www.ijecs.in
International Journal Of Engineering And Computer Science ISSN:2319-7242
Volume2 Issue 8 August, 2013 Page No. 2576-2610

Numerical Integration Over Polygonal Domains using Convex Quadrangulation and Gauss Legendre Quadrature Rules

H. T. Rathod[a,*], B. Venkatesh[b], Shivaram. K. T[c], Mamatha. T. M[b]

[a] Department of Mathematics, Central College Campus, Bangalore University, Bangalore-560 001, India
[b] Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Bangalore-560 035, India
[c] Department of Mathematics, Dayananda Sagar College of Engineering, Bangalore-560 078, India.

Abstract

This paper presents a numerical integration formula for the evaluation of $II_{\Omega}(f) = \iint_{\Omega} f(x,y)dxdy$, where $f \in C(\Omega)$ and Ω is any polygonal domain in $\Re^2$. That is a domain with boundary composed of piecewise straight lines. We then express $\text{II}_{P_N}(f) = \sum_{n=1}^{M} \text{II}_{T_n}(f) = \sum_{n=1}^{M}\left(\sum_{p=0}^{2} \text{II}_{Q_{3n-p}}(f)\right)$ in which P_N is a polygonal domain of N oriented edges l_{ik} $(k = i+1,\ i = 1,2,3,...,N)$, with end points (x_i, y_i), (x_k, y_k) and $(x_1, y_1) = (x_{N+1}, y_{N+1})$. We have also assumed that P_N can be discretised into a set of M triangles, T_n and each triangle T_n is further discretised into three special quadrilaterals Q_{3n-p} $(p = 0,1,2)$ which are obtained by joining the centroid to the midpoint of its sides. We choose $T_n = T_{pqr}^{xy}$ an arbitrary triangle with vertices $((x_\alpha, y_\alpha),\ \alpha = p,q,r)$ in Cartesian space (x, y). We have shown that an efficient formula for this purpose is given by

$$\text{II}_{T_n}(f) = (c_{pqr})\iint_S (4+\xi+\eta)\left(\sum_{e=1}^{3} f(x^{(e)}(u,v), y^{(e)}(u,v))\right)d\xi d\eta, \quad \text{where,}$$

$$z^{(e)}(u,v) = z_1^{(e)} + (z_2^{(e)} - z_1^{(e)})u + (z_3^{(e)} - z_1^{(e)})v,\ \ z = (x,y)$$

$$\left((z_1^{(e)}, z_2^{(e)}, z_3^{(e)}), e = 1,2,3\right) = \left((z_p, z_q, z_r), (z_q, z_r, z_p), (z_r, z_p, z_q)\right)$$

$$c_{pqr} = (area\ of\ T_n)/48,\ u = [1/3, 1/2, 0, 0][M_1, M_2, M_3, M_4]^T,\ v = [1/3, 0, 0, 1/2][M_1, M_2, M_3, M_4]^T,$$

$$M_\beta = M_\beta(\xi,\eta) = (1+\xi\xi_\beta)(1+\eta\eta_\beta)/4, \{(\xi_\beta,\eta_\beta), \beta = 1,2,3,4\} = \{(-1,-1),(1,-1),(1,1),(-1,1)\} \text{ and}$$

$S = \{(\xi,\eta) / -1 \le \xi, \eta \le 1\}$ is the standard 2- square in (ξ,η) space. Using Gauss Legendre Quadrature Rules of order 5(5)40, we obtain the weight coefficients and sampling points which can be used for any polygonal domain, $\Omega = P_N$ or T_n or Q_m $(m = 3n-2, 3n-1, 3n)$. Boundary integration methods are also proposed which are helpful in verifying the application of derived formulas to compute some typical integrals.

Keywords: Polygonal domain, Triangular and Convex Quadrilateral regions, Boundary integration, Green's theorem, Gauss Legendre Quadrature, Composite integration.

1 Introduction

The finite element method is one of the most powerful computational technique for approximate solution of a variety of "real world" engineering and applied science problems for over half a century since its inception in the mid 1960. Today, finite element analysis (FEA) has become an integral and major component in the design or modelling of a physical phenomenon in various disciplines. The triangular and quadrilateral elements with either straight sides or curved sides are very widely used in a variety of applications [1-3]. The basic problem of integrating a function of two variables over the surface of the triangle is the subject of extensive research by many authors [4-5]. Derivation of high precision formulas is now possible over the triangular region by application of product formulas based only on the sampling points and weights of the well known Gauss Legendre quadrature rules [6-8]. There are reasons which support the development of composite integration for practical applications. In some recent investigations composite integration is illustrated with reference to the standard triangle [9-10]. Recently, in [11] Green's integral formula is used in the numerical evaluation of $II_{\Omega}(f)=\iint\limits_{\Omega} f(x,y)dxdy$ by transforming a two dimensional problem into a one dimensional problem and by using univariate Gauss Legendre quadrature products. In [12], a cubature formula over polygons is proposed which is based on a 8-node spline finite elements. They use very dense meshes to prove the convergence of test function integrals for which error is shown to be in the range of 10^{-1} to 10^{-9}. In this paper we develop composite integration rules for polygonal domains which are fully discretised by special quadrilaterals and the test function integrals are shown to agree with the exact values up to 16 significant digits for smooth functions, this implies that the absolute error is of the order 10^{-16}. The composite integration rules of this paper as well as the cubature formulas of 8- node spline elements [12] converge to the exact values a little slowly for some nonsmooth functions. This again confirms the superiority of product formulas. In section 2 of this paper, we begin with a brief description of the special discretisation of arbitrary and the standard (right isosceles) triangular elements into a set of three special quadrilaterals which are obtained by joining the centroids to the midpoints of sides. In section 3 of this paper, we define some relevant linear transformations. In section 3.1, we prove lemma 1, which establishes the relation between the special quadrilaterals of an arbitrary triangle in (x,y) space and the special quadrilaterals of the standard triangle in (u,v) space by use of a single linear transformation between the global space (x,y) and the local parametric space (u,v). Then in section 3.2, we prove lemma 2 which establishes the relation between the three special quadrilaterals in (x,y) and a unique special quadrilateral interior to the standard triangle in (u,v) space by using three linear transformations. Section 4 of this paper is regarding the explicit form of the Jacobians. In section 4.1, we determine the explicit form of Jacobian when the arbitrary triangle in the global space (x,y) is mapped into a standard triangle in the local space (u,v) for the linear transformations used in lemma 1 and lemma 2. Section 4.2 of the paper begins with the derivation of explicit form of Jacobian for an arbitrary linear convex quadrilateral. In section 4.3, we determine the Jacobian for the special quadrilaterals $Q_e\,(e=1,2,3)$ in the global space (x,y) and the $\hat{Q}_e\,(e=1,2,3)$ in the local space (u,v), in either case we obtain the Jacobian as $c(4+\xi+\eta)$, where c is some appropriate constant. We prove this result in lemma 3 when the $\hat{Q}_e\,(e=1,2,3)$ are mapped into 2-squares $-1\le\xi,\eta\le+1$. In section 5, we establish two composite integration formulas which use lemmas 1, 2 and 3 proved in sections 3.1, 3.2 and 4.3. In section 5.1, we establish a composite integration formula which uses three bilinear transformations and a single linear transformation and in section 5.2, we also establish a composite integration formula which depends on three linear transformations and a single linear transformation. We see that composite integration formulas of section 5.1 are of the form $x(u^e(\xi,\eta),v^e(\xi,\eta)),\ y(u^e(\xi,\eta),v^e(\xi,\eta)),\ e=1,2,3$ and require the computation of three sets (u^e,v^e), $e=1,2,3$ whereas the composite formulas of section 5.2 are of the form $x^e(u^1(\xi,\eta),v^1(\xi,\eta))$, $y^e(u^1(\xi,\eta),v^1(\xi,\eta))$, and require the computation of one set $(u^1(\xi,\eta),v^1(\xi,\eta))$. Thus we prefer to use composite integration formula of section 5.2. In section 6, we present the composite numerical integration formulas. We may note that the problem domain must be discretised into special quadrilaterals. The problem domain must contain at least one triangle for this purpose. The composite integration formulas are then obtained by application of Gauss Legendre quadrature rules [4] to the formula established in section 5.2. In section 7, we derive boundary integration methods to compute the integrals over a polygonal domain. They are helpful in verifying the application of proposed composite integration formulas. In section 8, we consider the evaluation of some typical integrals. This demonstrates the efficiency of the derived formulas of last section. We have also appended the relevant and necessary computer codes.

2 A Special Discretization of Triangles

In this section, we describe a special discretisation scheme to generate quadrilaterals from triangles. In the proposed scheme, three unique quadrilaterals are obtained by joining the centroid of any triangle to the midpoints of its sides. We define such quadrilaterals as special quadrilaterals for our present investigations.

2.1 Special Quadrilaterals of an Arbitrary Triangle

We first consider an arbitrary triangle ΔPQR in the Cartesian space (x, y) with vertices $P(x_p, y_p)$, $Q(x_q, y_q)$ and $R(x_r, y_r)$. Let $Z((x_p + x_q + x_r)/3, (y_p + y_q + y_r)/3))$ be its centroid and also let S, T, U be the mid points of sides PQ, QR and RP respectively. Now by joining the centroid Z to the midpoints S, T, U by straight lines, we divide the triangle ΔPQR into three special quadrilaterals Q_1, Q_2 and Q_3 (say) which are spanned by vertices $\langle Z, U, P, S\rangle$, $\langle Z, S, Q, T\rangle$, and $\langle Z, T, R, U\rangle$ respectively.This is shown in Fig.1a

2.2 Special Quadrilaterals of a Standard Triangle

We next consider the triangle ΔABC in the Cartesian space (u, v) with vertices, centroid and midpoints: $A(1,0)$, $B(0,1)$, $C(0,0)$, $G(1/3,1/3)$, $D(1/2,1/2)$, $E(0,1/2)$ and $F(1/2,0)$. We now divide the triangle ΔABC into three special quadrilaterals $\hat{Q}_1, \hat{Q}_2$, and $\hat{Q}_3$ (say) which are spanned by vertices $\langle G, E, C, F\rangle$, $\langle G, F, A, D\rangle$, and $\langle G, D, B, E\rangle$ respectively. This is shown in Fig.1b

3 Linear Transformations

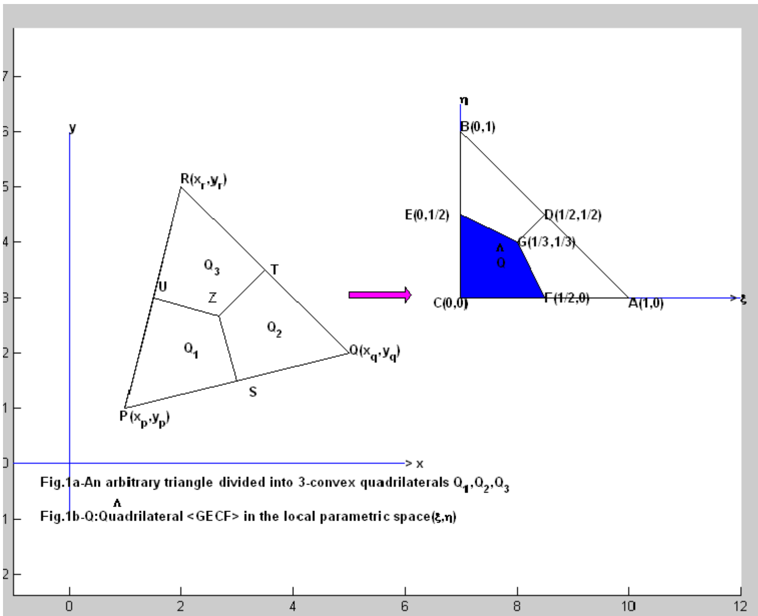
We apply linear transformations to map an arbitrary triangle into a triangle of our choice. In this section, we use the well known linear transformation which maps an arbitrary triangle into a standard triangle (a right isosceles triangle). We also assume the special discretization scheme of the previous section for the following developments.

3.1 Lemma 1. There exists a unique linear transformation which map the special quadrilaterals Q_i into $\hat{Q}_i$ (i =1, 2, 3) satisfying the conditions

(i) $\sum_{i=1}^{3} Q_i = \Delta PQR$, the arbitrary triangle in the (x, y) space.

(ii) $\sum_{i=1}^{3} \hat{Q}_i = \Delta ABC$, the standard triangle (right isosceles) in the (u, v) space.

Fig.1a and Fig.1b



Proof: We shall now refer to Fig.1a, 1b and consider the following linear transformation between (x, y) and (u,v) spaces.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} w + \begin{pmatrix} x_q \\ y_q \end{pmatrix} u + \begin{pmatrix} x_r \\ y_r \end{pmatrix} v, \quad w = 1 - u - v \tag{1}$$

We can verify that the linear transformation of eqn. (1) maps the arbitrary triangle ΔPQR into the standard triangle ΔABC. The points *P, Q, R, S, T, U* and *Z* are respectively mapped into the points *A, B, C, D, E, F* and *G* respectively. The quadrilaterals Q_i are mapped into quadrilaterals $\hat{Q}_i$. This proves the existence of the required transformation.

