*i-1,2*j)
Ke (2*i,2*j-1) Ke (2*i,2*j)]
where, (i,j=1,2,3,4,5,6,7,8)

ANALYTICAL VALUES

IntJdn1dn1uvrs = [-11/2-34*log (2)+27*log (3), -1/2-20*log (2)+27/2*log (3);...
-1/2-20*log (2)+27/2*log (3), -11/2-34*log (2)+27*log (3)]

IntJdn1dn2uvrs = [11/3+68/3*log (2)-18*log (3), 5/6+40/3*log (2)-9*log (3);...
1/3+40/3*log (2)-9*log (3), 25/6+68/3*log (2)-18*log (3)]

IntJdn1dn3uvrs = [-7/3-34/3*log (2)+9*log (3), -2/3-20/3*log (2)+9/2*log (3);...
-2/3-20/3*log (2)+9/2*log (3), -7/3-34/3*log (2)+9*log (3)]

IntJdn1dn4uvrs = [25/6+68/3*log (2)-18*log (3), 1/3+40/3*log (2)-9*log (3);...
5/6+40/3*log (2)-9*log (3), 11/3+68/3*log (2)-18*log (3)]

IntJdn2dn1uvrs = [11/3+68/3*log (2)-18*log (3), 1/3+40/3*log (2)-9*log (3);...
5/6+40/3*log (2)-9*log (3), 25/6+68/3*log (2)-18*log (3)]

IntJdn2dn2uvrs = [-22/9-136/9*log (2)+12*log (3), -5/9-80/9*log (2)+6*log (3);...
-5/9-80/9*log (2)+6*log (3), -22/9-136/9*log (2)+12*log (3)]

IntJdn2dn3uvrs =[14/9+68/9*log (2)-6*log (3), 4/9+40/9*log (2)-3*log (3);...

-1/18+40/9*log (2)-3*log (3), 19/18+68/9*log (2)-6*log (3)]

___ ___

IntJdn2dn4uvrs = [-25/9-136/9*log (2)+12*log (3),   -2/9-80/9*log (2)+6*log (3);...
-2/9-80/9*log (2)+6*log (3), -25/9-136/9*log (2)+12*log (3)]

___ ___

IntJdn3dn1uvrs = [-7/3-34/3*log (2)+9*log (3), -2/3-20/3*log (2)+9/2*log (3);...
-2/3-20/3*log (2)+9/2*log (3),   -7/3-34/3*log (2)+9*log (3)]

___ ___

IntJdn3dn2uvrs = [14/9+68/9*log (2)-6*log (3), -1/18+40/9*log (2)-3*log (3);...
4/9+40/9*log (2)-3*log (3), 19/18+68/9*log (2)-6*log (3)]

___ ___

IntJdn3dn3uvrs =[-5/18-34/9*log (2)+3*log (3), 5/18-20/9*log (2)+3/2*log (3);...
5/18-20/9*log (2)+3/2*log (3), -5/18-34/9*log (2)+3*log (3)]

___ ___

IntJdn3dn4uvrs = [19/18+68/9*log (2)-6*log (3),   4/9+40/9*log (2)-3*log (3);...
-1/18+40/9*log (2)-3*log (3), 14/9+68/9*log (2)-6*log (3)]

___ ___

IntJdn4dn1uvrs = [25/6+68/3*log (2)-18*log (3),   5/6+40/3*log (2)-9*log (3);...
1/3+40/3*log (2)-9*log (3), 11/3+68/3*log (2)-18*log (3)]

___ ___

IntJdn4dn2uvrs = -25/9-136/9*log (2)+12*log (3),   -2/9-80/9*log (2)+6*log (3);...
-2/9-80/9*log (2)+6*log (3), -25/9-136/9*log (2)+12*log (3)

___ ___

IntJdn4dn3uvrs = [19/18+68/9*log (2)-6*log (3), -1/18+40/9*log (2)-3*log (3);...
4/9+40/9*log (2)-3*log (3), 14/9+68/9*log (2)-6*log (3)]

___ ___

IntJdn4dn4uvrs = [-22/9-136/9*log (2)+12*log (3),   -5/9-80/9*log (2)+6*log (3);...
-5/9-80/9*log (2)+6*log (3), -22/9-136/9*log (2)+12*log (3)]

___ ___

Table-1.2
Values of integrals of the product of global derivatives over the quadrilateral {(xk,yk),k=1,2,3,4}={(1/3,1/3),(0,1/2),(0,0),(1/2,0)}, in the interior of the standard triangle (see eqn ()).

IntJdnidnjuvrs=[K_e (2*i-1,2*j-1) K_e (2*i-1,2*j)
K_e (2*i,2*j-1)   K_e (2*i,2*j)
where, (i,j=1,2,3,4,5,6,7,8)

NUMERICAL VALUES IN VARIABLE PRECISION ARITHMETIC

___ ___

intJdn1dn1uvrs = vpa (sym (' .595527655000821147485729267330')), vpa (sym ('.468322285820574645491168269290'));...
vpa (sym (' .468322285820574645491168269290')), vpa (sym ('.595527655000821147485729267330')) ;

intJdn1dn2uvrs = vpa (sym (' -.397018436667214098323819511552')), vpa (sym ('.187785142786283569672554487139 5'));...
vpa (sym (' -.312214857213716430327445512860 4')),   vpa (sym ('.102981563332785901676180488448')) ;
intJdn1dn3uvrs = vpa (sym (' -.301490781666392950838090244223 5')),vpa (sym (' -.343892571393141784836277243570 0'));...
vpa (sym (' -.343892571393141784836277243570 0')), vpa (sym ('-.301490781666392950838090244223 5')) ;
intJdn1dn4uvrs = vpa (sym ('   .102981563332785901676180488448')), vpa (sym ('-.312214857213716430327445512860 4'));...
vpa (sym (' .187785142786283569672554487139 5')), vpa (sym ('-.397018436667214098323819511552')) ;

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intJdn2dn1uvrs = vpa (sym (' -.397018436667214098323819511552')), vpa (sym ('-.312214857213716430327445512860 4'));...
vpa (sym (' .187785142786283569672554487139 5')), vpa (sym (' .102981563332785901676180488448')) ;
intJdn2dn2uvrs = vpa (sym ('   .264678957778142732215879674369')), vpa (sym ('-.125190095190855713115036324760 0'));...
vpa (sym (' -.125190095190855713115036324760 0')), vpa (sym (' .264678957778142732215879674369')) ;
intJdn2dn3uvrs = vpa (sym (' .200993854444261967225393496149 1')), vpa (sym ('.229261714262094523224184829046 6'));...
vpa (sym (' -.270738285737905476775815170953 4')), vpa (sym ('-.299006145555738032774606503850 9')) ;
intJdn2dn4uvrs = vpa (sym (' -.686543755551906011174536589 65e-1')),   vpa (sym ('.208143238142477620218297008573 4'));...
vpa (sym ('   .208143238142477620218297008573 4')), vpa (sym ('-.686543755551906011174536589 65e-1')) ;

___ ___

intJdn3dn1uvrs = vpa (sym (' -.301490781666392950838090244223 5')), vpa (sym ('-.343892571393141784836277243570 0'));...
vpa (sym (' -.343892571393141784836277243570 0')),vpa (sym (' -.301490781666392950838090244223 5')) ;
intJdn3dn2uvrs = vpa (sym (' .200993854444261967225393496149 1')), vpa (sym ('-.270738285737905476775815170953 4'));...
vpa (sym (' .229261714262094523224184829046 6')), vpa (sym ('-.299006145555738032774606503850 9')) ;
intJdn3dn3uvrs = vpa (sym (' .399503072777869016387303251925 4')), vpa (sym ('.385369142868952738387907585476 8'));...
vpa (sym (' .385369142868952738387907585476 8')), vpa (sym ('.399503072777869016387303251925 4')) ;
intJdn3dn4uvrs = vpa (sym (' -.299006145555738032774606503850 9')), vpa (sym (' .229261714262094523224184829046 6'));...
vpa (sym (' -.270738285737905476775815170953 4')), vpa (sym ('.200993854444261967225393496149 1')) ;

___ ___

intJdn4dn1uvrs = vpa (sym ('   .102981563332785901676180488448')), vpa (sym ('.187785142786283569672554487139 5'));...
vpa (sym (' -.312214857213716430327445512860 4')), vpa (sym (' -.397018436667214098323819511552')) ;
intJdn4dn2uvrs = vpa (sym (' -.686543755551906011174536589 65e-1')), vpa (sym (' .208143238142477620218297008573 4'));...
vpa (sym ('   .208143238142477620218297008573 4')), vpa (sym ('-.686543755551906011174536589 65e-1')) ;
intJdn4dn3uvrs = vpa (sym (' -.299006145555738032774606503850 9')), vpa (sym ('-.270738285737905476775815170953 4'));...
vpa (sym (' .229261714262094523224184829046 6')), vpa (sym ('.200993854444261967225393496149 1')) ;
intJdn4dn4uvrs = vpa (sym ('   .264678957778142732215879674369')), vpa (sym ('-.125190095190855713115036324760 0'));...
vpa (sym (' -.125190095190855713115036324760 0')), vpa (sym (' .264678957778142732215879674369')) ;

___ ___

Table 2.1
MESH-1
POISS0N BOUNDARY VALUE PROBLEM(EXAMPLE-1 FOR PENTAGONAL DOMAIN)
FEM MODEL:NODES=561,FOUR NODE QUADRILATERAL ELEMENTS=525
SOLUTION AT ELEMENT CENTROIDS

NODE NUMBER	FEM computed values	anlytical(theoretical)-values
72	0.956342038675229	0.972789205831714
73	0.845359429347379	0.861281226008774
74	0.652090747638962	0.665465038884934
75	0.395488719079333	0.404508497187474
76	0.10137243109474	0.10395584540888
77	0.877999178111196	0.893582297554377
78	0.713344881963615	0.726905328038456
79	0.479270076623328	0.489073800366903
80	0.198918190409272	0.2033683215379
81	0.777943989024293	0.791153573830373
82	0.600715455836758	0.611281226008774
83	0.36496600816973	0.371572412738697
84	0.0936803517666488	0.0954915028125263
85	0.632991881150565	0.643582297554377
86	0.42576939486992	0.433012701892219
87	0.176949249337238	0.180056805991955
88	0.489712369140768	0.497260947684137
89	0.297770601494129	0.302264231633827
90	0.0764195999964257	0.0776797865924606
91	0.32979341875372	0.334565303179429

92	0.137109461997713	0.139120075745983
93	0.200797202034953	0.2033683215379
94	0.051472150671726	0.0522642316338267
95	0.0834695926100803	0.0845653031794291
96	0.0211681475304891	0.0217326895365599
151	0.957095948127395	0.972789205831713
152	0.77848546830983	0.791153573830373
153	0.489964234227698	0.497260947684137
154	0.200852141918208	0.2033683215379
155	0.0211556498393284	0.0217326895365599
156	0.879325082183277	0.893582297554377
157	0.633772253149379	0.643582297554377
158	0.330054818688145	0.334565303179429
159	0.0834824164438746	0.0845653031794291
160	0.848079326227668	0.861281226008774
161	0.602280861089591	0.611281226008774
162	0.298262585170787	0.302264231633827
163	0.0514833998552369	0.0522642316338269
164	0.716081327799552	0.726905328038456
165	0.426953546226785	0.433012701892219
166	0.137297150304503	0.139120075745983
167	0.656026449725016	0.665465038884933
168	0.366525035220783	0.371572412738697
169	0.0765669376010333	0.0776797865924607
170	0.482355040428158	0.489073800366903
171	0.177617097590093	0.180056805991955
172	0.399334528727753	0.404508497187474
173	0.0941635770206533	0.0954915028125265
174	0.200744159277264	0.2033683215379
175	0.102713486598107	0.10395584540888
230	0.973700800497692	0.989073800366903
231	0.894840879010582	0.908540960039796
232	0.728787664059357	0.739073800366903
233	0.491277621416967	0.497260947684137
234	0.20498984636605	0.2067727288213
235	0.942159832614301	0.956772728821301
236	0.834968017375505	0.847100670886274
237	0.646233818736246	0.654508497187474
238	0.394142083170019	0.397848471555116
239	0.8948373376725	0.908540960039796
240	0.822950457455464	0.834565303179429
241	0.670887810071097	0.678896579685477
242	0.453239511003202	0.4567727288213
243	0.834964654998739	0.847100670886274
244	0.740556048518765	0.75
245	0.574301670036781	0.579484103556456
246	0.728787208636643	0.739073800366903
247	0.670888136428741	0.678896579685477

248	0.548067032358607	0.552264231633827
249	0.646230956976273	0.654508497187474
250	0.574300467568855	0.579484103556456
251	0.4912804840465	0.497260947684137
252	0.453241238683383	0.4567727288213
253	0.394139967368799	0.397848471555116
254	0.204993904379209	0.2067727288213
309	0.957079301954746	0.972789205831713
310	0.848070107485286	0.861281226008774
311	0.656022072196632	0.665465038884933
312	0.399334598711217	0.404508497187474
313	0.102716847623164	0.10395584540888
314	0.879315377666895	0.893582297554377
315	0.716079638452058	0.726905328038456
316	0.482360203173582	0.489073800366903
317	0.200755109571035	0.2033683215379
318	0.778472067681015	0.791153573830373
319	0.602275496981583	0.611281226008774
320	0.366528811355639	0.371572412738697
321	0.094177651088305	0.0954915028125265
322	0.633769505637659	0.643582297554377
323	0.4269607448309	0.433012701892219
324	0.177636329503656	0.180056805991955
325	0.489956982035174	0.497260947684137
326	0.298268295408819	0.302264231633827
327	0.0765901145211773	0.0776797865924607
328	0.330060431566015	0.334565303179429
329	0.137321072311919	0.139120075745983
330	0.200852697720788	0.2033683215379
331	0.0515105604322022	0.0522642316338269
332	0.0835012965779403	0.0845653031794291
333	0.0211695074800534	0.0217326895365599
388	0.956319689209778	0.972789205831714
389	0.777932082629173	0.791153573830373
390	0.489706975750355	0.497260947684137
391	0.200792186361779	0.2033683215379
392	0.0211538529789274	0.0217326895365599
393	0.877976074528338	0.893582297554377
394	0.632976404366267	0.643582297554377
395	0.329779975672512	0.334565303179429
396	0.0834489379635179	0.0845653031794291
397	0.845337749394084	0.861281226008774
398	0.6007034220094	0.611281226008774
399	0.297757780172745	0.302264231633827
400	0.0514438839369268	0.0522642316338267
401	0.713323037635475	0.726905328038456
402	0.425751100573011	0.433012701892219
403	0.137082447494272	0.139120075745983
404	0.652074163807158	0.665465038884934

