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Reliability Analysis of unreliable $M^X/G/1$ Retrial Queue with Second Optional Service, Setup and Discouragement

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ABSTRACT

Retrial queues have been widely used to model many problems arising in telephone switching systems, telecommunication networks, computer networks and computer systems, etc.. In this paper an $M^x/G/1$ retrial queue with two phase services, discouragement and general setup time is being studied where the server is subject to breakdown during service. Primary customers join the system according to Poisson process and receive the service immediately if the server is available upon arrival. Otherwise, they enter a retrial orbit with some probability and are queued in the orbit. They repeat their demand after some random interval of time. The customers are allowed to balk upon arrival. All the customers who join the queue have to undergo the first essential service, whereas only some of them demand for the second optional service. Using generating function approach and supplementary variable method, the steady state solutions for some queueing and reliability measures of the system are obtained. The sensitivity analysis has been carried out to explore the effects of system parameters on various performance measures.

Keywords: Retrial queue, Batch arrivals, Two phase service, Unreliable server, Balking, Setup time, Supplementary variable, Generating function, Reliability.

INTRODUCTION

Retrial queueing system is characterized by the feature that the arriving customers who find the server busy join the retrial queue (orbit) to try again for their requests in random order and at random interval or leave the service area immediately. During last two decades considerable attention has paid to the analysis of queueing systems with repeated calls or customers. A single server retrial queueing system in which each customer has discrete service time has considered by Wu and Ke (2007). The time dependent system size probabilities have been studied by Parthasarathy and Sudesh (2007) for a retrial queue by employing continued functions. Amar (2009) derived an explicit formula for the generating function of the number of the customers in orbit for $M^x/G/1$ retrial queue and exhibited explicit forms of stochastic decomposition property.

The queueing systems with second optional service are characterized by the feature that all arrivals demand the first essential service, whereas some of them require second optional service which can also be provided by the same server. Stability conditions and steady state analysis were investigated by Dimitrion and Langaris (2010) for a repairable queueing model with two phase service and retrial customers.

The study of a queueing model with unreliable server is an important interdisciplinary topic in queueing theory and reliability theory, as it considers not only the queueing characteristics for the system but also reliability indices for the server. An M/G/1 retrial queueing system with disasters and unreliable server was investigated by Wang et al. (2007). A discrete time Geo/G/1 retrial queue has been studied by Wang and Zhao (2007) where the server provides two types of services and also subject to breakdowns. Multi server retrial queue with the Batch Markovian Arrival Process (BMAP) was considered by Kim et al. (2007). Wang and Zhang (2009) analyzed a discrete time single server retrial queue with geometrical arrival for both positive and negative customers in which the server is subject o breakdowns and repairs.

The study of reliability indices for unreliable server queueing system has been done by many researchers. By using supplementary variable technique some queueing and reliability characteristics of the $M^x/(G_1,G_2)/1$ retrial queueing system have been derived by Ke and Chang (2009).

The customers are said to be impatient if they tend to join the queue only when a short wait is expected and tend to remain in line if the wait has been sufficiently small. When queue length is sufficiently long or due to some other reason, the arriving customers would not like to join the queue; this behavior of customers is known as balking. Operating characteristics of an $M^{[x]}/G/1$ queueing system under a variant vacation policy, where the server leaves for a vacation as soon as the system is empty was studied by Ke (2007).

The retrial queue with second optional service, balking, server failures, repairs and setup is the subject of investigation in the present study. The rest of paper structured as follows. The model under investigation is described along with notations and assumptions in section 2. In section 3, we establish the steady state equations after introducing supplementary variables corresponding to elapsed service time, setup time and repair time. Section 4 provides mathematical analysis to obtain joint probabilities and marginal probabilities. Some queueing performance indices are discussed in section 5. Section 6 is devoted to the reliability indices of the server. Some particular cases are deduced in section 7. The sensitivity analysis in order to validate the analytical results is given in section 8. Conclusions are outlined in the final section 9.

2. MODEL DESCRIPTION

$M^x/G/1$ retrial queueing system with unreliable server, balking, setup and second optional service is considered by making the following assumptions:

- The customers arrive at the system according to a compound Poisson process with random batch size.
- If an arriving customer finds the server idle, he may obtain service immediately. There is a single unreliable server who provides two kinds of general heterogeneous services to the customers on a first come first served (FCFS) basis.
- The first essential service is needed to all arriving customers; the essential service time has general distribution. As soon as the essential service of a customer is completed, he may opt for second optional service with probability r or else with probability (1-r), leaves the system.
- The arriving customer on finding the server under busy, setup or broken down state must leave the service area and repeats its demand with rate θ after some random interval of time. The inter arrival time and retrial times of batches are exponentially distributed.
- The server may breakdown while servicing the customers. We assume that the life time of the server is exponentially distributed with rate α_1 and α_2 in case when he is rendering first essential service and second optional service, respectively.
- When the server breaks down, it is sent for repair. The repair time distributions for both service phases are arbitrarily distributed with rate β_1 and β_2.
- The customers may balk from retrial queue with different balking rates on finding the server in busy, setup or broken down states due to impatience.
- The server is recovered after the completion of the repair and starts service of the customers immediately.