3.2 Lemma 2. There exists three linear transformations which map the special quadrilaterals Q_i (i =1,2,3) in ΔPQR into a unique special quadrilateral $\hat{Q} = \hat{Q}_1$ (say) of the standard triangle ΔABC satisfying the conditions

(i) $\sum_{i=1}^{3} Q_i = \Delta PQR$, the arbitrary triangle in the (x, y) space.

(ii) $\sum_{i=1}^{3} \hat{Q}_i = \Delta ABC$, the standard triangle (right isosceles) in the (u,v) space.

Proof: We again refer to Fig.1a, 1b and consider the following linear transformations between (x, y) and (u,v) spaces.

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} w + \begin{pmatrix} x_q \\ y_q \end{pmatrix} u + \begin{pmatrix} x_r \\ y_r \end{pmatrix} v, \quad w = 1 - u - v \tag{2}$$

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \end{pmatrix} w + \begin{pmatrix} x_r \\ y_r \end{pmatrix} u + \begin{pmatrix} x_p \\ y_p \end{pmatrix} v, \quad w = 1 - u - v \tag{3}$$

$$\begin{pmatrix} x^{(3)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \end{pmatrix} w + \begin{pmatrix} x_p \\ y_p \end{pmatrix} u + \begin{pmatrix} x_q \\ y_q \end{pmatrix} v, \quad w = 1 - u - v \tag{4}$$

It is quite clear that each of the above transformations map the arbitrary triangle ΔPQR into the standard triangle ΔABC. We may further note the following.

(i) The transformation of eqn. (2) maps the vertices *P, Q, R* in (x, y) space into vertices $C(0,0),\ A(1,0),\ B(0,1)$ in (u,v) space.

(ii) The transformation of eqn. (3) maps the vertices *Q, R, P* in (x, y) space into vertices $C(0,0),\ A(1,0),\ B(0,1)$ in (u,v) space.

(iii) The transformation of eqn. (4) maps the vertices *R, P, Q* in (x, y) space into vertices $C(0,0),\ A(1,0),\ B(0,1)$ in (u,v) space.

We can now verify that the linear transformation of eqn. (2) maps the quadrilateral Q_1 spanning the vertices $\langle Z, U, P, S \rangle$ in (x, y) space into the quadrilateral $\hat{Q} = Q_1$ spanning the vertices $\langle G, E, C, F \rangle$ in the (u,v) space. In a similar manner, we find that using the linear transformation of eqn. (3) the quadrilateral Q_2 spanned by vertices $\langle Z, S, Q, T \rangle$ in (x, y) space is mapped into the quadrilateral $\hat{Q} = Q_1$ spanning the vertices $\langle G, E, C, F \rangle$ in the (u,v) space. Finally on using the linear transformation of eqn. (4) the quadrilateral Q_3 spanned by vertices $\langle Z, T, R, U \rangle$ in (x, y) space is mapped into the quadrilateral $\hat{Q} = Q_1$ spanning the vertices $\langle G, E, C, F \rangle$ in the (u,v) space. This completes the proof of Lemma 2.

We may note here that the linear transformations $(x^{(1)}, y^{(1)})^T$ in eqn. (2) and $(x, y)^T$ in eqn. (1) are identical. We wish to say in advance that the application of the above lemmas will be of immense help in the development of this paper.

4 Explicit forms of the Jacobians

We have shown in the previous section that the quadrilaterals Q_e in Cartesian/global space (x, y) can be mapped into $\hat{Q}_e$ in the (u,v) space. Our ultimate aim is to find explicit integration formulas over the region Q_e. In this process, we first transform the integrals over Q_e into $\hat{Q}_e$, then the integrals over $\hat{Q}_e$ will be transformed to integrals over the 2-squares ($-1 \le \xi, \eta \le 1$) in (ξ,η) space using the bilinear transformations from (u,v) space to (ξ,η) space. The main reason in adopting this process is that, the integration over the quadrilaterals is independent of the nodal coordinates of the global/Cartesian space (x, y). This requires explicit form of the Jacobian which uses linear transformations to map Q_e into $\hat{Q}_e$ and the explicit form of the Jacobian which uses the bilinear transformation to map the $\hat{Q}_e$ into the 2-squares in (ξ,η) space.

4.1 Explicit form of the Jacobian using Linear Transformations

First, we consider the linear transformation $(x^{(e)}, y^{(e)})^T$ of eqns.(2-4) for lemma 2 which map the Q_e in (x, y) space into $\hat{Q}_e$ in (u,v) space.

Then it can be easily verified that

$$\frac{\partial(x^{(e)}, y^{(e)})}{\partial(u,v)} = \frac{\partial x^{(e)}}{\partial u}\frac{\partial y^{(e)}}{\partial v} - \frac{\partial x^{(e)}}{\partial v}\frac{\partial y^{(e)}}{\partial u}$$

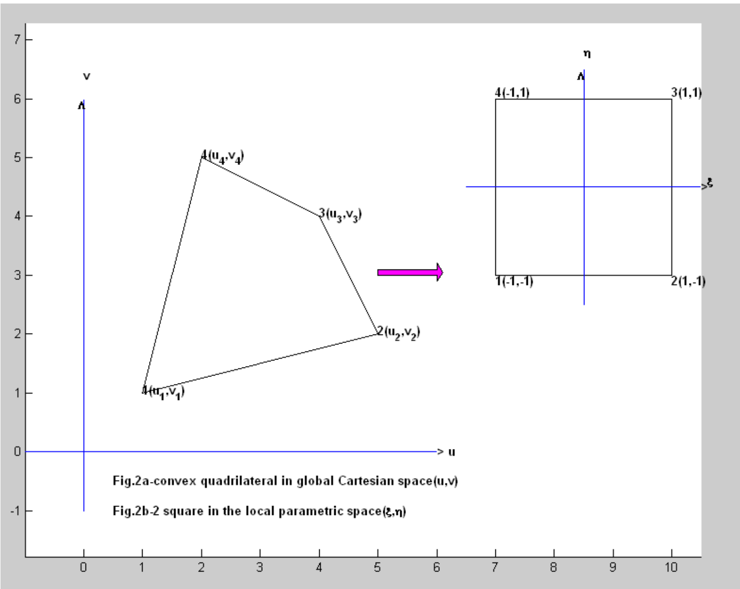
$$= 2 \times \text{area of the triangle } \Delta PQR$$

$$= \begin{vmatrix} 1 & x_p & y_p \\ 1 & x_q & y_q \\ 1 & x_r & y_r \end{vmatrix} = 2 \times \Delta_{pqr} \text{ (say)} \qquad (5)$$

We also note that for lemma1 $(x, y)^T = (x^{(1)}, y^{(1)})^T$. Hence again, we obtain the same value for J.

4.2 Explicit form of the Jacobian using Bilinear Transformations

Fig.2a and Fig.2b



Let us consider an arbitrary four noded linear convex quadrilateral element in the global Cartesian space (u,v) as shown in Fig.2a which is mapped into a 2-square in the local parametric space (ξ,η) as shown in Fig.2b. The necessary bilinear transformation is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k=1}^{4} \begin{pmatrix} u_k \\ vk \end{pmatrix} M_k(\xi,\eta) \tag{6}$$

where (u_k, v_k), $(k = 1,2,3,4)$ are the vertices of the quadrilateral element $Q^*{}_e$ in the (u,v) plane and $M_k(\xi,\eta)$ denotes the shape function of node k and they are expressed in the standard texts[1-3]:

$$M_k(\xi,\eta) = \frac{1}{4}(1+\xi\xi_k)(1+\eta\eta_k) \tag{7a}$$

$$\{(\xi_k,\eta_k), k = 1,2,3,4\} = \{(-1,-1),\ (1,-1),\ (1,1),\ (-1,1)\} \tag{7b}$$

From eqns. (6) and (7), we have

$$\frac{\partial u}{\partial \xi} = \sum_{k=1}^{4} u_k \frac{\partial M_k}{\partial \xi} = \frac{1}{4}[(-u_1+u_2+u_3-u_4)+(u_1-u_2+u_3-u_4)\eta] \tag{8a}$$

$$\frac{\partial u}{\partial \eta} = \sum_{k=1}^{4} u_k \frac{\partial M_k}{\partial \eta} = \frac{1}{4}[(-u_1-u_2+u_3+u_4)+(u_1-u_2+u_3-u_4)\xi] \tag{8b}$$

Similarly,

$$\frac{\partial v}{\partial \xi} = \sum_{k=1}^{4} v_k \frac{\partial M_k}{\partial \xi} = \frac{1}{4}[(-v_1+v_2+v_3-v_4)+(v_1-v_2+v_3-v_4)\eta] \tag{8c}$$

$$\frac{\partial v}{\partial \eta} = \sum_{k=1}^{4} v_k \frac{\partial M_k}{\partial \eta} = \frac{1}{4}[(-v_1-v_2+v_3+v_4)+(v_1-v_2+v_3-v_4)\xi] \tag{8d}$$

Hence, from eqns.(8), the Jacobian can be expressed as

$$J^* = \frac{\partial(u,v)}{\partial(\xi,\eta)} = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \alpha + \beta\xi + \gamma\eta \tag{9a}$$

where, $\alpha = [(u_4-u_2)(v_1-v_3)+(u_3-u_1)(v_4-v_2)]/8,$

$\beta = [(u_4-u_3)(v_2-v_1)+(u_1-u_2)(v_4-v_3)]/8,$

$$\gamma = [(u_4-u_1)(v_2-v_3)+(u_3-u_2)(v_4-v_1)]/8 \tag{9b}$$

4.3 Explicit form of Jacobian for Special Quadrilaterals

Lemma 3. Let ΔABC be an arbitrary triangle with vertices $A(1,0),\ B(0,1),\ C(0,0)$ and let $D(1/2,1/2), E(0,1/2)$ and $F(1/2,0)$ be midpoints of sides AB, BC and CA respectively and also let $G(1/3,1/3)$ be its centroid. Then the Jacobian of the three special quadrilaterals $\hat{Q}_e(e$ = 1, 2, 3) viz $\langle G, E, C, F\rangle, \langle G, F, A, D\rangle$ and $\langle G, D, B, E\rangle$ have the same expression given by:

$$\hat{J} = \frac{\partial(u,v)}{\partial(\xi,\eta)} = \hat{J}^e = \frac{1}{96}(4+\xi+\eta),\ (e = 1,2,3) \tag{10a}$$

Proof: We can immediately verify that eqn.(10a) is true by substituting the nodal values of $\hat{Q}_e$ in eqn. (9a-b).

The general result for special quadrilaterals $Q_e(e$ =1,2,3) follows by direct substitution of geometric coordinates of the vertices in eqns. (9a-9b) or by chain rule of partial differentiation and use of eqn.(1):

$$J = J^e = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(\xi,\eta)} = \left(2\Delta_{pqr}\right)\left(\frac{4+\xi+\eta}{96}\right) = \frac{\Delta_{pqr}}{48}\left(4+\xi+\eta\right) \tag{10b}$$

5 Problem Statement

In some physical applications, we are required to compute integrals of some functions which are expressed in explicit form. In finite element and boundary element method, evaluation of two dimensional integrals with explicit functions as integrands is of great importance. This is the subject matter of several investigations [4-15]. We now consider the evaluation of the integral

$$II_\Omega(f) = \iint_\Omega f(x,y)\,dxdy,\ \Omega: \text{polygonal domain} \tag{11}$$

$II_\Omega(f)$ can be computed as finite sum of linear integrals and this can be expressed as
$$II_\Omega(f)=\sum_i \iint_{\Delta_i} f(x,y)dxdy \tag{12}$$
where it is assumed that $\Omega=\bigcup_i \Delta_i$, Δ_i = an arbitrary triangle of the domain Ω.