405	0.364952707344858	0.371572412738697
406	0.076394671616776	0.0776797865924606
407	0.479251185459368	0.489073800366903
408	0.176925740322997	0.180056805991955
409	0.395477919147725	0.404508497187474
410	0.0936639904444136	0.0954915028125263
411	0.198902634989885	0.2033683215379
412	0.101366735055029	0.10395584540888
467	0.955778706883997	0.972789205831713
468	0.845027463179336	0.861281226008774
469	0.651909382898348	0.665465038884934
470	0.3954024792368	0.404508497187474
471	0.101351933786852	0.10395584540888
472	0.877217721137085	0.893582297554377
473	0.71290933095817	0.726905328038456
474	0.479050374708885	0.489073800366903
475	0.198840514207396	0.2033683215379
476	0.776734963640895	0.791153573830373
477	0.600066342184399	0.611281226008774
478	0.36466122186493	0.371572412738697
479	0.0936108760292304	0.0954915028125263
480	0.63203978602147	0.643582297554376
481	0.425296682415595	0.433012701892219
482	0.176784238326969	0.180056805991955
483	0.488806658418007	0.497260947684137
484	0.297352826783137	0.302264231633827
485	0.0763253729667889	0.0776797865924605
486	0.329233007149343	0.334565303179429
487	0.136916347822103	0.139120075745983
488	0.200422266262704	0.2033683215379
489	0.0513885370060442	0.0522642316338267
490	0.0833225628720519	0.0845653031794291
491	0.0211288108859121	0.0217326895365599
537	0.955775381841167	0.972789205831713
538	0.776735262405017	0.791153573830373
539	0.488807456213092	0.497260947684137
540	0.200419456501963	0.2033683215379
541	0.0211147341590103	0.0217326895365599
542	0.877209593047908	0.893582297554377
543	0.632032908218026	0.643582297554376
544	0.329223357317644	0.334565303179429
545	0.0833027823700829	0.0845653031794291
546	0.845019169165609	0.861281226008774
547	0.600062074950928	0.611281226008774
548	0.297343349266752	0.302264231633827
549	0.0513608078751592	0.0522642316338267
550	0.712897321665431	0.726905328038456
551	0.425283390367404	0.433012701892219
552	0.136890797076387	0.139120075745983

553	0.651901321267705	0.665465038884934
554	0.364652091868112	0.371572412738697
555	0.076301257720936	0.0776797865924605
556	0.479037191042077	0.489073800366903
557	0.176762652224923	0.180056805991955
558	0.395396250660319	0.404508497187474
559	0.0935955275526634	0.0954915028125263
560	0.19882715001752	0.2033683215379
561	0.101347347819409	0.10395584540888

Table 2.2
MESH-2
POISS0N BOUNDARY VALUE PROBLEM(EXAMPLE-1 FOR PENTAGONAL DOMAIN)
FEM MODEL:NODES=2171 , FOUR NODE QUADRILATERAL ELEMENTS=2400
SOLUTION AT ELEMENT CENTROIDS

NODE NUMBER	FEM computed values	anlytical(theoretical)-values
237	0.989071978987411	0.993158937674856
238	0.960362622219687	0.96460205851448
239	0.90808192844253	0.912293475342785
240	0.833476136985861	0.837521199079693
241	0.738373338403803	0.742126371321759
242	0.625115541777433	0.628457929325666
243	0.496498468830464	0.499314767377287
244	0.355705208959811	0.357876818725374
245	0.206237640370143	0.207626755071376
246	0.0518773349737633	0.0522642316338267
247	0.968639760774418	0.972789205831714
248	0.924310799544386	0.928466175238718
249	0.85725935804802	0.861281226008774
250	0.769126822588465	0.772888674565986
251	0.662081590914475	0.665465038884934
252	0.53876364385553	0.541655445394692
253	0.402220704026274	0.404508497187474
254	0.255838102969519	0.257401207292766
255	0.103267318142764	0.10395584540888
256	0.940813487711602	0.944818029471471
257	0.889651556601135	0.893582297554377
258	0.816609962282716	0.820343603841875
259	0.723485606836393	0.726905328038456
260	0.612571939633487	0.615568230598259
261	0.486604177136443	0.489073800366903
262	0.348693215969287	0.350536750027629
263	0.202249719503307	0.2033683215379
264	0.0508902319092544	0.0511922900311449

265	0.897901952943134	0.90176944487161
266	0.832813597159249	0.836516303737808
267	0.74724247116605	0.750665354967537
268	0.64329543667892	0.646330533842485
269	0.523533912229346	0.526080909926171
270	0.390911848882088	0.392877428045034
271	0.248702736695724	0.25
272	0.100413989764671	0.100966742252535
273	0.849275746165037	0.852868157970561
274	0.779589709017032	0.782966426513139
275	0.690728758060215	0.693785463118374
276	0.584880473373356	0.587521199079693
277	0.464651307139934	0.466790213248601
278	0.333002341265291	0.334565303179429
279	0.193173772529779	0.194102284987398
280	0.0486036812663231	0.0488598243504264
281	0.787820610016442	0.791153573830373
282	0.70691013947738	0.709958163014307
283	0.608612496369594	0.611281226008774
284	0.495346785518136	0.497552516492827
285	0.36990053102777	0.371572412738697
286	0.235359296339763	0.236442963004803
287	0.0950279611967388	0.0954915028125263
288	0.723332792360579	0.72631001724706
289	0.640915737821986	0.643582297554377
290	0.542732465230097	0.545007445768716
291	0.431196616690832	0.433012701892219
292	0.309048918929775	0.31035574820837
293	0.179287751049827	0.180056805991955
294	0.0451031905215681	0.0453242676377401
295	0.649126770386328	0.65176944487161
296	0.558892750555525	0.56118014566465
297	0.454907204095813	0.4567727288213
298	0.339724210022336	0.341118051452597
299	0.216170417965923	0.217063915551223
300	0.0872758450180442	0.0876649456551453
301	0.575276561599788	0.577531999897402
302	0.487171969509519	0.489073800366903
303	0.387074827490653	0.388572980728485
304	0.277440146772026	0.278504204704746
305	0.160954321890377	0.161577730858712
306	0.0404828379708958	0.0406726770331925
307	0.495364857860055	0.497260947684137
308	0.403218822995754	0.404745680624419
309	0.301138460563558	0.302264231633827
310	0.191624346698157	0.192340034102939
311	0.0773588558290928	0.0776797865924606
312	0.419570528390531	0.421097534857171
313	0.333378501373647	0.334565303179429

314	0.238962881912389	0.239794963378827
315	0.138633143043895	0.139120075745983
316	0.0348590650860893	0.0350195901352117
317	0.341558156391989	0.342752450496663
318	0.255099178057986	0.255967663274761
319	0.16233182943102	0.162880102675064
320	0.0655247417628147	0.0657818933794382
321	0.271431178986264	0.272319517507514
322	0.194566841337604	0.195181174220666
323	0.112875982035243	0.113236822655327
324	0.0283710049494732	0.0285042047047456
325	0.202740339270173	0.2033683215379
326	0.129014817328174	0.12940952255126
327	0.0520652750137041	0.0522642316338267
328	0.145342320823199	0.145761376784013
329	0.0843147453141511	0.0845653031794291
330	0.0211776311300555	0.0212869511544169
331	0.0924894500141357	0.092752450496663
332	0.0373093941169535	0.0374596510503502
333	0.0536449353360819	0.0538115052831031
334	0.0134524856057618	0.013545542219326
335	0.0216125961456229	0.0217326895365599
336	0.00535971699375773	0.00547059707971809
546	0.989168513491059	0.993158937674856
547	0.940903714854474	0.944818029471471
548	0.849352565935386	0.852868157970561
549	0.723392154262675	0.72631001724706
550	0.575317371235676	0.577531999897402
551	0.419594779839078	0.421097534857172
552	0.27144304466582	0.272319517507513
553	0.145346563304387	0.145761376784013
554	0.0536454902684243	0.053811505283103
555	0.00535879860193477	0.00547059707971812
556	0.968828829558917	0.972789205831713
557	0.898070149434932	0.90176944487161
558	0.787957226205733	0.791153573830373
559	0.649226533640733	0.65176944487161
560	0.495428788097215	0.497260947684137
561	0.341592798498049	0.342752450496663
562	0.202755068033581	0.2033683215379
563	0.0924932706115278	0.092752450496663
564	0.0216118227124735	0.0217326895365599
565	0.960744191463744	0.96460205851448
566	0.889989505055692	0.893582297554377
567	0.779863854833083	0.782966426513139
568	0.641115407080801	0.643582297554377
569	0.487299266762048	0.489073800366903
570	0.333446848386558	0.334565303179429
571	0.194595383182974	0.195181174220666

572	0.0843217585683347	0.0845653031794291
573	0.0134510583220761	0.0135455422193261
574	0.924764890250526	0.928466175238718
575	0.833199357513632	0.836516303737808
576	0.707206982059819	0.709958163014308
577	0.559095165425583	0.561180145664649
578	0.403337314935977	0.404745680624419
579	0.255155515880115	0.255967663274761
580	0.129033597562986	0.12940952255126
581	0.0373107958648573	0.0374596510503503
582	0.908724525495235	0.912293475342785
583	0.817153607410241	0.820343603841875
584	0.691145450570353	0.693785463118375
585	0.543014328816039	0.545007445768716
586	0.387237337785808	0.388572980728485
587	0.239037927301339	0.239794963378828
588	0.112899176663778	0.113236822655327
589	0.0211780682242638	0.0212869511544171
590	0.857944675750211	0.861281226008774
591	0.747795347110956	0.750665354967537
592	0.609010561990531	0.611281226008774
593	0.455155314185785	0.456772728821301
594	0.301266130478194	0.302264231633827
595	0.162380450421002	0.162880102675064
596	0.0520734171283086	0.0522642316338269
597	0.834344222346749	0.837521199079693
598	0.724181742796426	0.726905328038456
599	0.585377237245962	0.587521199079693
600	0.431500639124576	0.433012701892219
601	0.277591095234343	0.278504204704746
602	0.138685929606064	0.139120075745983
603	0.0283758852605683	0.0285042047047458
604	0.769996827606085	0.772888674565986
605	0.643954264813862	0.646330533842485
606	0.495781056656018	0.497552516492828
607	0.339962063222099	0.341118051452596
608	0.19172300264612	0.192340034102939
609	0.0655458352140635	0.0657818933794384
610	0.739417013995797	0.742126371321759
611	0.61335441554072	0.615568230598259
612	0.465157457162821	0.466790213248601
613	0.309315317229022	0.31035574820837
614	0.161055064375459	0.161577730858712
615	0.0348718789288993	0.035019590135212
616	0.663074724368381	0.665465038884933
617	0.524225100027525	0.526080909926171
618	0.370300810203354	0.371572412738697
619	0.216346945511906	0.217063915551224
620	0.077401448523751	0.0776797865924607

621	0.62626755754782	0.628457929325666
622	0.487391262743002	0.489073800366903
623	0.333439837978658	0.334565303179429
624	0.179462547559523	0.180056805991955
625	0.0405084165397245	0.0406726770331929
626	0.53979977865085	0.541655445394692
627	0.3915475389022	0.392877428045034
628	0.235654298106104	0.236442963004804
629	0.0873521051275679	0.0876649456551455
630	0.497669492004695	0.499314767377287
631	0.349384233564127	0.350536750027629
632	0.193462813794467	0.194102284987398
633	0.0451485779269482	0.0453242676377405
634	0.403195346659082	0.404508497187474
635	0.249180805419627	0.25
636	0.0951567019777165	0.0954915028125265
637	0.356775454508344	0.357876818725374
638	0.202722933245784	0.2033683215379
639	0.0486803024276766	0.0488598243504268
640	0.256611858982057	0.257401207292766
641	0.100628925216211	0.100966742252535
642	0.207038060788549	0.207626755071376
643	0.0510204280436817	0.0511922900311454
644	0.103646894068313	0.10395584540888
645	0.0521215395701683	0.0522642316338272
855	0.993386367943437	0.997260947684137
856	0.972960531578743	0.976807083442103
857	0.928689473666576	0.932300986968877
858	0.86158268415691	0.864838546066896
859	0.773269718335622	0.776080909926171
860	0.665910897886715	0.668213586118192
861	0.542136321727311	0.543892626146237
862	0.404978416624184	0.406179224642423
863	0.257795871416631	0.258464342596354
864	0.104179999010112	0.104385210641588
865	0.985192111097925	0.989073800366903
866	0.956899773222374	0.96063438354617
867	0.905089180313217	0.908540960039796
868	0.831014152242658	0.834076242822669
869	0.736483052796474	0.739073800366903
870	0.623810359686408	0.625872908100787
871	0.495756282275672	0.497260947684137
872	0.355456214764886	0.356404772421034
873	0.206349993566172	0.2067727288213
874	0.972960132845576	0.976807083442103
875	0.953082905836338	0.956772728821301
876	0.909744270343944	0.913179454270281
877	0.844026297111487	0.847100670886274
878	0.757539756122328	0.760163457619103

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879	0.652403402633683	0.654508497187474
880	0.531192703141431	0.532737365365924
881	0.396876311463867	0.397848471555116
882	0.252725344177156	0.253163227991246
883	0.956899482893147	0.96063438354617
884	0.929474565092452	0.933012701892219
885	0.879176846094529	0.882417151026054
886	0.807250404195257	0.810093561327006
887	0.71546001927026	0.717822779601698
888	0.606055364865995	0.607876818725374
889	0.48171538581046	0.482962913144534
890	0.345497014547649	0.346156857780064
891	0.928689113213019	0.932300986968877
892	0.909744144703313	0.913179454270281
893	0.868410095495528	0.871572412738697
894	0.805714805693361	0.808504365822488
895	0.723201554075878	0.725528258147577
896	0.622894970053053	0.624687236866834
897	0.507254211528693	0.508464342596354
898	0.379087883987651	0.379721368714746
899	0.9050888628107	0.908540960039796
900	0.879176728355692	0.882417151026054
901	0.831636764203677	0.834565303179429
902	0.763644976983998	0.766163687805082
903	0.676873447232183	0.678896579685477
904	0.573449844043997	0.574912784645444
905	0.455926533549453	0.4567727288213
906	0.861582422508033	0.864838546066896
907	0.844026169598884	0.847100670886274
908	0.805714766090407	0.808504365822488
909	0.747597584902235	0.75
910	0.671106602234271	0.673028145070219
911	0.578122069212497	0.579484103556456
912	0.470896076315735	0.471671240215656
913	0.831013878796477	0.834076242822669
914	0.807250258521815	0.810093561327006
915	0.763644922222067	0.766163687805082
916	0.701276098373329	0.7033683215379
917	0.621678085074288	0.623253692848829
918	0.526827092956106	0.527792489781403
919	0.773269571240199	0.776080909926171
920	0.757539677087922	0.760163457619103
921	0.723201524495232	0.725528258147577
922	0.671106598329204	0.673028145070219
923	0.602541645552638	0.60395584540888
924	0.519162621738687	0.520012148419041
925	0.736482847719668	0.739073800366903
926	0.715459887326437	0.717822779601698
927	0.67687337574893	0.678896579685477