NOTATIONS

λ	Mean arrival rate of the customers
X	Random variable denoting the batch size
a_i	Pr[X=i]
X(z)	Generating function for batch size X, i.e. $X(z)=\sum_{k=1}^{\infty}a_k z^k$
θ	Mean retrial rate of the customers
h_{P_i}, h_{S_i}, h_{R_i}	Joining probability of the customers from retrial queue when the server is busy, in setup and under repair state while rendering i^{th} phase (i=1,2) service
α_i (i=1,2)	Mean failure rate of the server in i^{th} phase service
μ_i, θ_i, β_i	Service rate, setup rate and repair rate (i=1,2)
$\mu_i(x)$, $\theta_i(y)$, $\beta_i(y)$	Hazard service rate, setup rate and repair rate
$w_i(x)$, $s_i(y)$, $b_i(y)$	Probability density functions for service time, setup time and repair time in case of i^{th} phase (i=1,2) service
$W_i(x)$, $S_i(x)$, $B_i(x)$	Distribution functions for i^{th} (i=1,2) phase of service time, setup time and repair time
$Q_n(t)$	Probability that there are n customers in the retrial queue at time t when the server is in idle state
$P_n^{(1)}(t,x)$	Probability that there are n customers in the retrial queue at time t when the server is busy with first essential service and elapsed service time lies in (x, x+dx)
$S_n^{(1)}(t,x,y)$	Probability that there are n customers in the retrial queue at time t when the server is in setup state for first essential service and the elapsed service time for the customer under service is equal to x, and elapsed setup time lies in (y, y+dy)
$R_n^{(1)}(t,x,y)$	Probability that there are n customers in the retrial queue at time t when the server is in repair state while broken down during first essential service and the elapsed service time for the customer under service is equal to x, and elapsed repair time lies in (y, y+dy)
$P_n^{(2)}(t)$	Probability that there are n customers in the retrial queue at time t when the server is busy in second optional service
$S_n^{(2)}(t,y)$	Probability that there are n customers in the retrial queue at time t when the server is in setup state, for second optional service and elapsed setup time lies in (y, y+dy)
$R_n^{(2)}(t,y)$	Probability that there are n customers in the retrial queue at time t when the server is broken down while rendering the second optional service and elapsed repair time lies in (y, y+dy)

The hazard rates are given by:

$$\mu_i(x)dx=\frac{dW_i(x)}{1-W_i(x)};\ \theta_i(y)dy=\frac{dS_i(y)}{1-S_i(y)};\ \beta_i(y)dy=\frac{dB_i(y)}{1-B_i(y)},\ i=1,2$$

3. GOVERNING EQUATIONS

We construct below the partial differential equations governing the model for the system state 'n' (n≥0) by assuming the elapsed service time, elapsed setup time and the elapsed repair time as supplementary variables:

$$\left(\frac{d}{dt}+\lambda+n\theta\right)Q_n(t)=\mu_2 P_0^{(2)}(t)+(1-r)\int_0^{\infty}P_0^{(1)}(t,x)\mu_1(x)dx \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)+\lambda h_{P_1}+\alpha_1\right)P_n^{(1)}(t,x)=\lambda h_{P_1}\sum_{k=1}^{n}a_k P_{n-k}^{(1)}(x)+\int_0^\infty R_n^{(1)}(t,x,y)\beta_1(y)dy,\quad n\geq 1 \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \varepsilon}+\theta_1(y)+\lambda h_{S_1}\right)S_n^{(1)}(t,x,y)=\lambda h_{S_1}\sum_{k=1}^{n}a_k S_{n-k}^{(1)}(t,x,y),\quad n\geq 1 \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta_1(y)+\lambda h_{R_1}\right)R_n^{(1)}(t,x,y)=\lambda h_{R_1}\sum_{k=1}^{n}a_k R_{n-k}^{(1)}(t,x,y),\quad n\geq 1 \tag{4}$$