5.1 Composite integration over an arbitrary triangle

Integration over a triangular domain is computed by use of linear transformation between Cartesian and area coordinates. We use the transformation
$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \end{pmatrix}\begin{pmatrix} 1-u-v \\ u \\ v \end{pmatrix} \tag{13}$$
to map the arbitrary triangle ΔPQR with vertices $((x_p,y_p),(x_q,y_q),(x_r,y_r))$ in (x,y) space into a standard triangle with vertices $(0,0),(1,0),(0,1)$ in (u,v) space. The original triangle ΔPQR in (x,y) space and the transformed triangle in (u,v) space are shown in Fig 1*a*,*b* and hence from eqn.(5) and above eqn.(13)
$$\frac{\partial(x,y)}{\partial(u,v)}=2\times \text{area of triangle } \Delta PQR = 2\Delta_{pqr}$$
$$=(x_q-x_p)(y_r-y_p)-(x_r-x_p)(y_q-y_p) \tag{14}$$
We now define
$$II_{\Delta PQR}(f)=\iint_{\Delta PQR} f(x,y)dxdy$$
$$=2\Delta_{pqr}\int_0^1\int_0^{1-u} f(x(u,v),y(u,v))dudv \tag{15}$$
We now divide the triangle ΔPQR into three special quadrilaterals Q_e as discussed in the previous section. By use of Lemma1 we know that the special quadrilaterals Q_e in (x,y) space are transformed into special quadrilaterals $\hat{Q}_e$ in (u,v) space. We use Lemma2 to transform each of these $\hat{Q}_e$ into a 2-square in (ξ,η) space by means of the following linear transformations between (x,y) and (u,v) spaces.
$$\begin{aligned} x(u^e,v^e)=x^{(e)}(\xi,\eta)&=\left(1-u^{(e)}-v^{(e)}\right)x_p+u^{(e)}x_q+x_r v^{(e)},(e=1,2,3) \\ y(u^e,v^e)=y^{(e)}(\xi,\eta)&=\left(1-u^{(e)}-v^{(e)}\right)y_p+u^{(e)}y_q+y_r v^{(e)},(e=1,2,3) \end{aligned} \tag{16}$$
and the bilinear transformation between (u,v) and (ξ,η) spaces
$$\begin{aligned} u^{(e)}=u^{(e)}(\xi,\eta)&=u_1^{(e)}M_1+u_2^{(e)}M_2+u_3^{(e)}M_3+u_4^{(e)}M_4 \\ v^{(e)}=v^{(e)}(\xi,\eta)&=v_1^{(e)}M_1+v_2^{(e)}M_2+v_3^{(e)}M_3+v_4^{(e)}M_4 \end{aligned} \tag{17}$$
where
$$\left((u_k^{(1)},v_k^{(1)}),k=1,2,3,4\right)=\left(\left(\frac{1}{3},\frac{1}{3}\right),\left(0,\frac{1}{2}\right),(0,0),\left(\frac{1}{2},0\right)\right),$$
$$\left((u_k^{(2)},v_k^{(2)}),k=1,2,3,4\right)=\left(\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{2},0\right),(1,0),\left(\frac{1}{2},\frac{1}{2}\right)\right),$$
$$\left((u_k^{(3)},v_k^{(3)}),k=1,2,3,4\right)=\left(\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{2},\frac{1}{2}\right),(0,1),\left(0,\frac{1}{2}\right)\right),$$
$$\begin{aligned} M_k=M_k(\xi,\eta)&=\frac{1}{4}(\xi+\xi\xi_k)(1+\eta\eta_k),k=1,2,3,4 \\ (\xi_k,\eta_k),k=1,2,3,4&=((-1,-1),(1,-1),(1,1),(-1,1)) \end{aligned} \tag{18}$$
We now verify that

$$u^{(1)}=u^{(1)}(\xi,\eta)=\frac{(1-\xi)(5+\eta)}{24},$$

$$v^{(1)}=v^{(1)}(\xi,\eta)=\frac{(1-\eta)(5+\xi)}{24},$$ We can

$$1-u^{(1)}-v^{(1)}=\frac{(7+2\xi+2\eta+\xi\eta)}{12}. \tag{19}$$

also further verify that

$$u^{(2)}=1-u^{(1)}-v^{(1)},\quad v^{(2)}=u^{(1)},\quad 1-u^{(2)}-v^{(2)}=v^{(1)},$$
$$u^{(3)}=v^{(1)},\quad v^{(3)}=1-u^{(1)}-v^{(1)},\quad 1-u^{(3)}-v^{(3)}=u^{(1)}. \tag{20}$$

Thus, we find the following three unique transformations which map the special quadrilaterals $Q_i,(i=1,2,3)$ in (x,y) space into a 2-square in (ξ,η) space:

$$x^{(1)}(\xi,\eta)=\left(1-u^{(1)}-v^{(1)}\right)x_p+u^{(1)}x_q+v^{(1)}x_r$$
$$y^{(1)}(\xi,\eta)=\left(1-u^{(1)}-v^{(1)}\right)y_p+u^{(1)}y_q+v^{(1)}y_r \tag{21}$$

$$x^{(2)}(\xi,\eta)=\left(1-u^{(1)}-v^{(1)}\right)x_q+u^{(1)}x_r+v^{(1)}x_p$$
$$y^{(2)}(\xi,\eta)=\left(1-u^{(1)}-v^{(1)}\right)y_q+u^{(1)}y_r+v^{(1)}y_p \tag{22}$$

$$x^{(3)}(\xi,\eta)=\left(1-u^{(1)}-v^{(1)}\right)x_r+u^{(1)}x_p+v^{(1)}x_q$$
$$y^{(3)}(\xi,\eta)=\left(1-u^{(1)}-v^{(1)}\right)y_r+u^{(1)}y_p+v^{(1)}y_q \tag{23}$$

The transformations of eqns.(21)- (23) are of the form stated in eqn.(2), (3), (4) and from eqn.(19), we now define

$$u^{(1)}=\lambda=\lambda(\xi,\eta),v^{(1)}=\mu=\mu(\xi,\eta),1-u^{(1)}-v^{(1)}=(1-\lambda-\mu) \tag{24}$$

The above findings again prove the hypothesis of Lemma 2.

5.2 Composite Integration Formula for the Arbitrary Triangle

We again consider the integral defined earlier in eqn.(15) and use Lemma1, 2 and our findings of section 5.1.

$$\begin{aligned}
II_{\Delta PQR}(f) &= \iint_{\Delta PQR} f(x,y)dxdy \\
&= 2\Delta_{pqr}\int_0^1\int_0^{1-u} f(x(u,v),y(u,v))dudv \\
&= \sum_{e=1}^{3}\iint_{Q_e} f(x,y)dxdy \\
&= 2\Delta_{pqr}\sum_{e=1}^{3}\iint_{\hat{Q}_e} f(x(u^{(e)},v^{(e)}),y(u^{(e)},v^{(e)}))du^{(e)}dv^{(e)} \\
&= 2\Delta_{pqr}\sum_{e=1}^{3}\iint_{\hat{Q}_e} f(x^{(e)}(u^{(1)},v^{(1)}),y^{(e)}(u^{(1)},v^{(1)}))du^{(1)}dv^{(1)} \\
&= 2\Delta_{pqr}\sum_{e=1}^{3}\iint_{\hat{Q}_e} f(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu))d\lambda d\mu \\
&= 2\Delta_{pqr}\int_{-1}^{1}\int_{-1}^{1}\sum_{e=1}^{3} f(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu))\frac{\partial(x^{(e)},y^{(e)})}{\partial(\xi,\eta)}d\xi d\eta \\
&= 2\Delta_{pqr}\int_{-1}^{1}\int_{-1}^{1}\frac{(4+\xi+\eta)}{96}\left(\sum_{e=1}^{3} f(x^{(e)}(\lambda,\mu),y^{(e)}(\lambda,\mu))\right)d\xi d\eta
\end{aligned} \tag{25}$$

where we have from eqns.(21) – (24)

$$x^{(1)}(\xi,\eta) = (1-\lambda-\mu)x_p + \lambda x_q + \mu x_r$$
$$y^{(1)}(\xi,\eta) = (1-\lambda-\mu)y_p + \lambda y_q + \mu y_r \quad (26)$$
$$x^{(2)}(\xi,\eta) = (1-\lambda-\mu)x_q + \lambda x_r + \mu x_p$$
$$y^{(2)}(\xi,\eta) = (1-\lambda-\mu)y_q + \lambda y_r + \mu y_p \quad (27)$$
$$x^{(3)}(\xi,\eta) = (1-\lambda-\mu)x_r + \lambda x_p + \mu x_q$$
$$y^{(3)}(\xi,\eta) = (1-\lambda-\mu)y_r + \lambda y_p + \mu y_q \quad (28)$$

with $\lambda = \lambda(\xi,\eta)$, $\mu = \mu(\xi,\eta)$ as given in eqns. (19) and (24).

6 Numerical Integration Formulas

6.1 Numerical integration over an Arbitrary Triangle ΔPQR

We could use either of the formulas in eqn.(16) or eqns.(26-28). We prefer to use eqn.(25), since it requires the computation of just one set of $(u,v) = (u^{(1)}, v^{(1)})$ for all the three quadrilaterals. The transformation formulas of eqns. (26-28) are easy to implement as a computer code, since the coordinates of ΔPQR are to be used in cyclic permutation in $(x^{(e)}, y^{(e)}),\ e = 1,2,3)$. Note that in $(x^{(1)},\ y^{(1)})^T$ the coefficients of w, u, v are $(x_p, y_p)^T$, $(x_q, y_q)^T$, $(x_r, y_r)^T$ respectively. In $(x^{(2)},\ y^{(2)})^T$ the coefficients of w, u, v are $(x_q, y_q)^T$, $(x_r, y_r)^T$, $(x_p, y_p)^T$ respectively and in $(x^{(3)},\ y^{(3)})^T$ the coefficients of w, u, v are $(x_r, y_r)^T$, $(x_p, y_p)^T$, $(x_q, y_q)^T$ respectively. We can use Gauss Legendre quadrature rule to evaluate eqn.(25). The resulting numerical integration formula can be written as

$$I\,I_{\Delta PQR}(f) \approx 2\Delta_{pqr} \sum_{k=1}^{N\times N} W_k^{(N)} \sum_{e=1}^{3} \left(f(x^{(e)}(U_k^{(N)},\ V_k^{(N)}), y^{(e)}(U_k^{(N)},\ V_k^{(N)})) \right) \quad (29)$$

The weights and sampling points in the above formula satisfy the relation

$$\left((W_k^{(N)},\ U_k^{(N)},\ V_k^{(N)})\right) = \left((4 + s_i^{(N)} + s_j^{(N)}) w_i^{(N)} w_j^{(N)} \big/ 96,\ u(s_i^{(N)},\ s_j^{(N)}),\ v(s_i^{(N)},\ s_j^{(N)})\right)$$
$$k = 1,2,3,\ldots..N\times N,\ \ i, j = 1,2,3,\ldots..N \quad (30)$$

and

$$u(s_i^{(N)},\ s_j^{(N)}) = (1-s_i^{(N)})(1-s_j^{(N)})\big/12 + (1-s_i^{(N)})(1+s_j^{(N)})\big/8,$$
$$v(s_i^{(N)},\ s_j^{(N)}) = (1-s_i^{(N)})(1-s_j^{(N)})\big/12 + (1+s_i^{(N)})(1-s_j^{(N)})\big/8 \quad (31)$$

for a *N*- point Gauss Legendre rule of order *N* with $\left((w_n^{(N)},\ s_n^{(N)}),\ n = 1,2,3,\ldots...N\right)$ as the weights and sampling points respectively.

We can compute the arrays $\left((W_k^{(N)},\ U_k^{(N)},\ V_k^{(N)}), k = 1,2,3,\ldots...N^2\right)$ for any available Gauss Legendre quadrature rule of order *N*. We have listed a code to compute the arrays $(W_k^{(N)},\ U_k^{(N)},\ V_k^{(N)}, k = 1,2,3,\ldots...N^2)$ for *N* = 5, 10, 15, 20, 25, 30, 35, 40. This is necessary since explicit list of $(W_k^{(N)},\ U_k^{(N)},\ V_k^{(N)}, N = 5, 10, 15, 20, 25, 30, 35, 40)$ will generate a large amount of values, viz: 25, 100, 225, 400, 625, 900, 1225, and 1600 for each of $W_k^{(N)},\ U_k^{(N)},\ V_k^{(N)}$. The computer code will be simple with few statements and it requires the input values of $\left((w_n^{(N)},\ s_n^{(N)},\ n = 1,2,\ldots,\ N), N = 5,10,15,20,25,30,35,40\right)$.