928	0.621678057487678	0.623253692848829
929	0.551257829267231	0.552264231633827
930	0.665910867564529	0.668213586118192
931	0.65240339019201	0.654508497187474
932	0.622894973966787	0.624687236866834
933	0.578122082792992	0.579484103556456
934	0.519162634792305	0.520012148419041
935	0.623810229361775	0.625872908100787
936	0.606055262579374	0.607876818725374
937	0.573449774559061	0.574912784645444
938	0.526827055044508	0.527792489781403
939	0.542136400368925	0.543892626146237
940	0.531192756091589	0.532737365365924
941	0.507254253752498	0.508464342596354
942	0.470896116276625	0.471671240215656
943	0.495756219620295	0.497260947684137
944	0.481715311157422	0.482962913144534
945	0.455926469598944	0.4567727288213
946	0.404978582659867	0.406179224642423
947	0.396876410586192	0.397848471555116
948	0.37908795300837	0.379721368714746
949	0.355456195266028	0.356404772421034
950	0.345496947395311	0.346156857780064
951	0.257796079836311	0.258464342596354
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2122	0.0520420024815229	0.0522642316338267
2123	0.833437455867595	0.837521199079693
2124	0.723375900332822	0.726905328038456
2125	0.584748921741657	0.587521199079693
2126	0.431076665620638	0.433012701892219
2127	0.277351154987251	0.278504204704746
2128	0.138582746098223	0.139120075745983
2129	0.0283578039433155	0.0285042047047456
2130	0.769064268017946	0.772888674565986
2131	0.64318602427363	0.646330533842485
2132	0.495230767674702	0.497552516492827
2133	0.339627750062283	0.341118051452596
2134	0.1915613306799	0.192340034102939
2135	0.0654990493187459	0.0657818933794383
2136	0.738344514152188	0.742126371321759

2137	0.612492481351269	0.615568230598259
2138	0.464559674860588	0.466790213248601
2139	0.30897111676614 8	0.31035574820837
2140	0.160905441817087	0.161577730858712
2141	0.0348452066404085	0.0350195901352117
2142	0.662035714968029	0.665465038884934
2143	0.52345638670232	0.526080909926171
2144	0.369822885437373	0.371572412738697
2145	0.21611322629087	0.217063915551223
2146	0.07733322846582	0.0776797865924605
2147	0.625094548132663	0.628457929325666
2148	0.48654866862622	0.489073800366903
2149	0.332942761604251	0.334565303179429
2150	0.179244990276529	0.180056805991955
2151	0.0404695757681872	0.0406726770331925
2152	0.53873099321708	0.541655445394692
2153	0.390859812700939	0.392877428045034
2154	0.235313172683076	0.236442963004803
2155	0.0872526537970532	0.0876649456551453
2156	0.496483733497188	0.499314767377287
2157	0.348657053794858	0.350536750027629
2158	0.193141003734482	0.194102284987398
2159	0.0450916846434457	0.0453242676377401
2160	0.402198695746848	0.404508497187474
2161	0.24867175889953	0.25
2162	0.0950092734087884	0.0954915028125263
2163	0.355695578416438	0.357876818725374
2164	0.20222981736586	0.2033683215379
2165	0.0485949089302559	0.0488598243504264
2166	0.255824940051354	0.257401207292766
2167	0.100401418193516	0.100966742252535
2168	0.206232335846344	0.207626755071376
2169	0.0508849180524333	0.0511922900311449
2170	0.103261923939942	0.10395584540888
2171	0.0518758872654712	0.0522642316338267

Table 3.1

POISS0N BOUNDARY VALUE PROBLEM(EXAMPLE-2 FOR A SQUARE DOMAIN)
FEM MODEL:NODES=643,FOUR NODE QUADRILATERAL ELEMENTS=600
SOLUTION AT ELEMENT CENTROIDS

NODE NUMBER	FEM computed values	anlytical(theoretical)-values
73	0.953482135754673	0.972789205831714
74	0.843161171273121	0.861281226008774

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75	0.650620087326386	0.665465038884934
76	0.394681801654656	0.404508497187474
77	0.101174463950614	0.10395584540888
78	0.875332012150061	0.893582297554377
79	0.711534774014504	0.726905328038456
80	0.478204029068659	0.489073800366903
81	0.1985066028557	0.2033683215379
82	0.775284911091137	0.791153573830373
83	0.599053315481233	0.611281226008774
84	0.364086776887519	0.371572412738697
85	0.0934682695292481	0.0954915028125263
86	0.63098601484579	0.643582297554377
87	0.424641288869316	0.433012701892219
88	0.176523669210805	0.180056805991955
89	0.488070174157289	0.497260947684137
90	0.296933348853692	0.302264231633827
91	0.0762209811250351	0.0776797865924606
92	0.328773598753415	0.334565303179429
93	0.136733242558403	0.139120075745983
94	0.200160059597621	0.2033683215379
95	0.0513233141659784	0.0522642316338267
96	0.0832180603536406	0.0845653031794291
97	0.021103084633879	0.0217326895365599
152	0.953475580166896	0.972789205831713
153	0.775284490266195	0.791153573830373
154	0.488070769846778	0.497260947684137
155	0.200157210243284	0.2033683215379
156	0.0210890068506871	0.0217326895365599
157	0.875321066900196	0.893582297554377
158	0.630978369734502	0.643582297554377
159	0.328763753767147	0.334565303179429
160	0.0831982579441417	0.0845653031794291
161	0.843148056993177	0.861281226008774
162	0.599047646706308	0.611281226008774
163	0.296923524544665	0.302264231633827
164	0.0512955590513467	0.0522642316338269
165	0.711519768408377	0.726905328038456
166	0.42462713138835	0.433012701892219
167	0.136707551662263	0.139120075745983
168	0.65060852875044	0.665465038884933
169	0.364076636397866	0.371572412738697
170	0.0761967569793705	0.0776797865924607
171	0.47818882371802	0.489073800366903
172	0.176501674085163	0.180056805991955
173	0.394673605078406	0.404508497187474
174	0.093452651964265	0.0954915028125265
175	0.198492418230007	0.2033683215379
176	0.101169394800268	0.10395584540888
231	0.953482135754673	0.972789205831714

232	0.843161171273121	0.861281226008774
233	0.650620087326386	0.665465038884934
234	0.394681801654654	0.404508497187474
235	0.101174463950613	0.10395584540888
236	0.875332012150061	0.893582297554377
237	0.711534774014504	0.726905328038456
238	0.478204029068657	0.489073800366903
239	0.198506602855699	0.2033683215379
240	0.775284911091137	0.791153573830373
241	0.599053315481232	0.611281226008774
242	0.364086776887518	0.371572412738697
243	0.0934682695292477	0.0954915028125265
244	0.63098601484579	0.643582297554377
245	0.424641288869315	0.433012701892219
246	0.176523669210804	0.180056805991955
247	0.488070174157289	0.497260947684137
248	0.296933348853691	0.302264231633827
249	0.076220981125035	0.0776797865924608
250	0.328773598753414	0.334565303179429
251	0.136733242558403	0.139120075745983
252	0.200160059597621	0.2033683215379
253	0.0513233141659783	0.0522642316338269
254	0.0832180603536405	0.0845653031794291
255	0.021103084633879	0.0217326895365599
310	0.953475580166895	0.972789205831714
311	0.775284490266194	0.791153573830373
312	0.488070769846778	0.497260947684137
313	0.200157210243284	0.2033683215379
314	0.021089006850687	0.0217326895365599
315	0.875321066900196	0.893582297554377
316	0.630978369734502	0.643582297554377
317	0.328763753767147	0.334565303179429
318	0.0831982579441415	0.0845653031794291
319	0.843148056993177	0.861281226008774
320	0.599047646706308	0.611281226008774
321	0.296923524544665	0.302264231633827
322	0.0512955590513466	0.0522642316338269
323	0.711519768408376	0.726905328038456
324	0.424627131388349	0.433012701892219
325	0.136707551662263	0.139120075745983
326	0.650608528750439	0.665465038884934
327	0.364076636397866	0.371572412738697
328	0.0761967569793705	0.0776797865924608
329	0.47818882371802	0.489073800366903
330	0.176501674085163	0.180056805991955
331	0.394673605078406	0.404508497187474
332	0.0934526519642648	0.0954915028125265
333	0.198492418230007	0.2033683215379
334	0.101169394800268	0.10395584540888

389	0.953482135754673	0.972789205831713
390	0.843161171273121	0.861281226008774
391	0.650620087326387	0.665465038884933
392	0.394681801654654	0.404508497187474
393	0.101174463950613	0.10395584540888
394	0.875332012150061	0.893582297554377
395	0.711534774014505	0.726905328038456
396	0.478204029068657	0.489073800366903
397	0.198506602855699	0.2033683215379
398	0.775284911091138	0.791153573830373
399	0.599053315481233	0.611281226008774
400	0.364086776887517	0.371572412738697
401	0.0934682695292476	0.0954915028125265
402	0.63098601484579	0.643582297554377
403	0.424641288869315	0.433012701892219
404	0.176523669210804	0.180056805991955
405	0.488070174157289	0.497260947684137
406	0.296933348853691	0.302264231633827
407	0.0762209811250349	0.0776797865924607
408	0.328773598753414	0.334565303179429
409	0.136733242558403	0.139120075745983
410	0.200160059597621	0.2033683215379
411	0.0513233141659784	0.0522642316338269
412	0.0832180603536406	0.0845653031794291
413	0.021103084633879	0.0217326895365599
468	0.953475580166897	0.972789205831714
469	0.775284490266195	0.791153573830373
470	0.488070769846778	0.497260947684137
471	0.200157210243284	0.2033683215379
472	0.0210890068506871	0.0217326895365599
473	0.875321066900198	0.893582297554377
474	0.630978369734501	0.643582297554377
475	0.328763753767147	0.334565303179429
476	0.0831982579441417	0.0845653031794291
477	0.843148056993178	0.861281226008774
478	0.599047646706307	0.611281226008774
479	0.296923524544666	0.302264231633827
480	0.0512955590513467	0.0522642316338267
481	0.711519768408378	0.726905328038456
482	0.42462713138835	0.433012701892219
483	0.136707551662263	0.139120075745983
484	0.650608528750441	0.665465038884934
485	0.364076636397867	0.371572412738697
486	0.0761967569793706	0.0776797865924606
487	0.478188823718021	0.489073800366903
488	0.176501674085163	0.180056805991955
489	0.394673605078407	0.404508497187474
490	0.0934526519642651	0.0954915028125263
491	0.198492418230008	0.2033683215379

492	0.101169394800269	0.10395584540888
547	0.953482135754673	0.972789205831713
548	0.843161171273121	0.861281226008774
549	0.650620087326387	0.665465038884934
550	0.394681801654656	0.404508497187474
551	0.101174463950613	0.10395584540888
552	0.875332012150061	0.893582297554377
553	0.711534774014506	0.726905328038456
554	0.478204029068659	0.489073800366903
555	0.1985066028557	0.2033683215379
556	0.775284911091137	0.791153573830373
557	0.599053315481233	0.611281226008774
558	0.364086776887519	0.371572412738697
559	0.093468269529248	0.0954915028125263
560	0.63098601484579	0.643582297554376
561	0.424641288869316	0.433012701892219
562	0.176523669210805	0.180056805991955
563	0.488070174157289	0.497260947684137
564	0.296933348853692	0.302264231633827
565	0.0762209811250352	0.0776797865924605
566	0.328773598753414	0.334565303179429
567	0.136733242558403	0.139120075745983
568	0.200160059597621	0.2033683215379
569	0.0513233141659784	0.0522642316338267
570	0.0832180603536405	0.0845653031794291
571	0.021103084633879	0.0217326895365599
617	0.953475580166896	0.972789205831713
618	0.775284490266194	0.791153573830373
619	0.488070769846778	0.497260947684137
620	0.200157210243284	0.2033683215379
621	0.021089006850687	0.0217326895365599
622	0.875321066900197	0.893582297554377
623	0.630978369734502	0.643582297554376
624	0.328763753767147	0.334565303179429
625	0.0831982579441414	0.0845653031794291
626	0.843148056993177	0.861281226008774
627	0.599047646706307	0.611281226008774
628	0.296923524544665	0.302264231633827
629	0.0512955590513465	0.0522642316338267
630	0.711519768408377	0.726905328038456
631	0.424627131388349	0.433012701892219
632	0.136707551662263	0.139120075745983
633	0.65060852875044	0.665465038884934
634	0.364076636397866	0.371572412738697
635	0.0761967569793706	0.0776797865924605
636	0.47818882371802	0.489073800366903
637	0.176501674085163	0.180056805991955
638	0.394673605078407	0.404508497187474
639	0.093452651964265	0.0954915028125263

640	0.198492418230008	0.2033683215379
641	0.101169394800269	0.10395584540888

Table 3.2

POISS0N BOUNDARY VALUE PROBLEM(EXAMPLE-2 FOR A SQUARE DOMAIN)
FEM MODEL:NODES=2481,FOUR NODE QUADRILATERAL ELEMENTS=2400
SOLUTION AT ELEMENT CENTROIDS

NODE NUMBER	FEM computed values	anlytical(theoretical)-values
238	0.988417102735794	0.993158937674856
239	0.959717112262044	0.96460205851448
240	0.907504096874246	0.912293475342785
241	0.83298056560894	0.837521199079693
242	0.737962689174309	0.742126371321759
243	0.624788084447998	0.628457929325666
244	0.496251014002483	0.499314767377287
245	0.355534498291853	0.357876818725374
246	0.206141060159078	0.207626755071376
247	0.0518532769561828	0.0522642316338267
248	0.967931080289033	0.972789205831714
249	0.923676783123244	0.928466175238718
250	0.856714061402572	0.861281226008774
251	0.768672563342312	0.772888674565986
252	0.661715907202288	0.665465038884934
253	0.538482463011842	0.541655445394692
254	0.402019886275129	0.404508497187474
255	0.255714234905616	0.257401207292766
256	0.103218122035525	0.10395584540888
257	0.940039092112261	0.944818029471471
258	0.888990021590203	0.893582297554377
259	0.816057094614254	0.820343603841875
260	0.723035365233011	0.726905328038456
261	0.612217381393755	0.615568230598259
262	0.486338676438818	0.489073800366903
263	0.34851122205529	0.350536750027629
264	0.202147166698121	0.2033683215379
265	0.0508647496796162	0.0511922900311449
266	0.897170725410378	0.90176944487161
267	0.832204079146261	0.836516303737808
268	0.746745025871177	0.750665354967537
269	0.642900797495705	0.646330533842485
270	0.523233680006328	0.526080909926171
271	0.390699043294539	0.392877428045034
272	0.248572127727066	0.25
273	0.10036226097908	0.100966742252535

274	0.848534395528583	0.852868157970561
275	0.778985859067914	0.782966426513139
276	0.690246081472839	0.693785463118374
277	0.584505568553036	0.587521199079693
278	0.464373382546206	0.466790213248601
279	0.332813168009446	0.334565303179429
280	0.193067644985375	0.194102284987398
281	0.0485773679258657	0.0488598243504264
282	0.787156432500507	0.791153573830373
283	0.706378893094704	0.709958163014307
284	0.608197394824003	0.611281226008774
285	0.495034535526922	0.497552516492827
286	0.369680993497343	0.371572412738697
287	0.235225274318541	0.236442963004803
288	0.0949750365793498	0.0954915028125263
289	0.722694092687376	0.72631001724706
290	0.640414000613426	0.643582297554377
291	0.542348002805854	0.545007445768716
292	0.43091449114091	0.433012701892219
293	0.308858263244099	0.31035574820837
294	0.179181276779572	0.180056805991955
295	0.0450768499873504	0.0453242676377401
296	0.648578401596787	0.65176944487161
297	0.558470433448355	0.56118014566465
298	0.454593042991271	0.4567727288213
299	0.339505111389888	0.341118051452597
300	0.216037382352943	0.217063915551223
301	0.0872234663946664	0.0876649456551453
302	0.574775302431038	0.577531999897402
303	0.486792781269151	0.489073800366903
304	0.386799290085722	0.388572980728485
305	0.277255243287368	0.278504204704746
306	0.160851518462883	0.161577730858712
307	0.0404574624706277	0.0406726770331925
308	0.494953071684126	0.497260947684137
309	0.402915731798129	0.404745680624419
310	0.300928720902451	0.302264231633827
311	0.191497653764516	0.192340034102939
312	0.0773091197163312	0.0776797865924606
313	0.419214837492182	0.421097534857171
314	0.33312242234375	0.334565303179429
315	0.238792166370687	0.239794963378827
316	0.138538625551592	0.139120075745983
317	0.0348357862826007	0.0350195901352117
318	0.341281289335076	0.342752450496663
319	0.254908971814031	0.255967663274761
320	0.162217489793882	0.162880102675064
321	0.0654799793879199	0.0657818933794382
322	0.271208594361582	0.272319517507514

323	0.194419341863302	0.195181174220666
324	0.112794628927132	0.113236822655327
325	0.0283510117997932	0.0285042047047456
326	0.202580286335157	0.2033683215379
327	0.128919017486617	0.12940952255126
328	0.0520278670515734	0.0522642316338267
329	0.145226812288235	0.145761376784013
330	0.0842512460022017	0.0845653031794291
331	0.021162060266308	0.0212869511544169
332	0.0924179370308632	0.092752450496663
333	0.0372815367832284	0.0374596510503502
334	0.0536032710262482	0.0538115052831031
335	0.0134422979227653	0.013545542219326
336	0.0215960502353594	0.0217326895365599
337	0.00535559321134433	0.00547059707971809
547	0.988416638768864	0.993158937674856
548	0.940038925856626	0.944818029471471
549	0.848534347235084	0.852868157970561
550	0.722694098795867	0.72631001724706
551	0.574775325112241	0.577531999897402
552	0.419214843237481	0.421097534857172
553	0.271208545754388	0.272319517507513
554	0.145226655994887	0.145761376784013
555	0.0536028960146885	0.053811505283103
556	0.00535464973643106	0.00547059707971812
557	0.967930418538892	0.972789205831713
558	0.897170361314643	0.90176944487161
559	0.787156164311979	0.791153573830373
560	0.6485781602077	0.65176944487161
561	0.49495281566464	0.497260947684137
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1552	0.138538625551593	0.139120075745983
1553	0.0348357862826008	0.035019590135212
1554	0.341281289335077	0.342752450496663
1555	0.254908971814032	0.255967663274761
1556	0.162217489793883	0.162880102675064
1557	0.0654799793879201	0.0657818933794384
1558	0.271208594361583	0.272319517507513
1559	0.194419341863303	0.195181174220666
1560	0.112794628927132	0.113236822655327
1561	0.0283510117997934	0.0285042047047458
1562	0.202580286335158	0.2033683215379
1563	0.128919017486618	0.12940952255126
1564	0.0520278670515736	0.0522642316338269
1565	0.145226812288235	0.145761376784013
1566	0.084251246002202	0.0845653031794291
1567	0.0211620602663081	0.0212869511544171
1568	0.0924179370308634	0.092752450496663
1569	0.0372815367832286	0.0374596510503503
1570	0.0536032710262483	0.053811505283103
1571	0.0134422979227653	0.0135455422193261
1572	0.0215960502353594	0.0217326895365599
1573	0.00535559321134433	0.00547059707971812
1783	0.988416638768866	0.993158937674856
1784	0.94003892585663	0.944818029471471
1785	0.848534347235088	0.852868157970561
1786	0.72269409879587	0.72631001724706
1787	0.574775325112243	0.577531999897402
1788	0.419214843237483	0.421097534857171
1789	0.271208545754389	0.272319517507514
1790	0.145226655994887	0.145761376784013
1791	0.0536028960146886	0.0538115052831031
1792	0.00535464973643106	0.00547059707971809
1793	0.967930418538895	0.972789205831714
1794	0.897170361314647	0.90176944487161
1795	0.787156164311982	0.791153573830373
1796	0.648578160207703	0.65176944487161
1797	0.494952815664642	0.497260947684137
1798	0.341280978356807	0.342752450496663
1799	0.20257986166334	0.2033683215379
1800	0.0924172755964509	0.092752450496663
1801	0.0215947576012202	0.0217326895365599
1802	0.959716104399745	0.96460205851448
1803	0.888989489384576	0.893582297554377
1804	0.778985568459131	0.782966426513139
1805	0.640413802512981	0.643582297554377
1806	0.486792568228332	0.489073800366903
1807	0.333122087025407	0.334565303179429
1808	0.194418744193771	0.195181174220666

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1809	0.0842501410653083	0.0845653031794291
1810	0.0134402526454918	0.013545542219326
1811	0.923675780853799	0.928466175238718
1812	0.832203448906097	0.836516303737808
1813	0.706378415365749	0.709958163014307
1814	0.558469979762597	0.56118014566465
1815	0.402915195240421	0.404745680624419
1816	0.254908227513152	0.255967663274761
1817	0.128917868176791	0.12940952255126
1818	0.0372795830710437	0.0374596510503502
1819	0.907503011739702	0.912293475342785
1820	0.816056416755256	0.820343603841875
1821	0.690245642650843	0.693785463118374
1822	0.542347617318762	0.545007445768716
1823	0.386798791286276	0.388572980728485
1824	0.238791364697046	0.239794963378827
1825	0.112793255720789	0.113236822655327
1826	0.0211598505203014	0.0212869511544169
1827	0.856712967113927	0.861281226008774
1828	0.746744265479101	0.750665354967537
1829	0.608196761245316	0.611281226008774
1830	0.454592365271101	0.4567727288213
1831	0.300927825913566	0.302264231633827
1832	0.162216153393314	0.162880102675064
1833	0.0520257568419518	0.0522642316338267
1834	0.83297951995283	0.837521199079693
1835	0.723034644858222	0.726905328038456
1836	0.584505023605276	0.587521199079693
1837	0.430913902134177	0.433012701892219
1838	0.277254382922407	0.278504204704746
1839	0.138537218656194	0.139120075745983
1840	0.0283488770996573	0.0285042047047456
1841	0.768671479863864	0.772888674565986
1842	0.642899975255595	0.646330533842485
1843	0.495033761831378	0.497552516492827
1844	0.339504170978073	0.341118051452597
1845	0.191496303797552	0.192340034102939
1846	0.0654779290391075	0.0657818933794382
1847	0.737961738218374	0.742126371321759
1848	0.61221666828636	0.615568230598259
1849	0.46437273970535	0.466790213248601
1850	0.308857431403526	0.31035574820837
1851	0.16085021229856	0.161577730858712
1852	0.0348338450960423	0.0350195901352117
1853	0.661714889429734	0.665465038884934
1854	0.523232827095319	0.526080909926171
1855	0.369680069682482	0.371572412738697
1856	0.216036124903196	0.217063915551223
1857	0.0773072526534659	0.0776797865924606

1858	0.624787262007608	0.628457929325666
1859	0.486337991734604	0.489073800366903
1860	0.33281241377086	0.334565303179429
1861	0.179180153646869	0.180056805991955
1862	0.0404557922710228	0.0406726770331925
1863	0.538481543641446	0.541655445394692
1864	0.390698168513626	0.392877428045034
1865	0.235224175222533	0.236442963004803
1866	0.0872218640840389	0.0876649456551453
1867	0.496250342180064	0.499314767377287
1868	0.348510569391933	0.350536750027629
1869	0.193066755281027	0.194102284987398
1870	0.04507550628685	0.0453242676377401
1871	0.402019084598443	0.404508497187474
1872	0.248571226326358	0.25
1873	0.0949737553694593	0.0954915028125263
1874	0.355533990709868	0.357876818725374
1875	0.20214653944413	0.2033683215379
1876	0.0485763903784784	0.0488598243504264
1877	0.255713560924141	0.257401207292766
1878	0.100361339269195	0.100966742252535
1879	0.206140722531811	0.207626755071376
1880	0.0508641644011485	0.0511922900311449
1881	0.103217592685841	0.10395584540888
1882	0.0518530954249094	0.0522642316338267
2092	0.988417102735794	0.993158937674856
2093	0.959717112262045	0.96460205851448
2094	0.907504096874247	0.912293475342785
2095	0.832980565608941	0.837521199079693
2096	0.73796268917431	0.742126371321759
2097	0.624788084447998	0.628457929325666
2098	0.496251014002482	0.499314767377287
2099	0.355534498291854	0.357876818725374
2100	0.206141060159079	0.207626755071376
2101	0.051853276956183	0.0522642316338267
2102	0.967931080289034	0.972789205831713
2103	0.923676783123244	0.928466175238718
2104	0.856714061402572	0.861281226008774
2105	0.768672563342312	0.772888674565986
2106	0.661715907202287	0.665465038884934
2107	0.538482463011841	0.541655445394692
2108	0.402019886275129	0.404508497187474
2109	0.255714234905616	0.257401207292766
2110	0.103218122035525	0.10395584540888
2111	0.940039092112263	0.944818029471471
2112	0.888990021590206	0.893582297554377
2113	0.816057094614255	0.820343603841875
2114	0.723035365233012	0.726905328038456
2115	0.612217381393755	0.615568230598259

2116	0.486338676438818	0.489073800366903
2117	0.34851122205529	0.350536750027629
2118	0.202147166698121	0.2033683215379
2119	0.0508647496796164	0.0511922900311449
2120	0.897170725410382	0.90176944487161
2121	0.832204079146263	0.836516303737808
2122	0.746745025871178	0.750665354967537
2123	0.642900797495706	0.646330533842485
2124	0.523233680006329	0.526080909926171
2125	0.390699043294539	0.392877428045034
2126	0.248572127727066	0.25
2127	0.100362260979081	0.100966742252535
2128	0.848534395528585	0.852868157970561
2129	0.778985859067916	0.782966426513139
2130	0.690246081472841	0.693785463118374
2131	0.584505568553037	0.587521199079693
2132	0.464373382546206	0.466790213248601
2133	0.332813168009446	0.334565303179429
2134	0.193067644985375	0.194102284987398
2135	0.0485773679258658	0.0488598243504264
2136	0.78715643250051	0.791153573830373
2137	0.706378893094706	0.709958163014307
2138	0.608197394824005	0.611281226008774
2139	0.495034535526923	0.497552516492827
2140	0.369680993497343	0.371572412738697
2141	0.235225274318541	0.236442963004803
2142	0.09497503657935	0.0954915028125263
2143	0.722694092687377	0.72631001724706
2144	0.640414000613427	0.643582297554376
2145	0.542348002805855	0.545007445768716
2146	0.43091449114091	0.433012701892219
2147	0.3088582632441	0.31035574820837
2148	0.179181276779572	0.180056805991955
2149	0.0450768499873505	0.0453242676377401
2150	0.648578401596788	0.65176944487161
2151	0.558470433448356	0.56118014566465
2152	0.454593042991272	0.4567727288213
2153	0.339505111389889	0.341118051452596
2154	0.216037382352944	0.217063915551223
2155	0.0872234663946665	0.0876649456551453
2156	0.574775302431038	0.577531999897402
2157	0.48679278126915	0.489073800366903
2158	0.386799290085722	0.388572980728485
2159	0.277255243287368	0.278504204704746
2160	0.160851518462884	0.161577730858712
2161	0.0404574624706278	0.0406726770331925
2162	0.494953071684126	0.497260947684137
2163	0.402915731798129	0.404745680624419
2164	0.300928720902451	0.302264231633827

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2165	0.191497653764516	0.192340034102939
2166	0.0773091197163313	0.0776797865924605
2167	0.419214837492183	0.421097534857172
2168	0.33312242234375	0.334565303179429
2169	0.238792166370687	0.239794963378828
2170	0.138538625551593	0.139120075745983
2171	0.0348357862826008	0.0350195901352117
2172	0.341281289335076	0.342752450496663
2173	0.254908971814031	0.255967663274761
2174	0.162217489793883	0.162880102675064
2175	0.06547997938792	0.0657818933794383
2176	0.271208594361582	0.272319517507513
2177	0.194419341863302	0.195181174220666
2178	0.112794628927132	0.113236822655327
2179	0.0283510117997933	0.0285042047047456
2180	0.202580286335157	0.2033683215379
2181	0.128919017486617	0.12940952255126
2182	0.0520278670515734	0.0522642316338267
2183	0.145226812288235	0.145761376784013
2184	0.0842512460022019	0.0845653031794291
2185	0.0211620602663081	0.0212869511544169
2186	0.0924179370308633	0.0927524504966629
2187	0.0372815367832285	0.0374596510503502
2188	0.0536032710262482	0.053811505283103
2189	0.0134422979227653	0.013545542219326
2190	0.0215960502353594	0.0217326895365599
2191	0.00535559321134433	0.00547059707971807
2382	0.988416638768864	0.993158937674856
2383	0.940038925856629	0.944818029471471
2384	0.848534347235087	0.852868157970561
2385	0.722694098795869	0.72631001724706
2386	0.574775325112242	0.577531999897402
2387	0.419214843237481	0.421097534857172
2388	0.271208545754388	0.272319517507513
2389	0.145226655994887	0.145761376784013
2390	0.0536028960146886	0.053811505283103
2391	0.00535464973643106	0.00547059707971807
2392	0.967930418538894	0.972789205831713
2393	0.897170361314646	0.90176944487161
2394	0.787156164311982	0.791153573830373
2395	0.648578160207701	0.65176944487161
2396	0.49495281566464	0.497260947684137
2397	0.341280978356806	0.342752450496663
2398	0.20257986166334	0.2033683215379
2399	0.0924172755964509	0.0927524504966629
2400	0.0215947576012202	0.0217326895365599
2401	0.959716104399744	0.96460205851448
2402	0.888989489384576	0.893582297554377
2403	0.778985568459131	0.782966426513139