6.2 Composite Integration over a polygonal Domain P_N

We now consider the evaluation of $II_\Omega(f) = \iint_\Omega f(x,y)dxdy$, where $f \in C(\Omega)$ and Ω is any polygonal domain in $\Re^2$. That is a domain with boundary composed of piecewise straight lines. We then write $\text{II}_{P_N}(f) = \sum_{n=1}^{M} \text{II}_{T_n}(f) = \sum_{n=1}^{M}\left(\sum_{p=0}^{2} \text{II}_{Q_{3n-p}}(f)\right)$ (32) in which, we define

P_N is a polygonal domain of *N* oriented edges l_{ik} $(k = i+1,\ i = 1,2,3,\ldots,N),$ with end points $(x_i, y_i),\ (x_k, y_k)$

and $(x_1, y_1) = (x_{N+1}, y_{N+1})$. We have assumed in the above eqn. (32) that P_N can be discretised into a set of M triangles, $T_n\,(n = 1, 2, 3, ..., M)$. In the numerical integration formula of section 6.1, we have $T_n = T_{pqr}^{xy} = \Delta PQR$, an arbitrary triangle with vertices $((x_\alpha, y_\alpha),\ \alpha = p, q, r)$ in Cartesian space (x, y). The numerical integration formula for $\text{II}_{T_n}(f) = \text{II}_{T_{pqr}^{xy}} = \text{II}_{\Delta PQR_n}$ is already explained in section 6.1. We can get higher accuracy for the integral $\text{II}_{P_N}(f)$ by using the refined triangular mesh of the polygonal domain. We have written a computer code in MATLAB for this purpose. We first decompose the given polygonal domain into a coarse mesh of M triangles T_n (say), as expressed in eqn.(32). We then refine the mesh containing M triangles into a new mesh with $n^2 \times M$ triangles, satisfying the relation $T_n = \sum_{p=1}^{n^2} t_p$. This division can be carried by using the linear transformations connecting the Cartesian space (x, y) and the local space (u, v).We divide the standard triangle (right isosceles) in the (u, v) space into n^2 right isosceles triangles and then use the linear transformation to obtain the corresponding Cartesian nodal coordinates using the linear transformation. The nodal connectivity data in (u, v) space is also determined for the subdivisions. This is prepared for each subdivision and incorporated into the computer code. The computer code can obtain integral values $II_\Omega(f_i(x, y)),\ \Omega = P_N$ by dividing the P_N into meshes with refinements. The first mesh is the coarse mesh with M triangles, the subsequent meshes will have $n^2 \times M$ ($n^2 = 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, ...$) triangles. We know that the numerical algorithm of section 6.1 is for an arbitrary triangle which is further divided into three special quadrilaterals. Thus the polygonal domain is divided into $3 \times n^2 \times M$ special quadrilaterals. This computer code written in MATLAB is appended for reference.

7 Boundary Integration over the Plane Polygon

In this section, we develop a numerical scheme to determine the integrals over a plane polygon. This will help us in verifying the computed values of some typical integrals considered in the next section. We have also developed a computer code based on MATLAB for this purpose. This will verify the computed values of integrals obtained by use of the numerical scheme developed in the previous section. It is well known that one of the remarkable theorems of multivariate calculus, Green's theorem for a plane region is of immense value for the computation of multiple integrals over the plane polygon. We know that $\iint_\Omega f(x, y)dxdy = \int_{\partial\Omega} F(x, y)dy,\ F(x, y) = \int f(x, y)dx$, where f is continuous on a domain Ω with piecewise smooth boundary $\partial\Omega$ described counterclockwise) gives in principle an appealing tool for numerical cubature, since it transforms a two-dimensional integration problem into a one-dimensional integration problem. Its practical use, however, requires the knowledge of a primitive of the integrand. However, this restriction can be overcome by proper application of the available theoretical knowledge on integration. We now show that by the application of Green's theorem, the integration over the polygonal domain with *n*-edges can be obtained as a sum of *n* triangles expanded with respect to the origin which does not require the knowledge of a primitive of the integrand. We now state and prove the following lemma 4 and theorems 1-2 which will help us in developing the computer codes.

Theorem 1. The integral $II_{\pi_{xy}}$ over a simple polygon with N - oriented edges $l_{ik}\ (k = i+1),\quad i = 1, 2, 3, \ldots, N$ each with end points (x_i, y_i) and (x_k, y_k) in the xy – plane is expressible as

$$II_{\pi_{xy}}(f) = \sum_{i=1}^{N} \iint_{T_{i0k}^{xy}} f(x, y)dxdy \text{ , (} f \text{ is a smooth function on the domain } \pi_{xy}) \qquad (33)$$

where, $x_{N+1} = x_1,\ y_{N+1} = y_1$, and T_{i0k}^{xy} refers to the triangle with vertices $(x_i, y_i),\ (0,0),\ (x_k, y_k), (k = i+1)$.

Proof: Let $\Phi(x, y) = \int f(x, y)dx$ and consider the following expression and apply the Green's theorem:

$$\left(\iint_{T_{j0i}^{xy}}+\iint_{T_{k0j}^{xy}}+\iint_{T_{i0k}^{xy}}\right)f(x,y)dxdy=\left(\iint_{T_{j0i}^{xy}}+\iint_{T_{k0j}^{xy}}+\iint_{T_{i0k}^{xy}}\right)\frac{\partial\Phi(x,y)}{\partial x}dxdy$$

$$=\left(\oint_{\partial T_{j0i}^{xy}}+\oint_{\partial T_{k0j}^{xy}}+\oint_{\partial T_{i0k}^{xy}}\right)\Phi(x,y)dy$$

$$=\left(\int_{l_{j0}}+\int_{l_{0i}}+\int_{l_{ij}}\right)\Phi(x,y)dy+\left(\int_{l_{k0}}+\int_{l_{0j}}+\int_{l_{jk}}\right)\Phi(x,y)dy+\left(\int_{l_{i0}}+\int_{l_{j0k}}+\int_{l_{ki}}\right)\Phi(x,y)dy$$

$$=\left(\int_{l_{ij}}+\int_{l_{jk}}+\int_{l_{ki}}\right)\Phi(x,y)dy=\oint_{\partial T_{ijk}^{xy}}\Phi(x,y)dy=\iint_{T_{ijk}^{xy}}f(x,y)dxdy \qquad (34)$$

Thus, we have proved that the triangle $T_{ijk}{}^{xy}$ expands into three new triangles with respect to the origin. We may note that in deriving the above result of eqn.(34), the following fact is used.

$$\int_{l_{0i}}\Phi(x,y)dy+\int_{l_{i0}}\Phi(x,y)dy=0$$

$$\int_{l_{j0}}\Phi(x,y)dy+\int_{l_{0j}}\Phi(x,y)dy=0$$

$$\int_{l_{k0}}\Phi(x,y)dy+\int_{l_{0k}}\Phi(x,y)dy=0$$

The general result of eqn.(33) can be readily proved on similar lines.

Theorem 2. Let us denote the triangle spanned by vertices (x_i,y_i), $(0,0)$, $(x_k,y_k),(k=i+1)$ as $T_{i0k}{}^{xy}$. Then the integral of a smooth function *f* over the region $T_{i0k}{}^{xy}$ is expressible as

$$II_{T_{i0k}{}^{xy}}(f)=(x_ky_i-x_iy_k)\int_0^1\int_0^1 rf(r(x_i+x_{ki}s),r(y_i+y_{ki}s))drds \qquad (35)$$

Proof: Let us consider the integral: $II_{T_{ijk}{}^{xy}}(f)=\iint_{T_{ijk}{}^{xy}}f(x,y)dxdy$ (36)

The parametric equations of the oriented triangle in the *xy* - plane with vertices spanned by (x_i,y_i), (x_j,y_j) and $(x_k,y_k),(k=i+1)$ which map this arbitrary triangle into a unit right isosceles triangle in the *uv* – plane are

$$x=x_i+x_{ji}u+x_{ki}v,\ y=y_i+y_{ji}u+y_{ki}v \qquad (37)$$

where, $0\le u,v\le 1,\ u+v\le 1,\ x_{ji}=x_j-x_i,\ x_{ki}=x_k-x_i,\ y_{ji}=y_j-y_i,\ y_{ki}=y_k-y_i$ (38)

We have then

$$dxdy=\frac{\partial(x,y)}{\partial(u,v)}dudv=(x_{ji}y_{ki}-x_{ki}y_{ji})dudv$$

$$=(2\Delta_{ijk}^{xy})dudv$$

$$=(2\times\text{area of the triangle } T_{ijk}^{xy})dudv \qquad (39)$$

and thus, we define

$$2\Delta_{ijk}^{xy}=(x_{ji}y_{ki}-x_{ki}y_{ji}) \qquad (40)$$

Using the above eqns.(37) –(39) in eqn.(36) gives us

$$II_{T_{i0k}{}^{xy}}=2\Delta_{ijk}^{xy}\int_0^1\int_0^{1-u}f((x_i+x_{ji}u+x_{ki}v),(y_i+y_{ji}u+y_{ki}v))dudv \qquad (41)$$

Using the transformations

$$u=1-r,\ v=rs \qquad (42)$$

we can map the above integral in eqn.(36) into an equivalent integral over the rectangle $\{(r,s)/0\le r,s\le 1\}$. This gives us

$$x = x_i + x_{ji}u + x_{ki}v = x_i + (x_j - x_i)(1-r) + x_{ki}rs$$
$$y = y_i + y_{ji}u + y_{ki}v = y_i + (y_j - y_i)(1-r) + y_{ki}rs \quad (43)$$
Letting $(x_j, y_j) = (0,0)$ in the above eqn.(43), we obtain
$$x_i + x_{ji}u + x_{ki}v = r(x_i + x_{ki}s)$$
$$y_i + y_{ji}u + y_{ki}v = r(y_i + y_{ki}s) \quad (44)$$
$$2\Delta_{i0k}^{xy} = (x_k y_i - x_i y_k) \quad (45)$$
$$dudv = \frac{\partial(u,v)}{\partial(r,s)} = -rdrds \quad (46)$$
and the limits of integration
$u = 0,\ u = 1,$ correspond to $r = 1,\ r = 0$
and $v = 0,\ v = 1-u,$ correspond to $s = 0,\ s = 1$
Thus from eqns.(41) - (46), we obtain
$$II_{T_{i0k}^{xy}}(f) = (x_k y_i - x_i y_k)\int_0^1\int_0^1 rf(r(x_i + x_{ki}s), r(y_i + y_{ki}s))drds$$
This completes the proof of Theorem 2.

Lemma 4

Let π_{xy} be a simple polygon in the $xy-$plane with N-oriented edges $l_{ik}\ (k = i+1, i = 1,2,3,..., N)$ each with end points (x_i, y_i) and (x_k, y_k), $(x_1, y_1) = (x_{N+1}, y_{N+1})$.

Then $\text{II}_{\pi_{xy}}(f) = \iint_{\pi_{xy}} f(x,y)dxdy = \sum_{i=1}^{N}\int_{l_{ik}} \Phi(x,y)dy$ (47) where l_{ik} refers to the like segment joining the points (x_i, y_i) and (x_k, y_k), and $\Phi(x,y) = \int f(x,y)dx$.

Proof:

Consider the integral of eqn. (47),
$$\text{II}_{\pi_{xy}}(f) = \iint_{\pi_{xy}} f(x,y)dxdy = \iint_{\pi_{xy}} \frac{\partial\Phi}{\partial x}dy, \Phi = \int f(x,y)dx$$
$$= \oint_{\partial\pi_{xy}} \Phi(x,y)dy, \text{ on using Green's theorem}$$
$$= \sum_{i=1}^{N}\int_{l_{ik}} \Phi(x,y)dy, \text{ (where } \partial\pi_{xy} \text{ refers to the boundary of } \pi_{xy})$$
This completes the proof of Lemma 4.

The line integrals in eqn. (47) can be computed by using the parametric equation of the line segment l_{ik} in $xy-$plane given by
$$x = x(t) = x_i + (x_{i+1} - x_i)t = x_i + (x_k - x_i)t$$
$$y = y(t) = y_i + (y_{i+1} - y_i)t = y_i + (y_k - y_i)t\ ,\ 0 \le t \le 1 \quad (48)$$

Using eqn. (48), we write eqn. (47) as
$$\text{II}_{\pi_{xy}}(f) = \sum_{i=1}^{N}(y_k - y_1)\int_0^1 \Phi(x(t), y(t)dt \quad (49)$$
We can use Lemma 4 to integrate over a closed domain with piecewise linear boundary, if the indefinite integral $\int f(x,y)dx$ is available. We can apply Theorems1-2 even when the primitive $\int f(x,y)dx$ is not available. We have applied these results to the typical examples considered in this paper. We have also given MATLAB code which use the Lemma 4 (direct application of Green's theorem) and Theorems 1-2 (indirect application of Green's theorem).

8 Numerical Examples

In our earlier works the composite integration methods were applied to integrals over standard triangular region while in this paper, the proposed method is applied to a variety of regions including the triangular regions.

8.1 Integrals over Standard Triangular Domains

In this section, we consider some typical integrals which were experimented for the first time over the standard triangular domains in [13].

$$I_1 = \int_0^1 \int_0^{1-y} (x+y)^{\frac{1}{2}} dxdy = 0.4$$

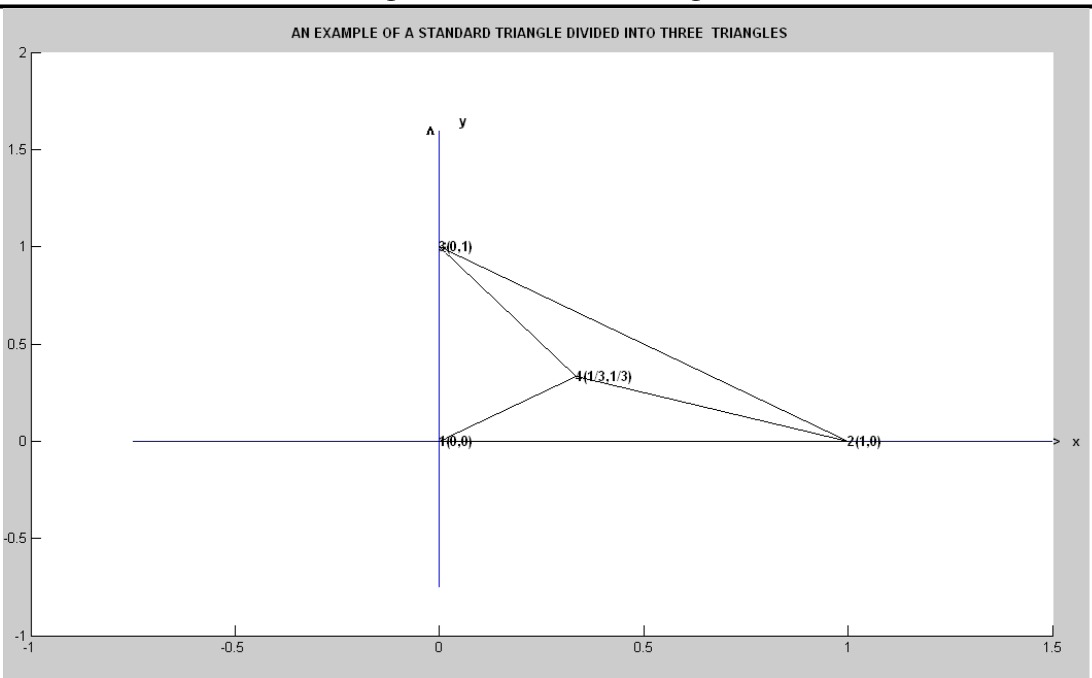
$$I_2 = \int_0^1 \int_0^{1-y} (x+y)^{-\frac{1}{2}} dxdy = \frac{2}{3}$$

$$I_3 = \int_0^1 \int_0^{y} (x^2+y^2)^{\frac{1}{2}} dxdy = \ln\left(\frac{1}{\sqrt{2}-1}\right)$$

$$I_4 = \int_0^{\frac{\pi}{2}} \int_0^{y} \sin(x+y) dxdy = 1$$

$$I_5 = \int_0^1 \int_0^{y} e^{|x+y-1|} dxdy = -2+e$$

We find the numerical solution to the above by joining the centroid of the standard triangle i.e, (1/3, 1/3) to the three vertices which creates three triangles.This is shown in Fig.3



8.2 Integrals over Quadrilaterals and Standard 2-Squares

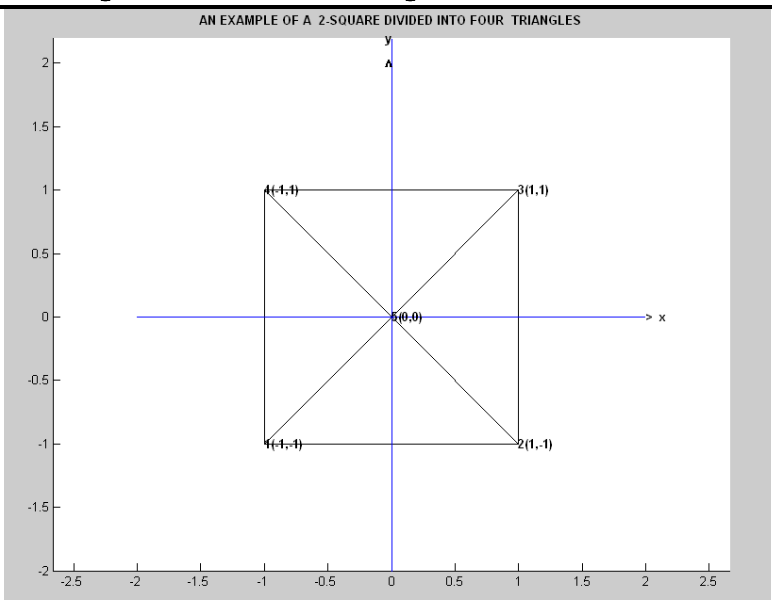
In this section, we consider two typical integrals which are considered in [14, 15] over the quadrilateral and standard 2-square.
We first consider [15]

$$I_6 = \int_0^{\frac{\pi}{4}} \int_0^{\sin y} \frac{dxdy}{\sqrt{1-x^2}} = \frac{\pi^2}{32} \approx 0.3084251375340424 3.....$$

In order to solve I_6, by using the present method. We write:

$$\int_0^{\frac{\pi}{4}}\int_0^{\sin y}\frac{dxdy}{\sqrt{1-x^2}}=\int_{-1}^{1}\int_{-1}^{1}\frac{\frac{1}{2}\sin\left(\frac{\pi}{8}(1+s)\right)\frac{\pi}{8}\,dsdt}{\left\{1-\frac{1}{2}\left(\sin\left(\frac{\pi}{8}(1+s)\right)\right)^2(1+t)^2\right\}^{\frac{1}{2}}}$$

We find the numerical solution to the above by joining the centroid of the 2-square i.e, (0, 0) to the four vertices which creates four triangles.This is shown in Fig.4



We next consider [14]

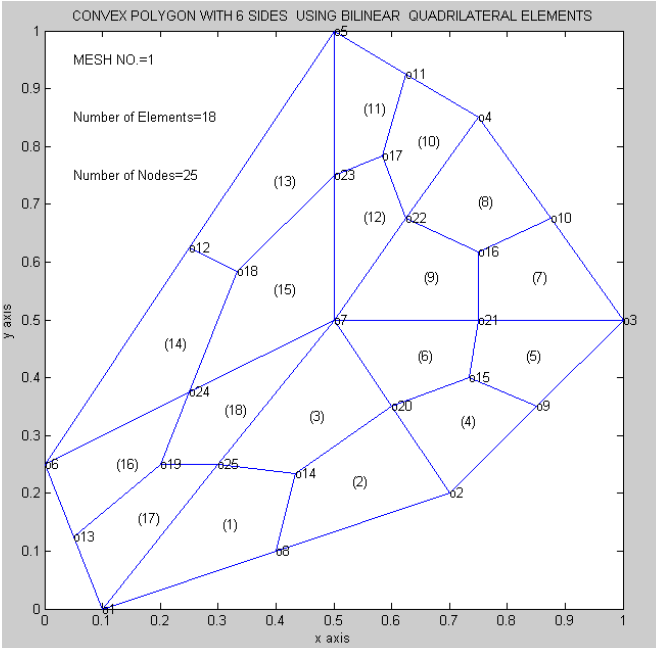
$$I_7=\iint_Q (x+y)^{\frac{-1}{2}}\,dxdy=\frac{2}{3}\left(2-7\sqrt{3}-15\sqrt{5}+20\sqrt{6}\right)\approx 0.35496130267897116975410$$

where Q is the quadrilateral region connecting the points $(-1,\ 2),(2,1),(3\ ,\ 3),(1,4)$.

We solve this integral by connecting the centroid of the quadrilateral $\left(\frac{5}{4},\frac{5}{2}\right)$ to the vertices of the quadrilateral which creates four triangles.

The computed values of integrals $I_N\left(N=1,\ 2,\ 3,4,\ 5,\ 6,7\right)$ are given in Table II which use the numerical scheme developed in sections 6.1 and 6.2.

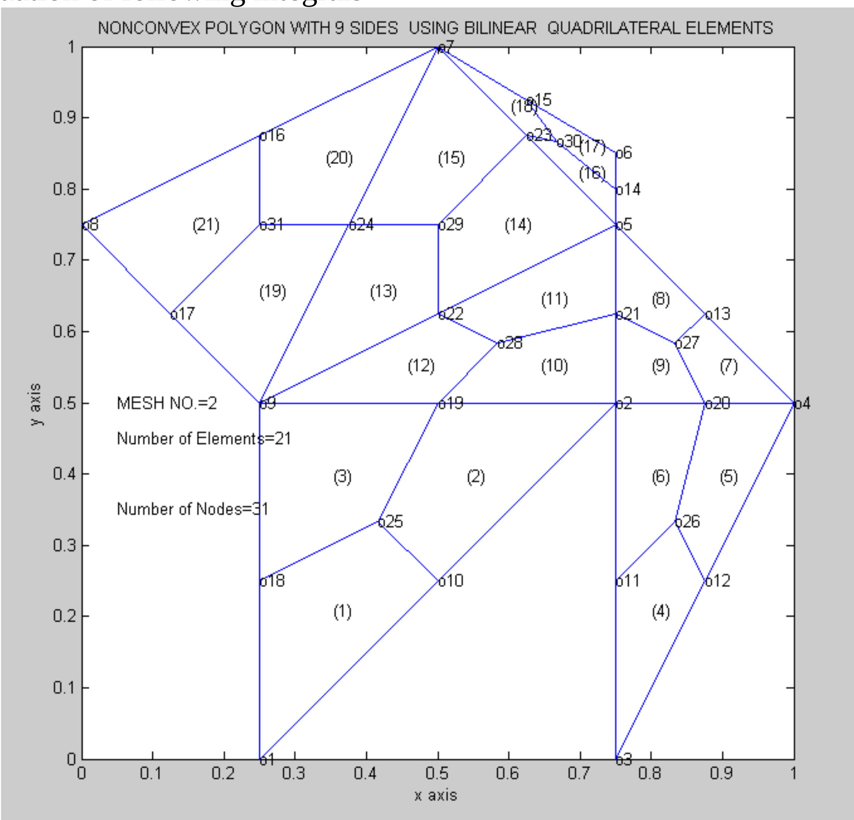
This quadrilateral is shown in Fig.5.



8.3 Some Integrals over the Polygonal Domains

In this section, integration of some typical examples is presented. The integration domains are the same as the ones considered in [11, 12]. Fig.5 shows the convex polygon with six sides which is discretised by six composite triangles, whose coordinates of vertices are $1(0.1,0)$, $2(0.7,0.2)$, $3(1,0.5)$, $4(0.75,0.85)$, $5(0.5,1)$, $6(0,0.25)$ Fig.6 shows the nonconvex polygon with nine sides which is discretised by seven composite triangles whose coordinates of vertices are $1(0.25,0)$, $2(0.75,0.5)$, $3(0.75,0)$, $4(1,0.5)$, $5(0.75,0.75)$, $6(0.75,0.85)$, $7(0.5,1)$, $8(0,0.75)$, $9(0.25,0.5)$.

We consider the evaluation of following integrals



$$\text{II}_{\Omega}(f_i) = \iint_{\Omega} f_i(x,y)dxdy,\ i = 1\,(1)9\ ,\quad \Omega = \text{P}_6\,,\text{P}_9$$

where, $f_1 = (x+y)^{19}$, $f_2 = \cos(x+y)$, $f_3 = \sqrt{\left((x-1/2)^2 + (y-1/2)^2\right)}$, $f_4 = \exp\left\{-(x-0.5)^2 - (y-0.5)^2\right\}$, $f_5 = \exp\left\{-100\,(x-0.5)^2 - 100\,(y-0.5)^2\right\}$,

$$f_6=\frac{3}{4}\exp\left\{-\frac{1}{4}(9x-2)^2+(9y-2)^2\right\}+\frac{3}{4}\exp\left\{-\frac{1}{49}(9x+1)^2-\frac{1}{10}(9y+1)\right\}+\frac{1}{2}\exp\left\{-\frac{1}{4}(9x-7)^2+(9y-3)^2\right\}$$
$$-\frac{1}{5}\exp\left\{-(9x-4)^2+(9y-7)^2\right\},\ f_7=\left|x^2+y^2-1/4\right|,\ f_8=\sqrt{\left|3-4x-4y\right|},$$
$$f_9=\exp\left\{-(5-10x)^2/2\right\}+0.75*\exp\left\{-(5-10x)^2/2-(5-10y)^2/2\right\}+(x+y)^3(x-0.6)_+,$$

Table -1

Exact values of test integrals $=\iint_{P_N} f_i(x,y)dxdy = II_{P_N}(f_i),\ i=1\ (1)9,\ \ N=6$ (convex polygon with sides), $N=9$ (Non-convex polygon with nine sides). Using MATLAB symbolic method, Greens theorem and Boundary integration method.
Section 7of the present work

f_i	$II_{P_6}(f_i)$	$II_{P_9}(f_i)$
f_1	169.7043434031279086481893	130.8412349867964988121030
f_2	0.84211809414899477639648664e-2	0.1422205098151202880410645e-1
f_3	0.1568251255860885374289978	0.1393814567714511086304939
f_4	0.4850601470247113893333032	0.4374093366938112280464110
f_5	0.3141452863239333834537669e-1	0.3122083897153926942995164
f_6	0.2663307419125152728165545	0.1829713239189687908913098
f_7	0.199062549435189053162	0.20842559601611674
f_8	0.545386805005417548157	0.4545305519051566
f_9	0.4492795032617593831499270	0.04115120322110287586271420

Table-I I-a COMPUTED VALUES OF INTEGRAL I_1

II(F_1)= I_1, F_1= sqrt(x+y),ST = standard triangle, I_1=∫ ∫ sqrt(x+y)dxdy;