2404	0.64041380251298	0.643582297554376
2405	0.48679256822833	0.489073800366903
2406	0.333122087025405	0.334565303179429
2407	0.194418744193771	0.195181174220666
2408	0.0842501410653084	0.0845653031794291
2409	0.0134402526454918	0.013545542219326
2410	0.9236757808538	0.928466175238718
2411	0.832203448906096	0.836516303737808
2412	0.706378415365749	0.709958163014307
2413	0.558469979762595	0.56118014566465
2414	0.402915195240419	0.404745680624419
2415	0.254908227513151	0.255967663274761
2416	0.128917868176791	0.12940952255126
2417	0.0372795830710437	0.0374596510503502
2418	0.907503011739702	0.912293475342785
2419	0.816056416755254	0.820343603841875
2420	0.690245642650841	0.693785463118374
2421	0.54234761731876	0.545007445768716
2422	0.386798791286275	0.388572980728485
2423	0.238791364697046	0.239794963378828
2424	0.112793255720789	0.113236822655327
2425	0.0211598505203015	0.0212869511544169
2426	0.856712967113928	0.861281226008774
2427	0.746744265479099	0.750665354967537
2428	0.608196761245315	0.611281226008774
2429	0.4545923652711	0.4567727288213
2430	0.300927825913565	0.302264231633827
2431	0.162216153393314	0.162880102675064
2432	0.0520257568419519	0.0522642316338267
2433	0.832979519952829	0.837521199079693
2434	0.72303464485822	0.726905328038456
2435	0.584505023605274	0.587521199079693
2436	0.430913902134176	0.433012701892219
2437	0.277254382922407	0.278504204704746
2438	0.138537218656194	0.139120075745983
2439	0.0283488770996573	0.0285042047047456
2440	0.768671479863862	0.772888674565986
2441	0.642899975255594	0.646330533842485
2442	0.495033761831377	0.497552516492827
2443	0.339504170978071	0.341118051452596
2444	0.191496303797552	0.192340034102939
2445	0.0654779290391076	0.0657818933794383
2446	0.737961738218373	0.742126371321759
2447	0.612216668286359	0.615568230598259
2448	0.464372739705349	0.466790213248601
2449	0.308857431403525	0.31035574820837
2450	0.16085021229856	0.161577730858712
2451	0.0348338450960423	0.0350195901352117
2452	0.661714889429733	0.665465038884934

2453	0.523232827095318	0.526080909926171
2454	0.369680069682481	0.371572412738697
2455	0.216036124903196	0.217063915551223
2456	0.0773072526534659	0.0776797865924605
2457	0.624787262007607	0.628457929325666
2458	0.486337991734604	0.489073800366903
2459	0.33281241377086	0.334565303179429
2460	0.179180153646869	0.180056805991955
2461	0.0404557922710228	0.0406726770331925
2462	0.538481543641446	0.541655445394692
2463	0.390698168513626	0.392877428045034
2464	0.235224175222533	0.236442963004803
2465	0.087221864084039	0.0876649456551453
2466	0.496250342180064	0.499314767377287
2467	0.348510569391932	0.350536750027629
2468	0.193066755281027	0.194102284987398
2469	0.04507550628685	0.0453242676377401
2470	0.402019084598442	0.404508497187474
2471	0.248571226326357	0.25
2472	0.0949737553694593	0.0954915028125263
2473	0.355533990709868	0.357876818725374
2474	0.20214653944413	0.2033683215379
2475	0.0485763903784784	0.0488598243504264
2476	0.25571356092414	0.257401207292766
2477	0.100361339269195	0.100966742252535
2478	0.206140722531811	0.207626755071376
2479	0.0508641644011485	0.0511922900311449
2480	0.103217592685841	0.10395584540888
2481	0.0518530954249094	0.0522642316338267

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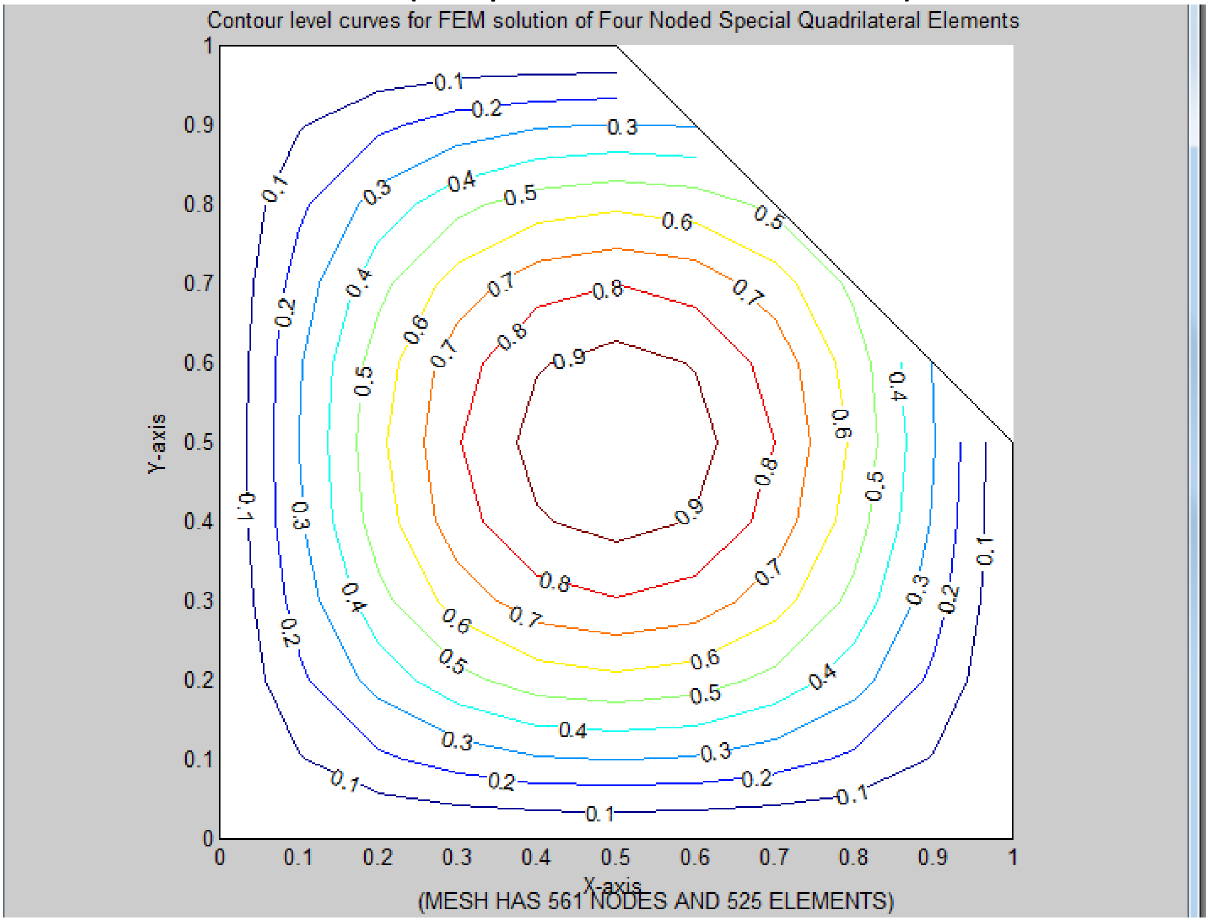
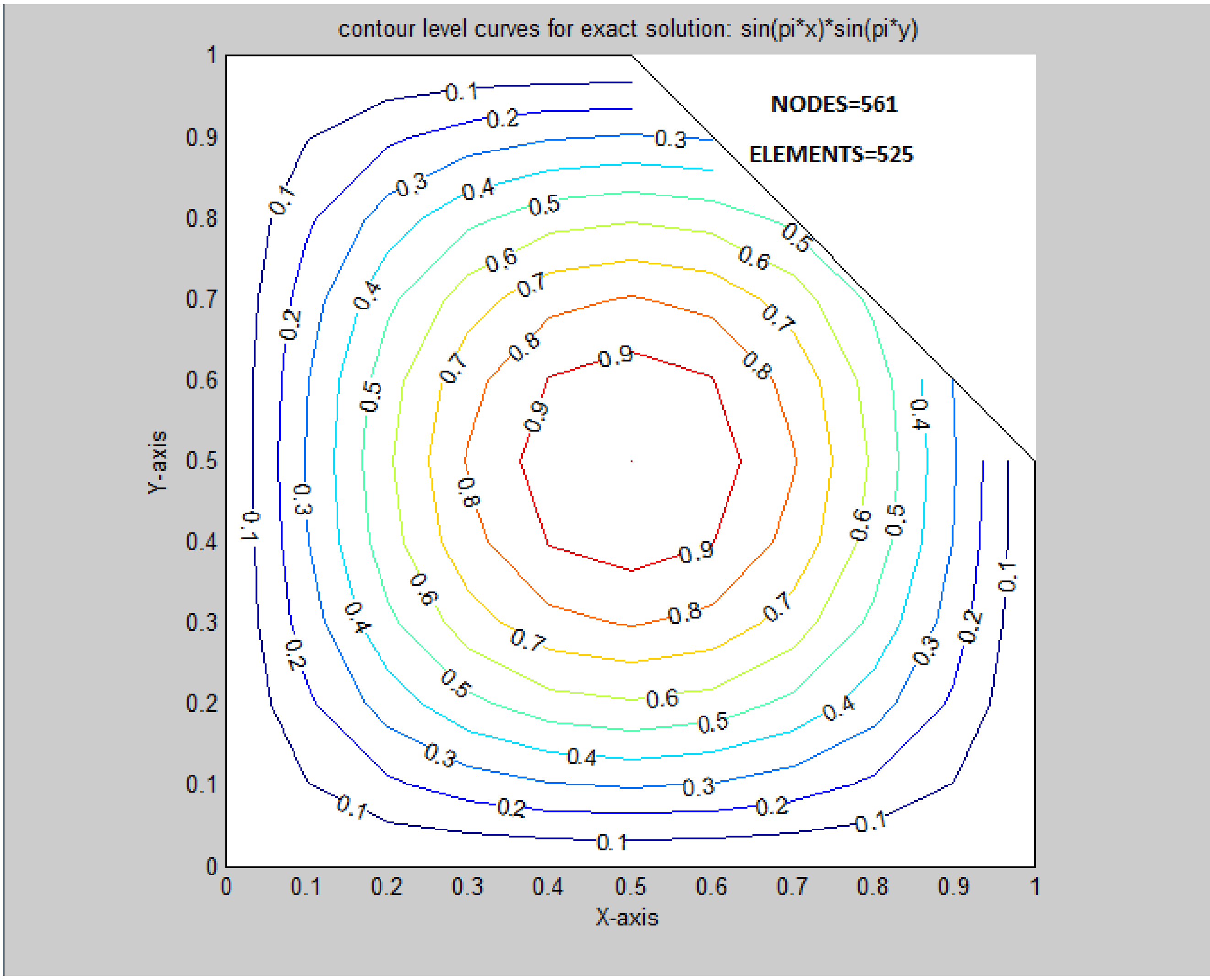
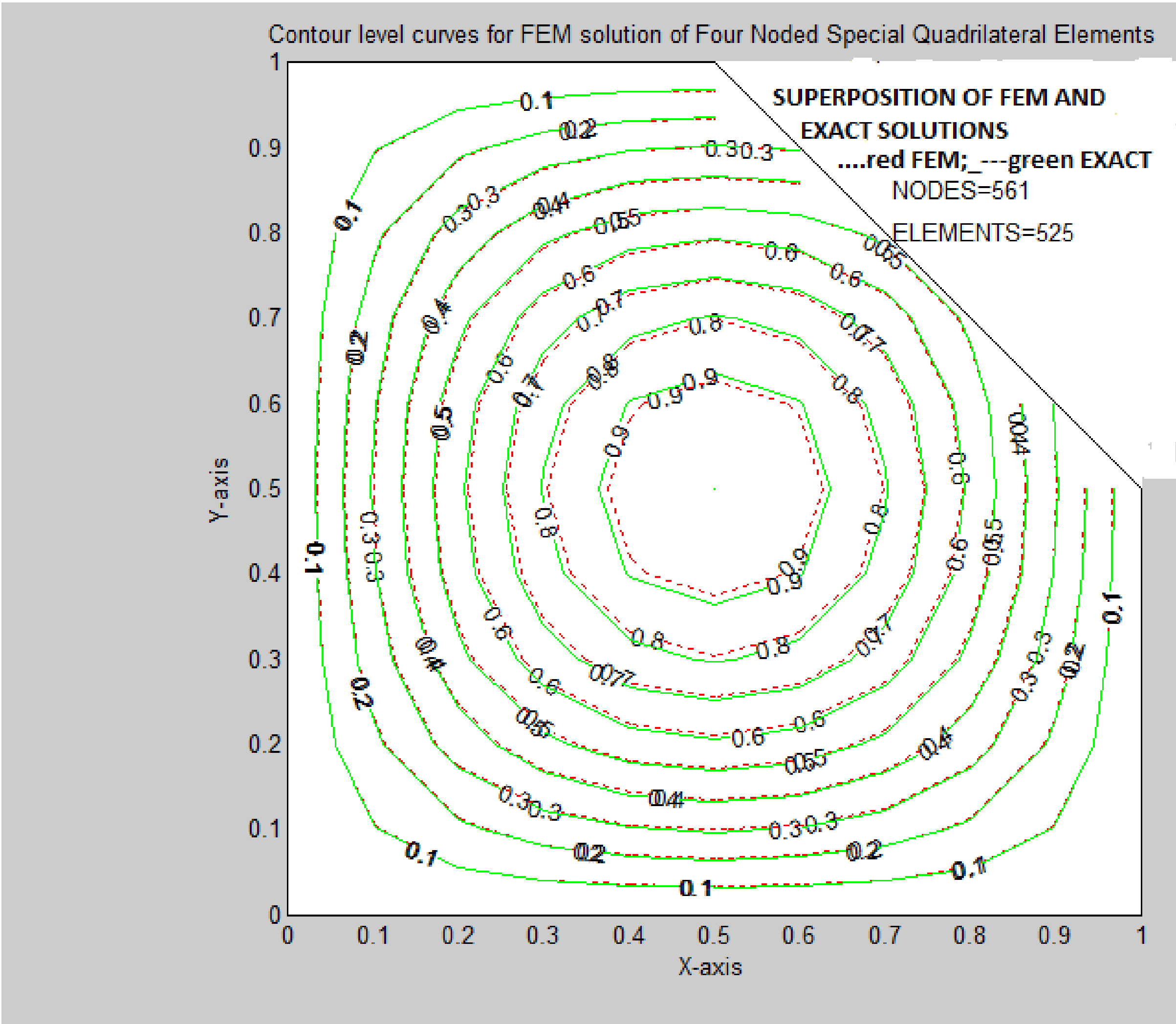
FIGURES(Example-1:A PENTAGONAL DOMAIN)