ST

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order GLQ ng	NUMBER OF SPECIAL QUADRILATERALS (3*n^2) 3*1^2 (n=1)	3*2^2 (n=2)	3*3^2 (n=3)	3*4^2 (n=4)	3*5^2 (n=5)	3*6^2 (n=6)	3*7^2 (n=7)
5	**0.40000323165681**	**0.40000057128175**	**0.40000020731093**	**0.40000010098930**	**0.40000005780965**	**0.40000003664774**	**0.40000002492759**
10	**0.40000012339593**	**0.40000002181352**	**0.40000000791585**	**0.40000000385612**	**0.40000000220737**	**0.40000000139934**	**0.40000000095182**
15	**0.40000001749070**	**0.40000000309195**	**0.40000000112203**	**0.40000000054658**	**0.40000000031288**	**0.40000000019835**	**0.40000000013492**
20	**0.40000000431214**	**0.40000000076229**	**0.40000000027662**	**0.40000000013475**	**0.40000000007714**	**0.40000000004890**	**0.40000000003326**
25	**0.40000000144640**	**0.40000000025569**	**0.40000000009279**	**0.40000000004520**	**0.40000000002587**	**0.40000000001640**	**0.40000000001116**

ng	3*1^2 (n=1)	3*2^2 (n=2)	3*3^2 (n=3)	3*4^2 (n=4)	3*5^2 (n=5)	3*6^2 (n=6)	3*7^2 (n=7)
30	0.40000000059051	0.40000000010439	0.40000000003788	0.40000000001845	0.40000000001056	0.40000000000670	0.40000000000455
35	0.40000000027633	0.40000000004885	0.40000000001773	0.40000000000864	0.40000000000494	0.40000000000313	0.40000000000213
40	0.40000000014295	0.40000000002527	0.40000000000917	0.40000000000447	0.40000000000256	0.40000000000162	0.40000000000110

Table-I I-b COMPUTED VALUES OF INTEGRAL I_2

II(F_2)= I_2 ,ST = standard triangle, F_2 = 1/sqrt(x+y); $I_2 = \int\int_{ST}$ (1/sqrt(x+y))dxdy;

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order	NUMBER OF SPECIAL QUADRILATERALS (3*n^2)						
GLQ ng	3*1^2 (n=1)	3*2^2 (n=2)	3*3^2 (n=3)	3*4^2 (n=4)	3*5^2 (n=5)	3*6^2 (n=6)	3*7^2 (n=7)
5	0.66605408228510	0.66645008537746	0.66654877474524	0.66659009361752	0.66661187545286	0.66662498557812	0.66663359021848
10	0.66657892893273	0.66663564669335	0.66664978153190	0.66665569944992	0.66665881916518	0.66666069687002	0.66666192927433
15	0.66663941005368	0.66665702999873	0.66666142112905	0.66666325959004	0.66666422876109	0.66666481208906	0.66666519494790
20	0.66665488782359	0.66666250221676	0.66666439982726	0.66666519431128	0.66666561313491	0.66666586521791	0.66666603066892
25	0.66666054745524	0.66666450319872	0.66666548902388	0.66666590176524	0.66666611934776	0.66666625030707	0.66666633626031
30	0.66666309072814	0.66666540238148	0.66666597847698	0.66666621967435	0.66666634682500	0.66666642335487	0.66666647358413
35	0.66666439895176	0.66666586490837	0.66666623024473	0.66666638320230	0.66666646383608	0.66666651236821	0.66666654422157
40	0.66666513945943	0.66666612671737	0.66666637275550	0.66666647576576	0.66666653006910	0.66666656275338	0.66666658420523

Table-I I-c COMPUTED VALUES OF INTEGRAL I_3

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

II(F_3)= I_3 ,ST = standard triangle, F_3 =1/sqrt(x^2+(1-y)^2) ;); $I_3 = \int\int_{ST}$ (1/sqrt(x^2+(1-y)^2))dxdy;

Gauss

Order **NUMBER OF SPECIAL QUADRILATERALS (3*n^2)**

GLQ ng	3*1^2 (n=1)	3*2^2 (n=2)	3*3^2 (n=3)	3*4^2 (n=4)	3*5^2 (n=5)	3*6^2 (n=6)	3*7^2 (n=7)
5	0.87049395646700	0.87593377172323	0.87774704348860	0.87865367937133	0.87919766090097	0.87956031525407	0.87981935407771
10	0.87839000134351	0.87988179418153	0.88037905846087	0.88062769060054	0.88077686988434	0.88087632274020	0.88094736049440
15	0.88000456146178	0.88068907424066	0.88091724516695	0.88103133063010	0.88109978190799	0.88114541609325	0.88117801193986
20	0.88059096482503	0.88098227592229	0.88111271295471	0.88117793147092	0.88121706258064	0.88124314998712	0.88126178384890
25	0.88086779444421	0.88112069073188	0.88120498949443	0.88124713887571	0.88127242850448	0.88128928825699	0.88130133093735
30	0.88102003831133	0.88119681266544	0.88125573745014	0.88128519984249	0.88130287727790	0.88131466223484	0.88132308006123
35.	0.88111261744741	0.88124310223347	0.88128659716216	0.88130834462651	0.88132139310512	0.88133009209085	0.88133630565209
40	0.88117307749940	0.88127333225947	0.88130675051283	0.88132345963951	0.88133348511552	0.88134016876619	0.88134494280238

Table-I I-d COMPUTED VALUES OF INTEGRAL I_4

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

II(F_4)= I_4 ,ST = standard triangle; F_4 =pi^2/4*sin(((pi*(x-y+1)))/2);

I_4 =∫ ∫ pi^2/4*sin(((pi*(x-y+1)))/2) dxdy;

ST

Gauss Order **NUMBER OF SPECIAL QUADRILATERALS (3*n^2)**

GLQ ng	3*1^2 (n=1)	3*2^2 (n=2)	3*3^2 (n=3)	3*4^2 (n=4)	3*5^2 (n=5)	3*6^2 (n=6)	3*7^2 (n=7)
5	1.00000000001688	1.00000000000002	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
10	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
15	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000

20	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
25	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
30	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
35	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
40	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000

Table-I I-e COMPUTED VALUES OF INTEGRAL I_5
(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

II(F_5)= I_5 ,ST = standard triangle; F_5 = exp(abs(x-y));); $I_5 = \int\int_{ST}$ exp(abs(x-y))dxdy;

Gauss Order — NUMBER OF SPECIAL QUADRILATERALS (3*n^2)

GLQ ng	3*1^2 (n=1)	3*2^2 (n=2)	3*3^2 (n=3)	3*4^2 (n=4)	3*5^2 (n=5)	3*6^2 (n=6)	3*7^2 (n=7)	3*8^2 (n=8)
5	0.71752519643000	0.71809263001681	0.71819773693315	0.71823452632021	0.71825155489600	0.71826080507812	0.71826638267734	0.71827000276630
10	0.71807453267703	0.71823000157704	0.71825879404755	0.71826887155499	0.71827353602635	0.71827606981991	0.71827759761978	0.71827858922156
15	0.71818672111906	0.71825805101003	0.71827126065339	0.71827588405841	0.71827802403969	0.71827918650005	0.71827988742745	0.71828034235649
20	0.71822746073088	0.71826823632684	0.71827578749492	0.71827843041348	0.71827965370892	0.71828031821554	0.71828071889223	0.71828097894687
25	0.71824669214957	0.71827304429817	0.71827792438067	0.71827963241361	0.71828042298956	0.71828085243842	0.71828111138260	0.71828127944736
30	0.71825726834761	0.71827568839041	0.71827909953630	0.71828029343934	0.71828084604624	0.71828114622785	0.71828132722793	0.71828144470396
35	0.71826369964134	0.71827729623241	0.71827981413427	0.71828069540100	0.71828110330179	0.71828132487758	0.71828145848081	0.71828154519445
40	0.71826789965915	0.71827834624596	0.71828028080771	0.71828095790496	0.71828127130436	0.71828144154605	0.71828154419642	0.71828161082047

………..continued

GLQ ng	3*9^2 (n=9)	3*10^2 (n=10)
5	0.71827248469308	0.71827426000355

10	0.71827926906089	0.71827975534617
15	0.71828065425442	0.71828087735323
20	0.71828115723951	0.71828128477120
25	0.71828139467202	0.71828147709154
30	0.71828152524514	0.71828158285578
35	0.71828160464504	0.71828164716970
40	0.71828165649770	0.71828168917036

………continued

Table-I I-e COMPUTED VALUES OF INTEGRAL I_5 (GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **NUMBER OF SPECIAL QUADRILATERALS (3*10^2*n^2)**

GLQ ng	3*10^2*1^2 (n=1)	3*10^2*2^2 (n=2)	3*10^2*3^2 (n=3)	3*10^2*4^2 (n=4)	3*10^2*5^2 (n=5)	3*10^2*6^2 (n=6)	3*10^2*7^2 (n=7)	3*10^2*8^2 (n=8)
5	0.71827426000355	0.71827993634113	0.71828098751741	0.71828135542931	0.71828152571999	0.71828161822358	0.71828167400036	0.71828171020159
10	0.71827975534617	0.71828131018053	0.71828159811301	0.71828169888940	0.71828174553447	0.71828177087252	0.71828178615061	0.71828179606663
15	0.71828087735323	0.71828159068253	0.71828172278058	0.71828176901491	0.71828179041480	0.71828180203942	0.71828180904873	0.71828181359801
20	0.71828128477120	0.71828169253706	0.71828176804927	0.71828179447855	0.71828180671152	0.71828181335659	0.71828181736339	0.71828181996392
25	0.71828147709154	0.71828174061716	0.71828178941820	0.71828180649857	0.71828181440434	0.71828181869882	0.71828182128830	0.71828182296892
30	0.71828158285578	0.71828176705822	0.71828180116979	0.71828181310884	0.71828181863491	0.71828182163672	0.71828182344675	0.71828182462149
35	0.71828164716970	0.71828178313671	0.71828180831578	0.71828181712846	0.71828182120747	0.71828182342322	0.71828182475928	0.71828182562639
40	0.71828168917036	0.71828179363687	0.71828181298252	0.71828181975350	0.71828182288750	0.71828182458990	0.71828182561644	0.71828182628265

………..continued

GLQ ng	3*10^2*9^2 (n=9)	3*10^2*10^2 (n=10)
5	0.71828173502107	0.71828175277428
10	0.71828180286506	0.71828180772791
15	0.71828181671701	0.71828181894799
20	0.71828182174686	0.71828182302217
25	0.71828182412119	0.71828182494537

30	0.71828182542692	0.71828182600302
35	0.71828182622092	0.71828182664616
40	0.71828182673945	0.71828182706616

………..continued

Table-I I-e COMPUTED VALUES OF INTEGRAL I_5

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — NUMBER OF SPECIAL QUADRILATERALS (3*20^2*n^2)

GLQ ng	3*20^2*1^2 (n=1)	3*20^2*2^2 (n=2)	3*20^2*3^2 (n=3)	3*20^2*4^2 (n=4)	3*20^2*5^2 (n=5)	3*20^2*6^2 (n=6)	3*20^2*7^2 (n=7)	3*20^2*8^2 (n=8)
5	0.71827993634113	0.71828135542931	0.71828161822358	0.71828171020160	0.71828175277427	0.71828177590017	0.71828178984436	0.71828179889464
10	0.71828131018053	0.71828169888940	0.71828177087253	0.71828179606664	0.71828180772790	0.71828181406241	0.71828181788193	0.71828182036090
15	0.71828159068253	0.71828176901491	0.71828180203943	0.71828181359801	0.71828181894798	0.71828182185414	0.71828182360646	0.71828182474374
20	0.71828169253706	0.71828179447855	0.71828181335660	0.71828181996392	0.71828182302216	0.71828182468343	0.71828182568513	0.71828182633522
25	0.71828174061716	0.71828180649857	0.71828181869883	0.71828182296893	0.71828182494537	0.71828182601899	0.71828182666635	0.71828182708647
30	0.71828176705822	0.71828181310884	0.71828182163673	0.71828182462150	0.71828182600301	0.71828182675346	0.71828182720597	0.71828182749961
35	0.71828178313671	0.71828181712846	0.71828182342323	0.71828182562640	0.71828182664615	0.71828182720009	0.71828182753410	0.71828182775084
40	0.71828179363687	0.71828181975350	0.71828182458991	0.71828182628266	0.71828182706616	0.71828182749176	0.71828182774839	0.71828182791490

……continued

GLQ ng	3*20^2*9^2 (n=9)	3*20^2*10^2 (n=10)
5	0.71828180509951	0.71828180953786
10	0.71828182206051	0.71828182327626
15	0.71828182552350	0.71828182608129
20	0.71828182678096	0.71828182709983
25	0.71828182737454	0.71828182758063
30	0.71828182770097	0.71828182784504
35	0.71828182789947	0.71828182800583
40	0.71828182802910	0.71828182811083