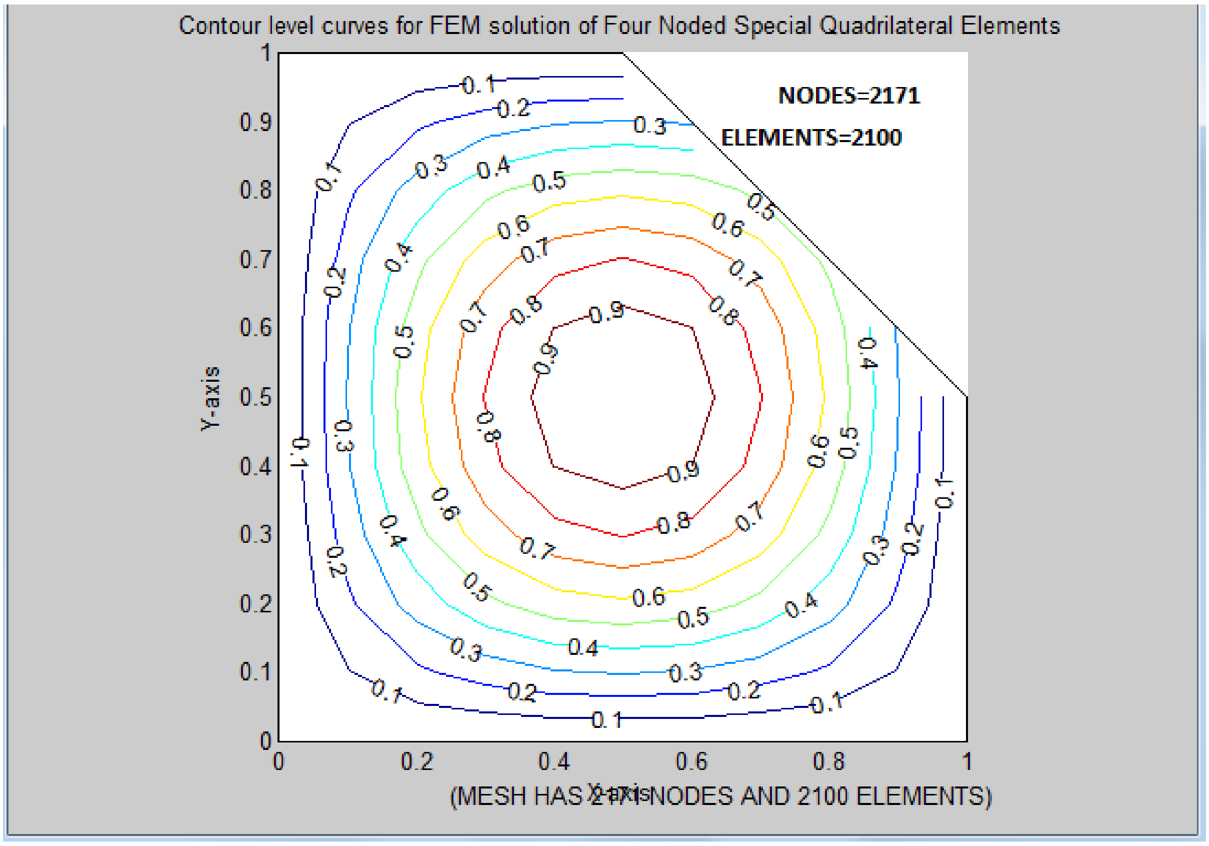
DOI: 10.18535/ijecs/v5i8.08

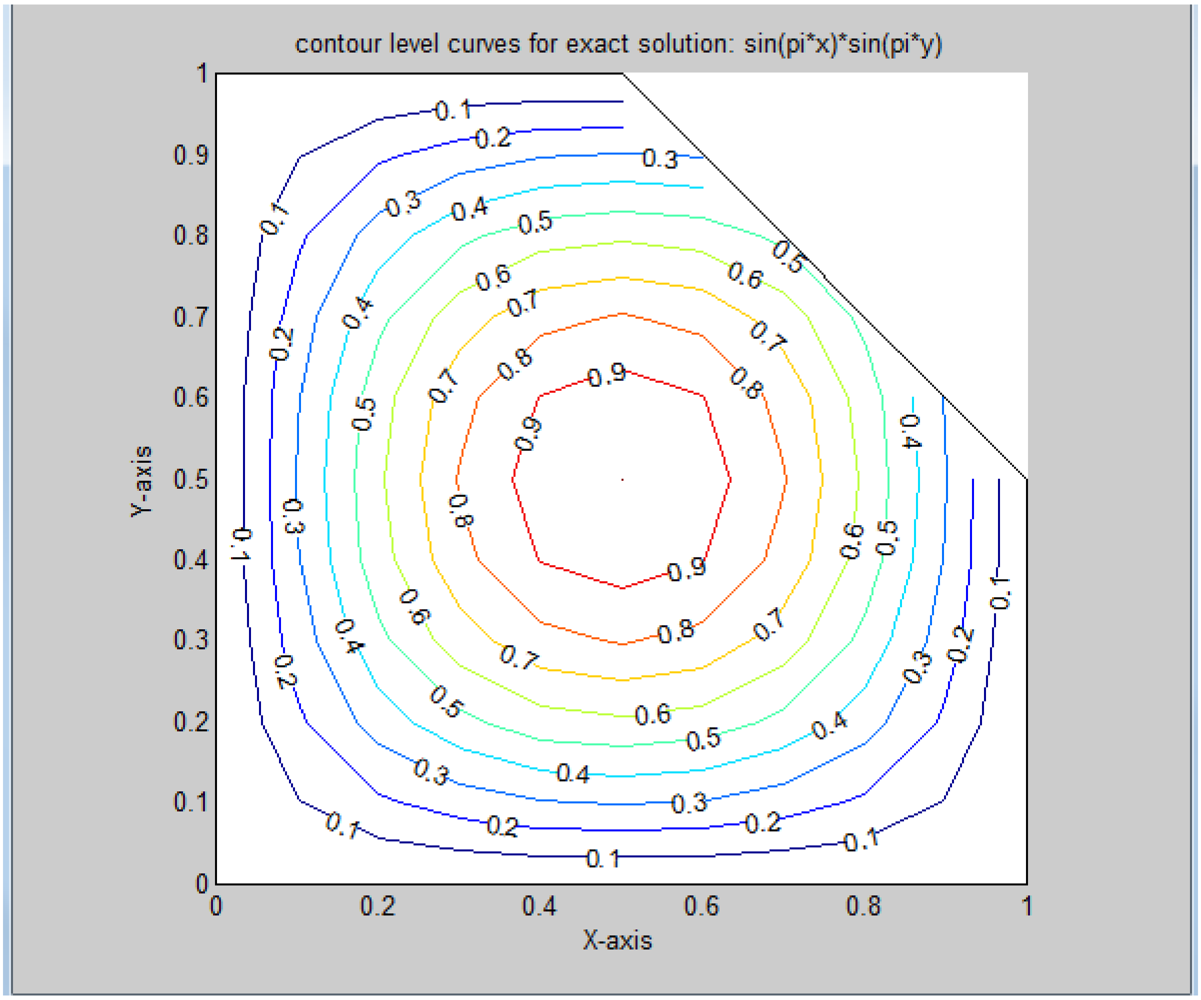


f

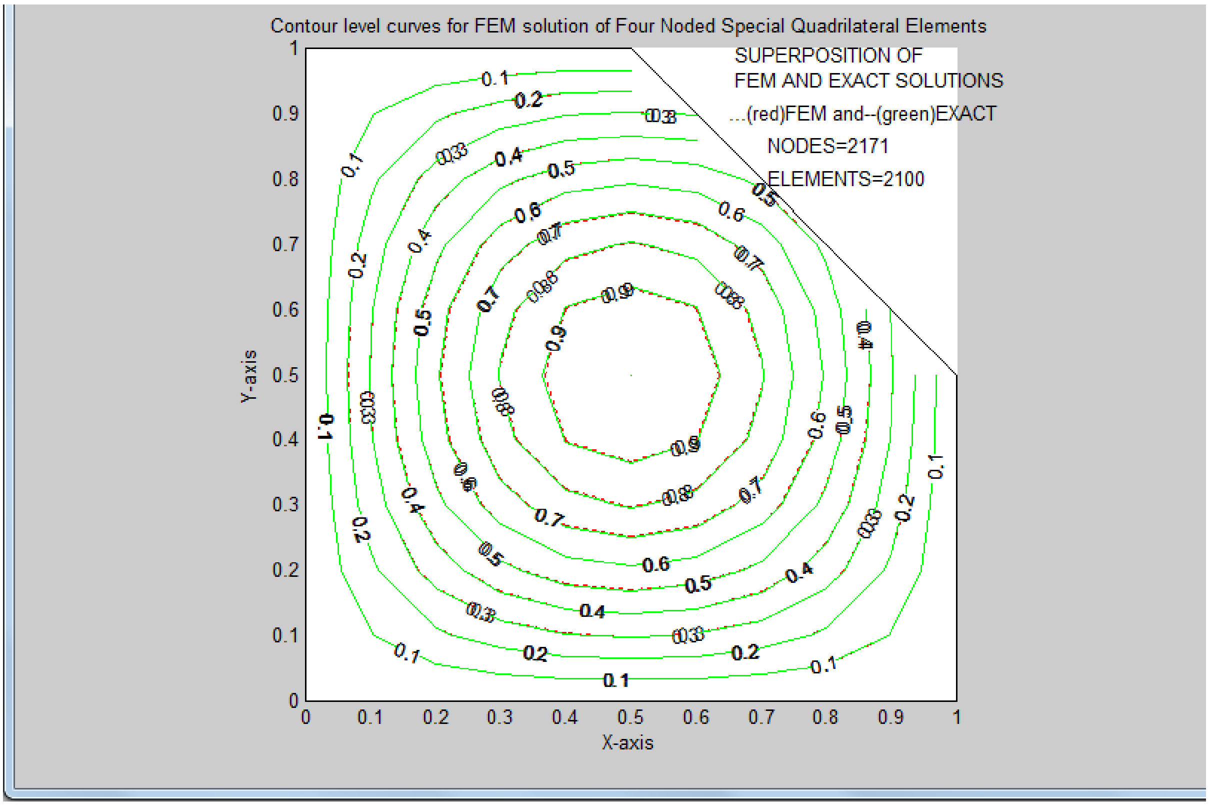

Contour level curves for FEM solution of Four Noded Special Quadrilateral Elements
SUPERPOSITION OF FEM AND
EXACT SOLUTIONS
....red FEM;_---green EXACT
NODES=561
ELEMENTS=525
Y-axis
X-axis

MESH-2


Contour level curves for FEM solution of Four Noded Special Quadrilateral Elements
NODES=2171
ELEMENTS=2100
Y-axis
X-axis
(MESH HAS 2171 NODES AND 2100 ELEMENTS)