……..continued(GLQ:ng=40)

Divisions	I_5 (computed)
3* 20^2*11^2	0.71828182817121

3* 20^2*12^2	**0.71828182821718**
3* 20^2*13^2	**0.71828182825303**
3* 20^2*14^2	**0.71828182828132**
3* 20^2*15^2	**0.71828182830426**
3* 20^2*16^2	**0.71828182832298**
3* 20^2*17^2	**0.71828182833842**
3* 20^2*18^2	**0.71828182835157**
3* 20^2*19^2	**0.71828182836269**
3* 20^2*20^2	**0.71828182837216**
3*21^2*20^2	**0.71828182838000**
3*21^2*21^2	**0.71828182838723**
3*21^2*22^2	**0.71828182839376**
3*22^2*22^2	**0.71828182839963**
3*23^2*22^2	**0.71828182840479**
3*23^2*22^2	**0.71828182840923**

Table-I I-f COMPUTED VALUES OF INTEGRAL I_6
(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **NUMBER OF SPECIAL QUADRILATERALS (4*3*n^2)**

GLQ ng	4* 3*1^2 (n=1)	4* 3*2^2 (n=2)	4*3*3^2 (n=3)	4* 3*4^2 (n=4)	4*3*5^2 (n=5)	4* 3*6^2 (n=6)	4*3*7^2 (n=7)
5	0.30842513715380	0.30842513753309	0.30842513753402	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
10	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
15	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
20	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
25	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
30	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
35	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404
40	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404	0.30842513753404

Table-I I-g COMPUTED VALUES OF INTEGRAL I_7
(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order **NUMBER OF SPECIAL QUADRILATERALS (4*3*n^2)**

GLQ ng	4*3*1^2 (n=1)	4*3*2^2 (n=2)	4*3*3^2 (n=3)	4*3*4^2 (n=4)	4*3*5^2 (n=5)	4* 3*6^2 (n=6)	4*3*7^2 (n=7)
5	3.54961298700924	3.54961302661193	3.54961302678461	3.54961302678936	3.54961302678967	3.54961302678971	3.54961302678972
10	3.54961302678969	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678971	3.54961302678972
15	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678972	3.54961302678972
20	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678971	3.54961302678972
25	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678972
30	3.54961302678972	3.54961302678971	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678971	3.54961302678972
35	3.54961302678972	3.54961302678971	3.54961302678971	3.54961302678972	3.54961302678971	3.54961302678972	3.54961302678972
40	3.54961302678971	3.54961302678972	3.54961302678972	3.54961302678972	3.54961302678971	3.54961302678971	3.54961302678972

Table-I I I-a : COMPUTED VALUES OF INTEGRAL 1.0e+002 * I I (f_1), f_1=(x+y)^19

P_6

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order **P6:CONVEX POLYGN WITH SIX SIDES**
NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)

5	**1.69704243836483**	**1.69704343258906**	**1.69704343400375**	**1.69704343402967**	**1.69704343403110**	**1.69704343403125**	**1.69704343403127**
10	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**
15	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**
20	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**
25	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**
30	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**
35	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**
40	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**	**1.69704343403128**

Table-I I I-b:COMPUTED VALUES OF INTEGRAL I I (f_2), f_2 =cos(30*(x+y))

P_6

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_6 :CONVEX POLYGN WITH SIX SIDES**
NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)

GLQ ng	**6*3*1^2 (n=1)**	**6*3*2^2 (n=2)**	**6*3*3^2 (n=3)**	**6*3*4^2 (n=4)**	**6*3*5^2 (n=5)**	**6* 3*6^2 (n=6)**	**6*3*7^2 (n=7)**
5	**0.00616553081729**	**0.00843351498452**	**0.00842132120140**	**0.00842118222025**	**0.00842118091305**	**0.00842118089885**	**0.00842118094044**
10	**0.00842119971021**	**0.00842118094140**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**
15	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**
20	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**
25	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**
30	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**	**0.00842118094149**

35	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149
40	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149	0.00842118094149

Table-I I I-c: COMPUTED VALUES OF INTEGRAL I I (f_3), f_3 =sqrt((x-1/2)^2+(y-1/2)^2)

P_6

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_6 :CONVEX POLYGN WITH SIX SIDES**
NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)
5	0.15682395100710	0.15682497878680	0.15682508209042	0.15682510723635	0.15682511619102	0.15682512014913	0.15682512216223
10	0.15682509968849	0.15682512234889	0.15682512462692	0.15682512518144	0.15682512537891	0.15682512546619	0.15682512551059
15	0.15682512304214	0.15682512526810	0.15682512549187	0.15682512554634	0.15682512556574	0.15682512557431	0.15682512557867
20	0.15682512510784	0.15682512552631	0.15682512556838	0.15682512557862	0.15682512558226	0.15682512558387	0.15682512558469
25	0.15682512545660	0.15682512556990	0.15682512558129	0.15682512558407	0.15682512558505	0.15682512558549	0.15682512558571
30	0.15682512554179	0.15682512558055	0.15682512558445	0.15682512558540	0.15682512558573	0.15682512558588	0.15682512558596
35	0.15682512556826	0.15682512558386	0.15682512558543	0.15682512558581	0.15682512558595	0.15682512558601	0.15682512558604
40	0.15682512557799	0.15682512558508	0.15682512558579	0.15682512558596	0.15682512558602	0.15682512558605	0.15682512558606

Table-I I I-d: COMPUTED VALUES OF INTEGRAL I I (f_4), f_4 =exp(-((x-1/2)^2+(y-1/2)^2))

P_6

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_6 :CONVEX POLYGN WITH SIX SIDES**
NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)
5	0.48506014702487	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471
10	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471
15	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471
20	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471
25	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471
30	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471
35	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471
40	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471	0.48506014702471

Table-I I I-e:COMPUTED VALUES OF INTEGRAL I I (f_5), f_5 =exp(-100*((x-1/2)^2+(y-1/2)^2)) P_6

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_6 :CONVEX POLYGN WITH SIX SIDES**

NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)
5	0.03141726522717	0.03141451384185	0.03141452806619	0.03141452862618	0.03141452863366	0.03141452863267	0.03141452863245
10	0.03141452863620	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239
15	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239
20	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239
25.	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239
30	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239	0.03141452863239

35	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**
40	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**	**0.03141452863239**

Table-I I I-f: COMPUTED VALUES OF INTEGRAL I I (f_6) P_6

f_6 **=0.75*exp(-0.25*(9*x-2)^2-0.25*(9*y-2)^2)+0.75*exp((-1/49)*(9*x+1)^2-0.1*(9*y+1))+0.5*exp(-0.25*(9*x-7)^2-0.25*(9*y-3)^2)**
-0.2*exp(-(9*y-4)^2-(9*y-7)^2)

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — P_6 **:CONVEX POLYGN WITH SIX SIDES**
NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)
5	0.26633070116829	0.26633074175184	0.26633074191385	0.26633074191256	0.26633074191251	0.26633074191251	0.26633074191252
10	0.26633074191521	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252
15	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252
20	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252
25	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252
30	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191251	0.26633074191252	0.26633074191252
35	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191251	0.26633074191252	0.26633074191252
40	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252	0.26633074191252

Table-I I I-g: COMPUTED VALUES OF INTEGRAL I I (f_7), f_7 **=abs(x^2+y^2-1/4)** P_6

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss — P_6 **:CONVEX POLYGN WITH SIX SIDES**

Order **NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)**

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)
5	0.19903352981736	0.19905979007635	0.19906371151222	0.19906229930019	0.19906330529342	0.19906270803522	0.19906247033217
10	0.19905369536629	0.19906280241181	0.19906213483435	0.19906242921932	0.19906261550509	0.19906273223668	0.19906261381848
15	0.19906366619666	0.19906218929993	0.19906256861516	0.19906258978774	0.19906250696727	0.19906252882143	0.19906255079581
20	0.19906340050366	0.19906274081207	0.19906256072161	0.19906259031151	0.19906254027666	0.19906256978306	0.19906253265968
25	0.19906308590290	0.19906235360574	0.19906251623488	0.19906252632592	0.19906253084970	0.19906254409790	0.19906254680830
30	0.19906251425581	0.19906262148723	0.19906253828244	0.19906254582550	0.19906255540059	0.19906254142025	0.19906254869375
35	0.19906254121229	0.19906255247052	0.19906256806165	0.19906255037190	0.19906254770044	0.19906254783626	0.19906254833208
40	0.19906253275998	0.19906255241666	0.19906255587398	0.19906254388559	0.19906255318369	0.19906255104895	0.19906254996347

Table-I I I-h: COMPUTED VALUES OF INTEGRAL I I (f_8), f_8 =sqrt(abs(3-4*x-3*y))

P_6

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss **P_6 :CONVEX POLYGN WITH SIX SIDES**

Order **NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)**

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)
5	0.54633818559135	0.54546982770485	0.54534953707627	0.54535392660196	0.54539804447400	0.54538588560013	0.54537469853148
10	0.54542368668244	0.54537162036858	0.54536633379583	0.54539574674233	0.54538814490198	0.54538427937946	0.54538204489828
15	0.54541254462994	0.54540535861609	0.54539469828563	0.54539127591383	0.54538360018250	0.54538422833215	0.54538457730304

20	0.54541595958491	0.54539256393809	0.54539428588983	0.54538629369310	0.54538518446605	0.54538818982076	0.54538806529291
25	0.54538547658571	0.54538709099086	0.54538501324984	0.54538522383986	0.54538795815734	0.54538665689145	0.54538662536620
30	0.54538260936147	0.54539022667259	0.54538624958045	0.54538840455852	0.54538703622612	0.54538712218941	0.54538731682337
35	0.54538928045384	0.54538260515738	0.54538441421465	0.54538644599553	0.54538707383407	0.54538667076345	0.54538649443127
40	0.54538286708749	0.54538520706955	0.54538738481764	0.54538696373953	0.54538690962984	0.54538661216464	0.54538669942990

Table-I I I-i: COMPUTED VALUES OF INTEGRAL I I (f_9), P_6

f_9 **=exp(-((5-10*x)^2)/2)+0.75*exp(-((5-10*y)^2)/2)+0.75*(exp(-((5-10*x)^2)/2-((5-10*y)^2)/2))+((x+y)^3)* max((x-0.6),0)**

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_6 :CONVEX POLYGN WITH SIX SIDES**
NUMBER OF SPECIAL QUADRILATERALS (6*3*n^2)

GLQ ng	6*3*1^2 (n=1)	6*3*2^2 (n=2)	6*3*3^2 (n=3)	6*3*4^2 (n=4)	6*3*5^2 (n=5)	6* 3*6^2 (n=6)	6*3*7^2 (n=7)
5	0.44929239347399	0.44927849261302	0.44927921726830	0.44927971858606	0.44927905444872	0.44927979388787	0.44927933173744
10	0.44927850797339	0.44927982114288	0.44927920763720	0.44927957158401	0.44927941733087	0.44927953466290	0.44927947793699
15	0.44927856770153	0.44927924012897	0.44927942836383	0.44927950629033	0.44927943257098	0.44927950426225	0.44927947868845
20	0.44927880164607	0.44927966942102	0.44927954155272	0.44927952777386	0.44927947613846	0.44927950657505	0.44927951195894
25	0.44927930624472	0.44927947930239	0.44927950500342	0.44927950231950	0.44927948080638	0.44927950295114	0.44927950072699
30	0.44927940468002	0.44927951107224	0.44927948937268	0.44927949669120	0.44927948938762	0.44927950170795	0.44927950144357
35	0.44927957847553	0.44927948696086	0.44927949327452	0.44927950418417	0.44927949172925	0.44927950475090	0.44927950121250
40	0.44927953740260	0.44927952167553	0.44927950172921	0.44927950815835	0.44927949542373	0.44927950539333	0.44927950354651

-

Table-I V-a : COMPUTED VALUES OF INTEGRAL 1.0e+002 * I I (f_1), f_1 =(x+y)^19

P_9

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order	P_9 :NONCONVEX POLYGN WITH NINE SIDES NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)						
GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)
5	1.30841223947876	1.30841234971968	1.30841234986522	1.30841234986781	1.30841234986795	1.30841234986796	1.30841234986797
10	1.30841234986797	1.30841234986797	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986797
15	1.30841234986796	1.30841234986797	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986797
.20	1.30841234986797	1.30841234986797	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986797
25	1.30841234986797	1.30841234986797	1.30841234986796	1.30841234986796	1.30841234986797	1.30841234986796	1.30841234986797
30	1.30841234986796	1.30841234986797	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986797
35	1.30841234986796	1.30841234986797	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986797
40	1.30841234986796	1.30841234986797	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986796	1.30841234986797

Table-I V-b: COMPUTED VALUES OF INTEGRAL I I (f_2), f_2 =cos(30*(x+y))

P_9

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order	P_9 :NONCONVEX POLYGN WITH NINE SIDES NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)						
GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)