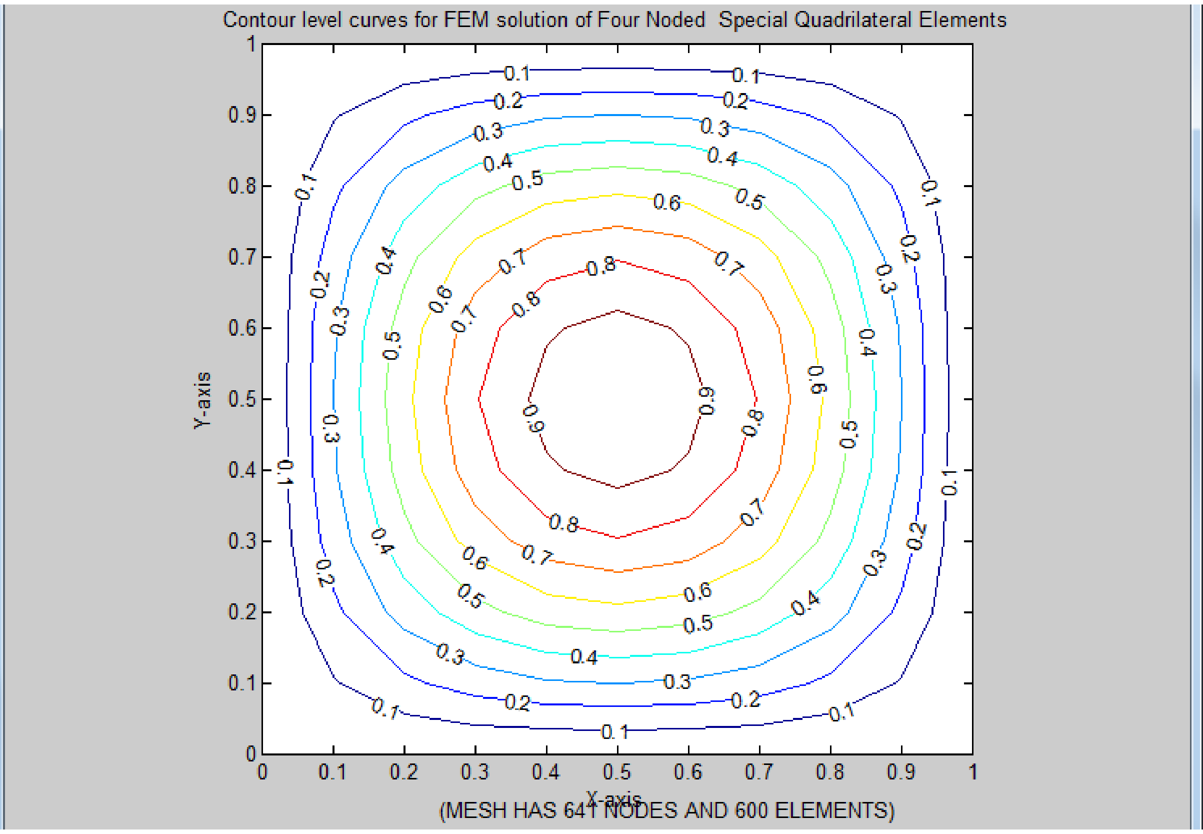

contour level curves for exact solution: sin(pi*x)*sin(pi*y)
Y-axis
X-axis

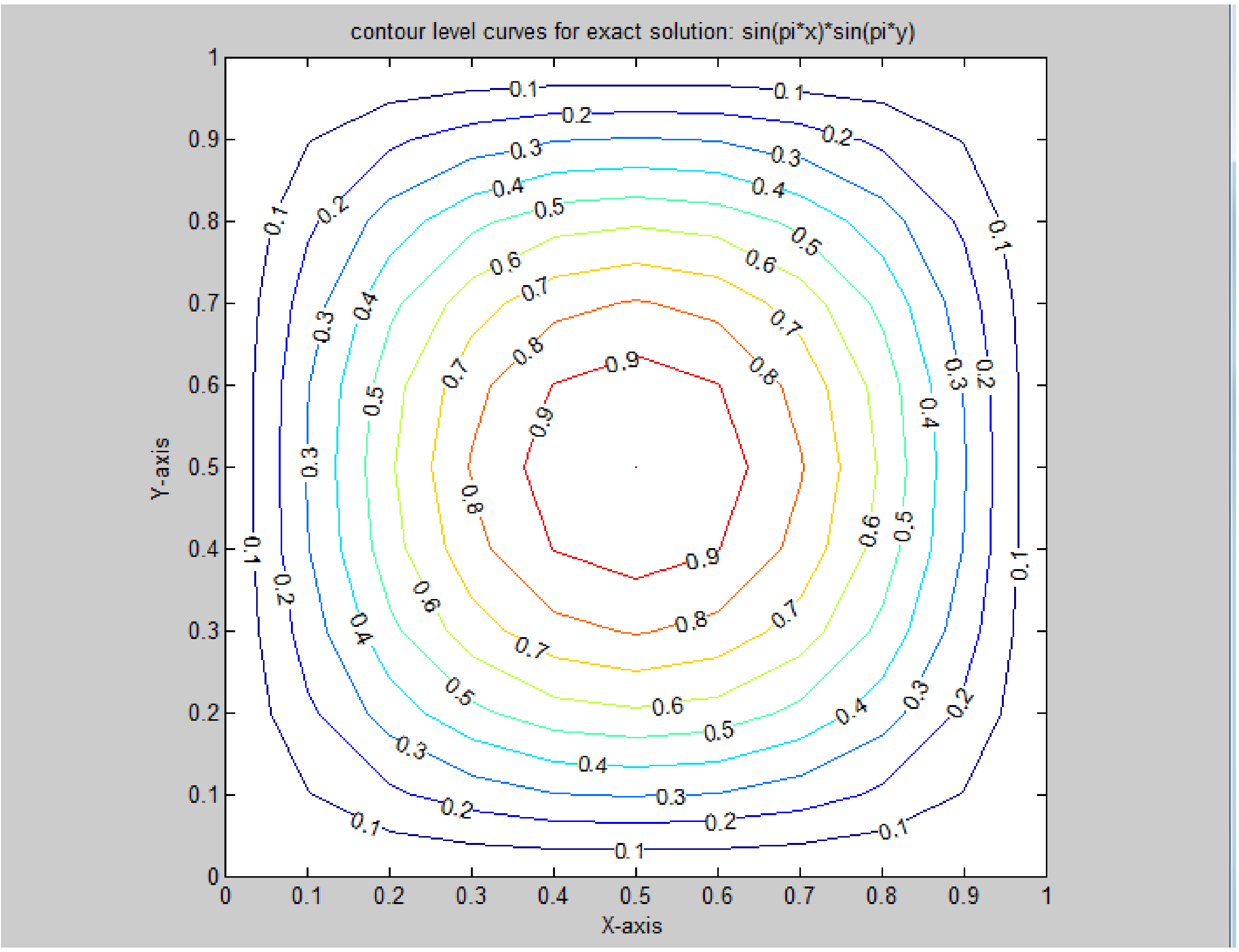
DOI: 10.18535/ijecs/v5i8.08

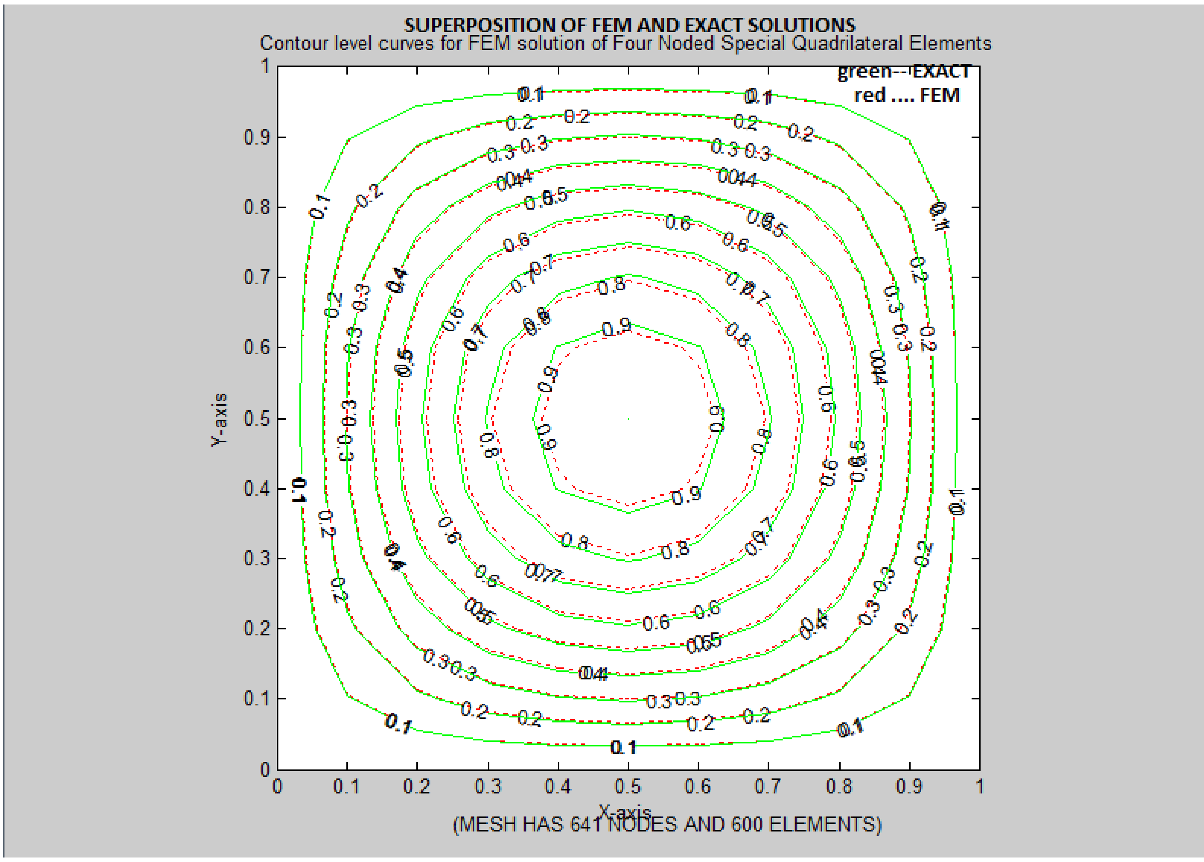


FIGURES(Example-2: A SQUARE DOMAIN)

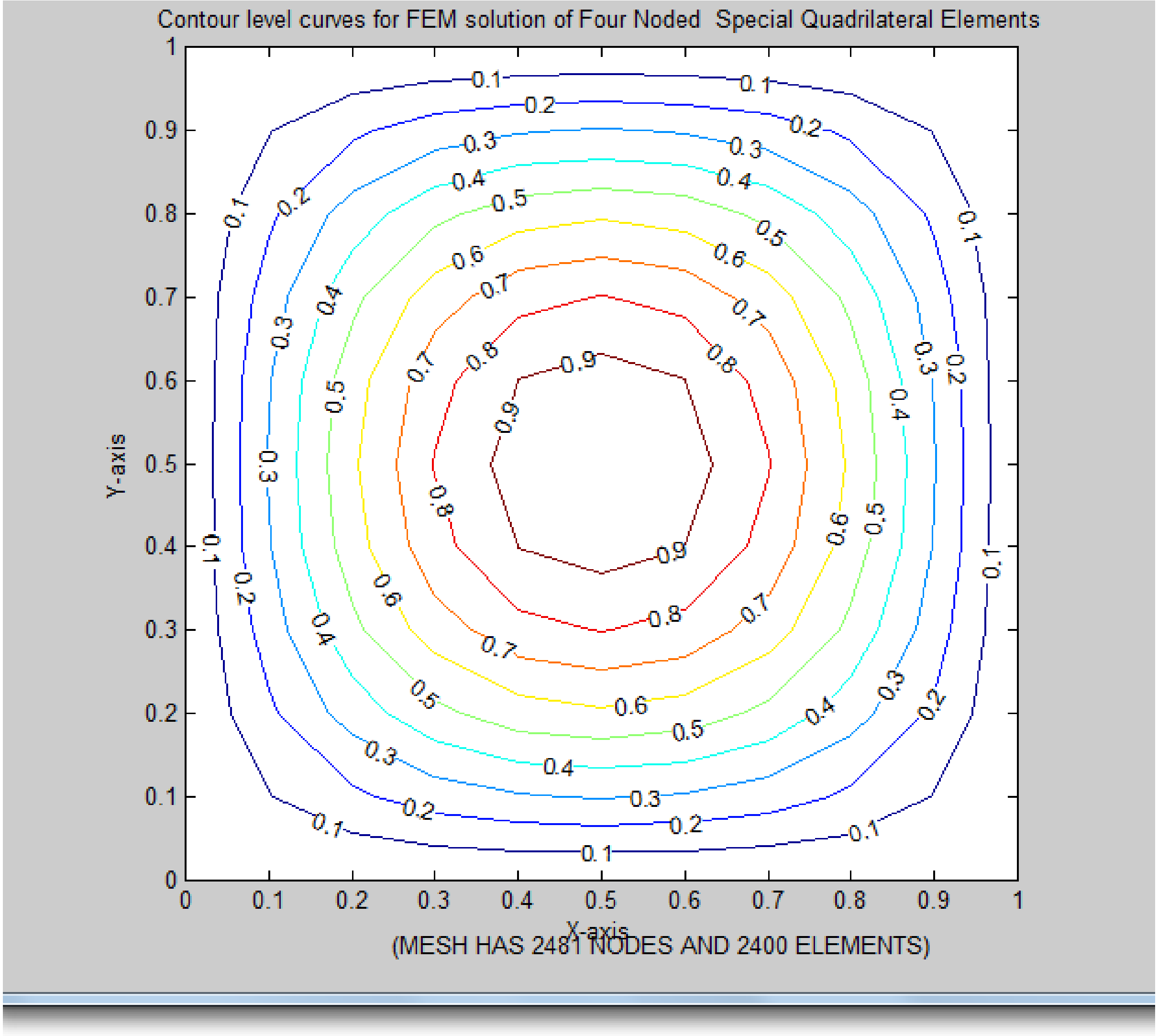
MESH-1

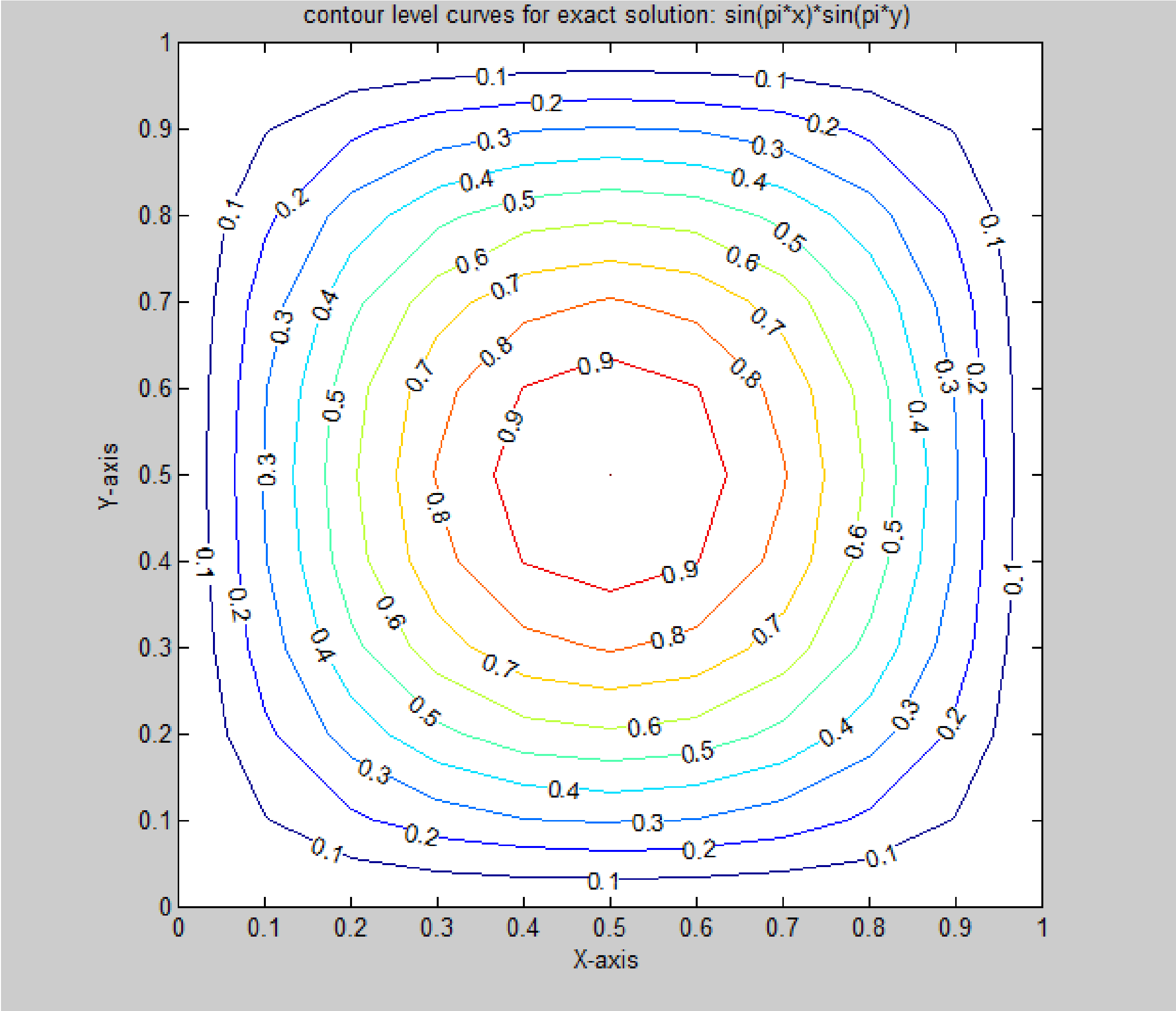

Contour level curves for FEM solution of Four Noded Special Quadrilateral Elements
Y-axis
X-axis
(MESH HAS 641 NODES AND 600 ELEMENTS)
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


contour level curves for exact solution: sin(pi*x)*sin(pi*y)
Y-axis
X-axis


SUPERPOSITION OF FEM AND EXACT SOLUTIONS
Contour level curves for FEM solution of Four Noded Special Quadrilateral Elements
green---EXACT
red .... FEM
Y-axis
X-axis
(MESH HAS 641 NODES AND 600 ELEMENTS)