5	0.01307915142055	0.01422421442506	0.01422204725615	0.01422205140690	0.01422205099109	0.01422205098216	0.01422205098158
10	0.01422205185946	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151
15	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151
20	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151
25	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151
30	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151
35	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151
40	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151	0.01422205098151

Table-I V-c : COMPUTED VALUES OF INTEGRAL I I (f_3), f_3 =sqrt((x-1/2)^2+(y-1/2)^2)

P_9

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_9 :NONCONVEX POLYGN WITH NINE SIDES**
NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)

GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)
5	0.13938109504746	0.13938134884632	0.13938144299375	0.13938144328308	0.13938145379547	0.13938145277490	0.13938145568691
10	0.13938145081036	0.13938145457976	0.13938145655067	0.13938145649749	0.13938145672376	0.13938145669028	0.13938145675407
15	0.13938145615861	0.13938145656036	0.13938145674875	0.13938145674507	0.13938145676655	0.13938145676363	0.13938145676966
20	0.13938145665953	0.13938145673206	0.13938145676731	0.13938145676653	0.13938145677056	0.13938145676999	0.13938145677112
25	0.13938145674171	0.13938145676082	0.13938145677035	0.13938145677012	0.13938145677121	0.13938145677106	0.13938145677136
30	0.13938145676139	0.13938145676782	0.13938145677108	0.13938145677100	0.13938145677137	0.13938145677132	0.13938145677142
35	0.13938145676743	0.13938145676999	0.13938145677130	0.13938145677127	0.13938145677142	0.13938145677140	0.13938145677144

40	0.13938145676963	0.13938145677079	0.13938145677138	0.13938145677137	0.13938145677144	0.13938145677143	0.13938145677145

Table-I V-d : COMPUTED VALUES OF INTEGRAL I I (f_4), f_4 =exp(-((x-1/2)^2+(y-1/2)^2))

P_9

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_9 :NONCONVEX POLYGN WITH NINE SIDES**
NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)

GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)
5	0.43740933669383	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381
10	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381
15	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381
20	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381
25	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381
30	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381
35	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381
40	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381	0.43740933669381

Table-I V-e : COMPUTED VALUES OF INTEGRAL I I (f_5), f_5 =exp(-100*((x-1/2)^2+(y-1/2)^2))

P_9

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_9 :NONCONVEX POLYGN WITH NINE SIDES**
NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)

GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)
5	0.03122065218856	0.03122083627879	0.03122083906797	0.03122083896860	0.03122083897156	0.03122083897154	0.03122083897154
10	0.03122083897079	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154
15	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154
20	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154
25	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154
30	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154
35	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154
40	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154	0.03122083897154

Table-I V-f : COMPUTED VALUES OF INTEGRAL I I (f_6), P_9

f_6 **= 0.75*exp(-0.25*(9*x-2)^2-0.25*(9*y-2)^2)+ 0.75*exp((-1/49)*(9*x+1)^2-0.1*(9*y+1))+ 0.5*exp(-0.25*(9*x-7)^2-0.25*(9*y-3)^2)**
-0.2*exp(-(9*y-4)^2-(9*y-7)^2)

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — P_9 **:NONCONVEX POLYGN WITH NINE SIDES**
NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)

GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)
5	0.18297132077204	0.18297132392682	0.18297132391907	0.18297132391898	0.18297132391897	0.18297132391897	0.18297132391897
10	0.18297132391896	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897
15	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897
20	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897

25	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897
30	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897
35	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897
40	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897	0.18297132391897

Table-I V-g : COMPUTED VALUES OF INTEGRAL I I (f_7), f_7 =abs(x^2+y^2-1/4)

P_9

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — **P_9 :NONCONVEX POLYGN WITH NINE SIDES**
NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)

GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)
5	0.20844394584995	0.20841905917766	0.20842713849810	0.20842421160301	0.20842525391083	0.20842491620520	0.20842582234677
10	0.20841672541165	0.20842594637462	0.20842533478894	0.20842576884722	0.20842553191086	0.20842555226767	0.20842558304590
15.	0.20842927458162	0.20842608299204	0.20842561604705	0.20842559532913	0.20842559977298	0.20842561806445	0.20842557701480
20	0.20842641550708	0.20842533658672	0.20842553189402	0.20842554818046	0.20842560255710	0.20842559082207	0.20842559488963
25	0.20842359931350	0.20842565665399	0.20842559902491	0.20842558536091	0.20842560032992	0.20842559629247	0.20842559516647
30	0.20842675295346	0.20842555635770	0.20842560972240	0.20842560602327	0.20842559481829	0.20842559430733	0.20842559575745
35.	0.20842508704774	0.20842561788372	0.20842559503307	0.20842559961901	0.20842559484565	0.20842559755288	0.20842559696447
40.	0.20842532578467	0.20842557375430	0.20842559095752	0.20842558734079	0.20842559548916	0.20842559702284	0.20842559567410

Table-I V-h : COMPUTED VALUES OF INTEGRAL I I (f_8), f_8 =sqrt(abs(3-4*x-3*y))

P_9

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — P_9 **:NONCONVEX POLYGN WITH NINE SIDES**
NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)

GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)
5	0.45473375289789	0.45439567363063	0.45438942383074	0.45455561519849	0.45456223233319	0.45453697553741	0.45451876476068
10	0.45437625782090	0.45451370636829	0.45452618805866	0.45451706880383	0.45451784256629	0.45452580570362	0.45452889586418
15	0.45459487441197	0.45452620291738	0.45453165772159	0.45453555465212	0.45453408312801	0.45453001547066	0.45453016134679
20	0.45455609981153	0.45453289561831	0.45452937568765	0.45452932679218	0.45453098370806	0.45453135769017	0.45453048746875
25	0.45454468599761	0.45452764734405	0.45452680025276	0.45453004885994	0.45452992875562	0.45452987616887	0.45452965869560
30	0.45452631265161	0.45452990028673	0.45452959981279	0.45452980283071	0.45452980655493	0.45452972784178	0.45453002569471
35	0.45454119087496	0.45453427186766	0.45453256802524	0.45452971195168	0.45453015505917	0.45453076548393	0.45453082199840
40	0.45452935778447	0.45452851457437	0.45453122078446	0.45453087295082	0.45453100328713	0.45453058363618	0.45453059236939

Table-I V-i : COMPUTED VALUES OF INTEGRAL I I (f_9), P_9

f_9 **=exp(-((5-10*x)^2)/2)+0.75*exp(-((5-10*y)^2)/2)+0.75*(exp(-((5-10*x)^2)/2-((5-10*y)^2)/2))+((x+y)^3)*max((x-0.6),0)**

(GLQ=GAUSS LEGENDRE QUADRATURE,ng=ORDER OF RULE)

Gauss Order — P_9 **:NONCONVEX POLYGN WITH NINE SIDES**
NUMBER OF SPECIAL QUADRILATERALS (7*3*n^2)

GLQ ng	7*3*1^2 (n=1)	7*3*2^2 (n=2)	7*3*3^2 (n=3)	7*3*4^2 (n=4)	7*3*5^2 (n=5)	7* 3*6^2 (n=6)	7*3*7^2 (n=7)

5	**0.41152417226537**	**0.41151302056156**	**0.41150660153031**	**0.41151163548226**	**0.41151049969047**	**0.41151165401960**	**0.41151111936157**
10	**0.41151356381615**	**0.41151136865499**	**0.41151242596435**	**0.41151162939723**	**0.41151178904929**	**0.41151183695832**	**0.41151208661500**
15	**0.41151211334202**	**0.41151208351452**	**0.41151175724372**	**0.41151194627916**	**0.41151194004453**	**0.41151199917496**	**0.41151198976921**
20	**0.41151256005506**	**0.41151209541408**	**0.41151217468024**	**0.41151203569019**	**0.41151197154428**	**0.41151204307143**	**0.41151205423525**
25	**0.41151247839751**	**0.41151198864253**	**0.41151198154234**	**0.41151203556092**	**0.41151198247729**	**0.41151203658115**	**0.41151202440118**
30	**0.41151200014248**	**0.41151195077017**	**0.41151204808848**	**0.41151201901763**	**0.41151198995654**	**0.41151202557716**	**0.41151203462344**
35	**0.41151204820854**	**0.41151204986742**	**0.41151201479456**	**0.41151202270352**	**0.41151199830968**	**0.41151202703843**	**0.41151202866201**
40	**0.41151204957091**	**0.41151205661548**	**0.41151204733799**	**0.41151203177144**	**0.41151200671352**	**0.41151203096742**	**0.41151203474553**

The computed values of integrals $\mathrm{II}_{P_N}(f_i)$ $(N = 6,9)$, $i = 1\,(1)9$ are given in Tables III- IV which use the numerical scheme developed in sections 6.1 and 6.2.

Conclusions:

The purpose of this paper is to develop efficient numerical integration schemes for arbitrary linear polygons in a 2-space which are very useful in finite element method, boundary integration method and mathematical modelling of several phenomena in science and engineering. The present study concentrates on those phenomena which may require integrating arbitrary functions over linear polygons which may be either convex or nonconvex. We can discretise these domains by using either triangles or quadrilaterals. In this paper, we propose to discretise the polygonal domain into triangles and these triangles are then divided into three special quadrilaterals by joining the centroid of the triangle to the midpoints of the three sides. This discretises the entire polygonal domain into a finite number of special quadrilaterals. The composite integration scheme is developed by discretising the arbitrary triangle into n^2,(n=1,2,3,4,5,……) triangles and then each of these triangles is divided further into three special quadrilaterals. We map each of these special quadrilaterals of the arbitrary triangle into a unique special quadrilateral of the standard triangle. Thus we are able to find sampling points and weight coefficients applicable for the entire polygonal domain. The composite integration scheme is tested on examples of integrals over convex and nonconvex polygons with complicated integrands. The necessary and relevant MATLAB codes are also appended.The MATLAB codes are listed below:
special_convexquadrilateral_domain_gausslegendrequadrature.m
nodel_address_rtisosceles_triangle.m
coordinates_stdtriangle.m
glsampleptsweights.m
newpolygon_boundary_n.m
newnonconvexpolygon _boundary_n.m

References:

[1] O. C. Zienkiewicz, R. L. Taylor and J. Z. Zhu, The Finite Element Method, its basis and fundamentals, Sixth edition, Butterworth-Heinemann, An Imprint of Elsevier,(2000).

[2] K. J. Bathe, Finite Element Procedures, Prentice Hall, Inc. Englewood Cliffs, N.J (1996).
[3] R. D. Cook, Concepts and Applications of Finite Element Analysis, 2nd Ed. John Wiley & Sons, Inc. New York (1981).
[4] G. R. Cowper, Gaussian Quadrature formulas for triangles, Int. J. Numer. Methods. Eng 7 (1973) 405 - 408.
[5] D. A. Dunavant, High degree efficient symmetrical Gaussian Quadrature rules for triangle, Int. J. Numer. Methods Eng 21 (1985) 1129-1148.
[6] A. H. Stroud, Approximate Calculation of Multiple integrals, in: Prentice Hall series in Automatic Computation, Prentice Hall, Inc. Englewood Cliffs. N. J (1971).
[7] G. Lague, R. Baldur, Extended numerical integration method for triangular surfaces, Int. J. Numer. Methods Eng 11 (1977) 388-392.
[8] H. T. Rathod, K. V. Nagaraja, B. Venkatesudu, N. L. Ramesh, Gauss Legendre Quadrature over a triangle, J. Indian Inst. Sci 84 (2004) 183-188.
[9] H. T. Rathod, K. V. Nagaraja, B. Venkatesudu, Symmetric Gauss Legendre Quadrature formulas for composite numerical integration over a triangular surface, Applied Mathematics and Computation 188 (2007) 865-876.
[10] H. T. Rathod, K.V. Nagaraja, B. Venkatesudu, On the application of two symmetric Gauss Legendre quadrature rules for composite numerical integration over s triangular surface, Applied Mathematics and Computation 190 (2007) 21-39.
[11] A. Sommariva, M. Vianello, Product Gauss cubature over polygons based on Green's Integration formula, BIT Numerical Mathematics, 47 (2007) 441-453.
[12] C. J. Li, P. Lamberti, C. Dagnino, Numerical integration over polygons using an eight-node quadrilateral finite element, Journal of Computational and Applied Mathematics, 233 (2009) 279-292.
[13] C. T. Reddy, D. J. Shippy, Alternative integration formulae for triangular finite elements, Int. J. Numer. Methods Eng 17 (1981) 1890-1896.
[14] Md. Shafiqual Islam and M. Alamgir Hossain, Numerical integration over an arbitrary quadrilateral region, Appl. Math. Computation 210 (2009) 515-524.
[15] Siraj-ul-Islam, Imran Aziz and Fazal Haq, A comparative study of numerical integration based on haar wavelets and hybrid functions, Computers and Mathematics with Applications, 59 (2010) No.6, 2026-2036.