MESH-2


Contour level curves for FEM solution of Four Noded Special Quadrilateral Elements
Y-axis
X-axis
(MESH HAS 2481 NODES AND 2400 ELEMENTS)
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


contour level curves for exact solution: sin(pi*x)*sin(pi*y)
Y-axis
X-axis
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

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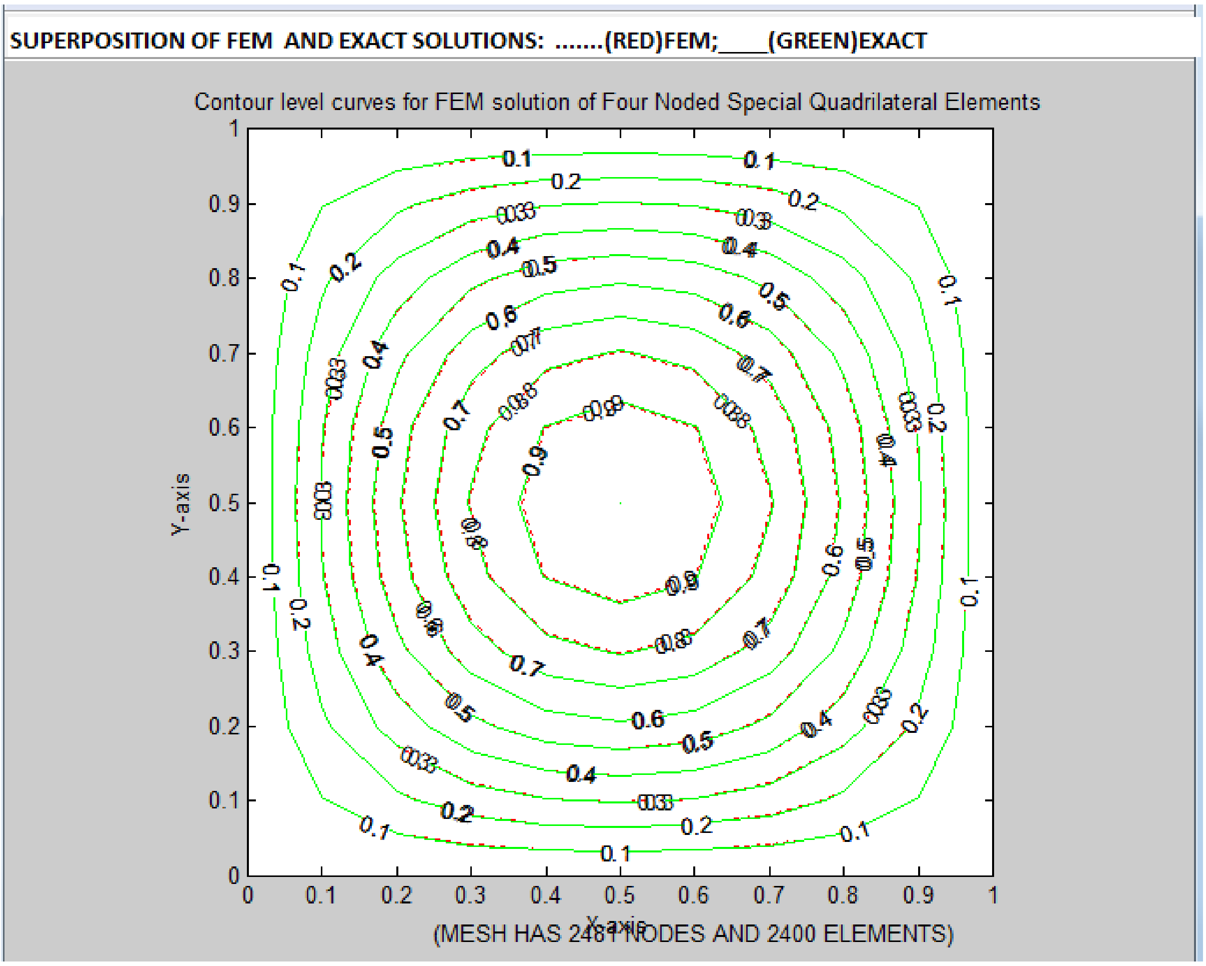
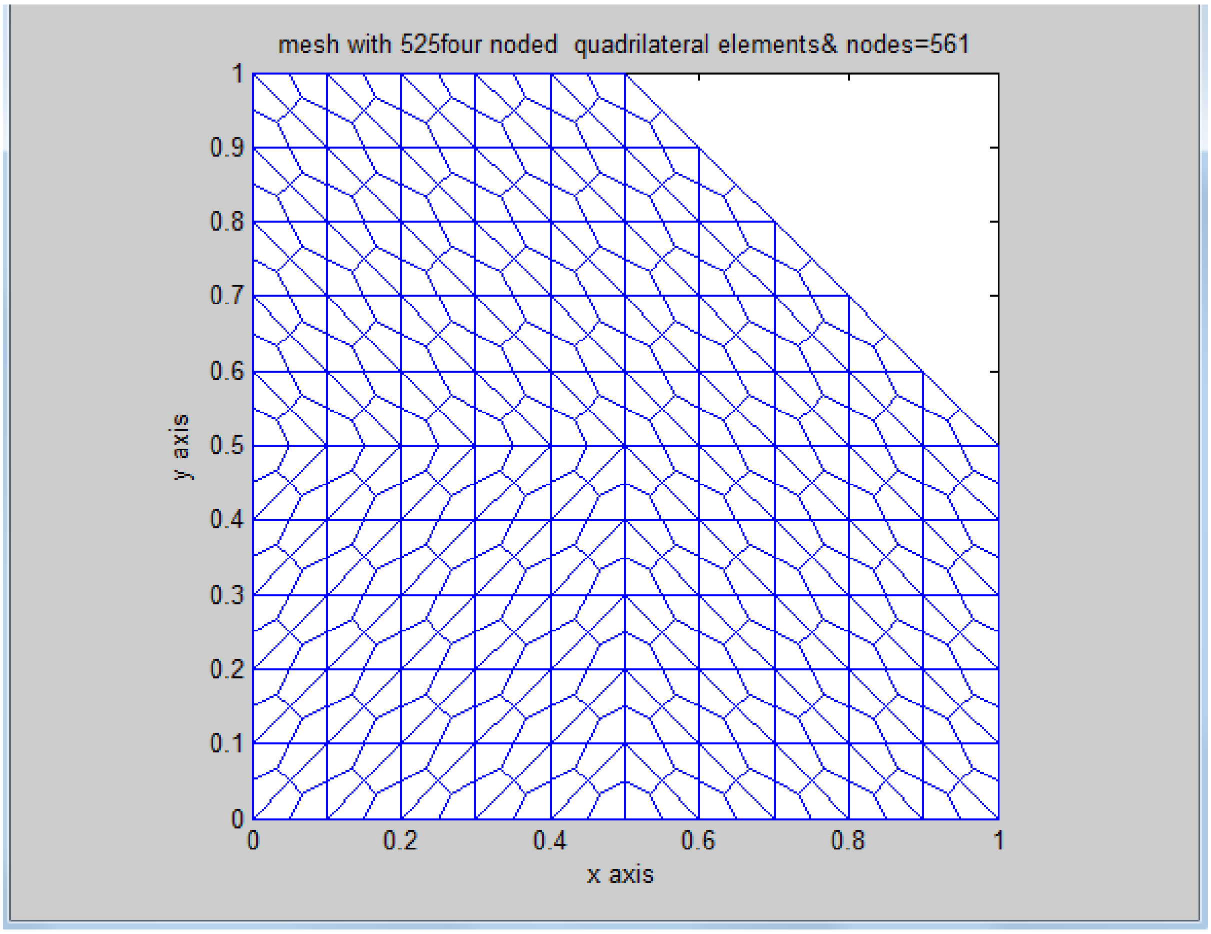
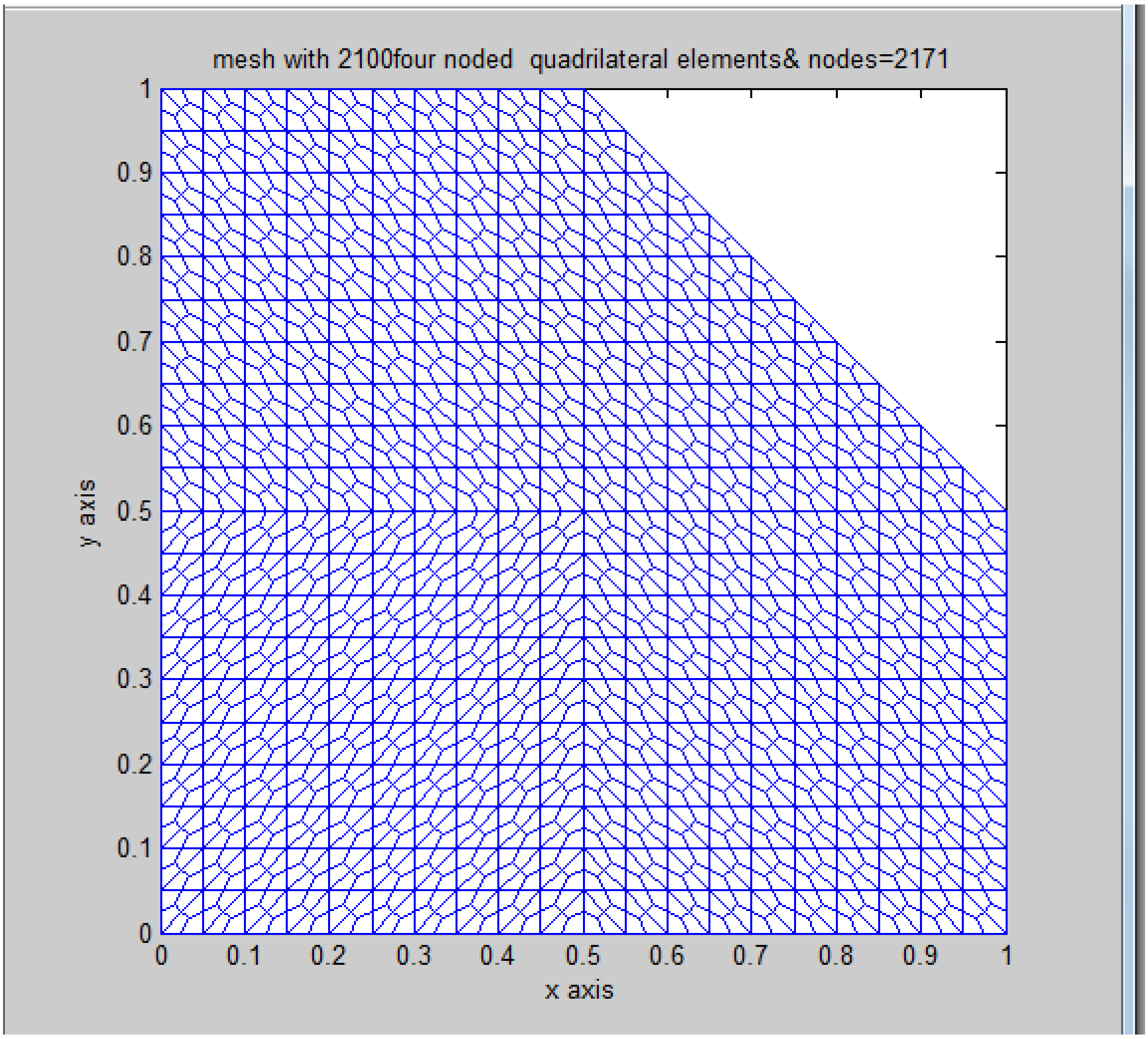


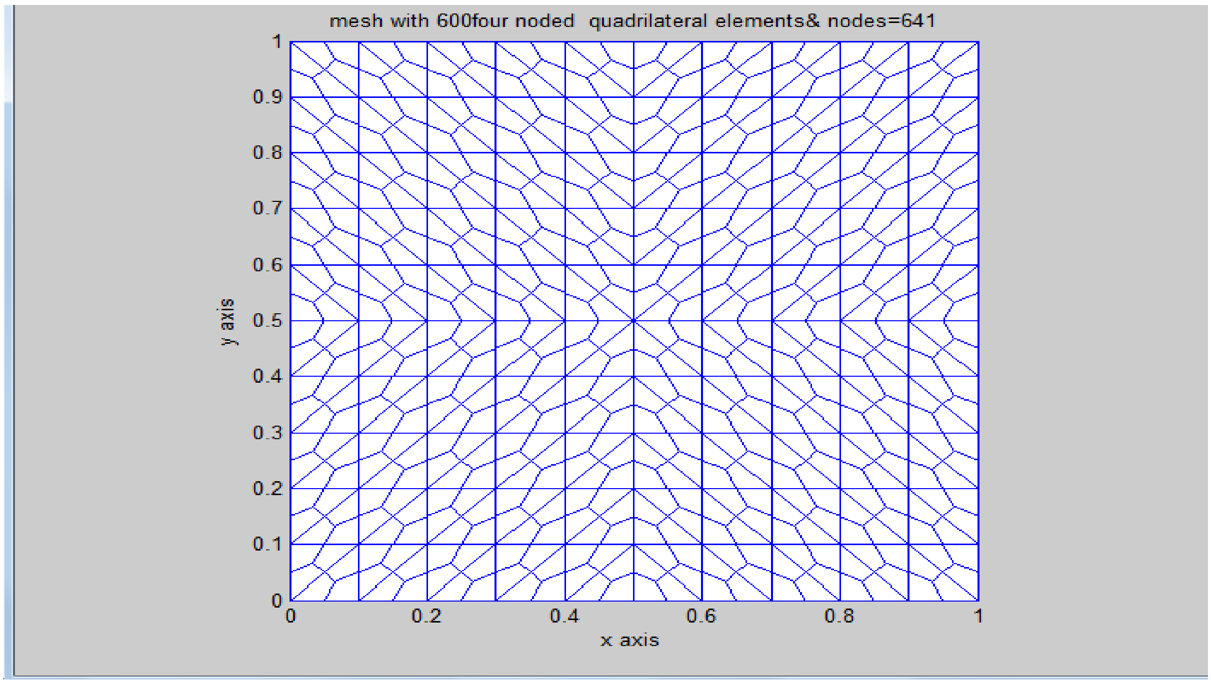
FEM MESHES USED IN THE ABOVE PRESENTATIONS

(1)MESH GENERATION:PENTAGONAL DOMAIN


mesh with 525four noded quadrilateral elements& nodes=561
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
y axis
0
0.2
0.4
0.6
0.8
1
x axis



(2)MESH GENERATION:SQUARE DOMAIN



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