$$\left(\frac{d}{dt}+\mu_2+\lambda h_{P_2}+\alpha_2\right)P_n^{(2)}(t)=\lambda h_{P_2}\sum_{k=1}^{n}a_k P_{n-k}^{(2)}(t)+\int_0^\infty R_n^{(2)}(t,y)\beta_2(y)dy+r\int_0^\infty P_n^{(1)}(t,x)\mu_1(x)dx,\quad n\geq 1 \tag{5}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \varepsilon}+\theta_2(y)+\lambda h_{S_2}\right)S_n^{(2)}(t,y)=\lambda h_{S_2}\sum_{k=1}^{n}a_k S_{n-k}^{(2)}(t,y),\quad n\geq 1 \tag{6}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta_2(y)+\lambda h_{R_2}\right)R_n^{(2)}(t,y)=\lambda h_{R_2}\sum_{k=1}^{n}a_k R_{n-k}^{(2)}(y),\quad n\geq 1 \tag{7}$$

The following boundary conditions are taken into consideration

$$P_n^{(1)}(t,0)=\mu_2 P_{n+1}^{(2)}(t)+(1-r)\int_0^\infty P_{n+1}^{(1)}(t,x)\mu_1(x)dx+(n+1)\theta Q_{n+1}(t),\quad n\geq 1 \tag{8}$$

$$P_0^{(1)}(t,0)=\mu_2 P_1^{(2)}(t)+(1-r)\int_0^\infty P_1^{(1)}(t,x)\mu_1(x)dx+\lambda\sum_{k=1}^{n}a_k Q_{n-k}(t) \tag{9}$$

$$S_n^{(1)}(t,x,0)=\alpha_1 P_n^{(1)}(t,x),\quad n\geq 1 \tag{10}$$

$$S_n^{(2)}(t,0)=\alpha_2 P_n^{(2)}(t),\quad n\geq 1 \tag{11}$$

$$R_n^{(1)}(x,0)=\int_0^\infty S_n^{(1)}(x,y)\theta_1(y)dy,\quad n\geq 1 \tag{12}$$

$$R_n^{(2)}(0)=\int_0^\infty S_n^{(2)}(y)\theta_2(y)dy,\quad n\geq 1 \tag{13}$$

For steady state equations, we define the probabilities as follows:

$$Q_n=\lim_{t\to 0}Q_n(t),\ P_n^{(1)}(x)=\lim_{t\to 0}P_n^{(1)}(t,x),\ P_n^{(2)}=\lim_{t\to 0}P_n^{(2)}(t),$$
$$R_n^{(1)}(x,y)=\lim_{t\to 0}R_n^{(1)}(t,x,y),\ R_n^{(2)}(y)=\lim_{t\to 0}R_n^{(2)}(t,y)$$

The steady state equations for the system state 'n' corresponding to equations. (1)- (7) are given by

$$(\lambda+n\theta)Q_n=\mu_2 P_0^{(2)}+(1-r)\int_0^\infty P_0^{(1)}(x)\mu_1(x)dx \tag{14}$$

$$\left(\frac{\partial}{\partial x}+\mu_1(x)+\lambda h_{P_1}+\alpha_1\right)P_n^{(1)}(x)=\lambda h_{P_1}\sum_{k=1}^{n}a_k P_{n-k}^{(1)}(x)+\int_0^\infty R_n^{(1)}(x,y)\beta_1(y)dy,\quad n\geq 1 \tag{15}$$

$$\left(\frac{\partial}{\partial \varepsilon}+\theta_1(y)+\lambda h_{S_1}\right)S_n^{(1)}(x,y)=\lambda h_{S_1}\sum_{k=1}^{n}a_k S_{n-k}^{(1)}(x,y),\quad n\geq 1 \tag{16}$$

$$\left(\frac{\partial}{\partial y}+\beta_1(y)+\lambda h_{R_1}\right)R_n^{(1)}(x,y)=\lambda h_{R_1}\sum_{i=1}^{n}a_k R_{n-k}^{(1)}(x,y),\quad n\geq 1 \tag{17}$$

$$(\mu_2+\lambda h_{P_2}+\alpha_2)P_n^{(2)}=\lambda h_{P_2}\sum_{k=1}^{n}a_k P_{n-k}^{(2)}+\int_0^\infty R_n^{(2)}(y)\beta_2(y)dx+r\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx,\quad n\geq 1 \tag{18}$$

$$\left(\frac{\partial}{\partial\varepsilon}+\theta_2(y)+\lambda h_{S_2}\right)S_n^{(2)}(y)=\lambda h_{S_2}\sum_{k=1}^{n}a_i S_{n-k}^{(2)}(y),\quad n\geq 1 \tag{19}$$

$$\left(\frac{\partial}{\partial y}+\beta_2(y)+\lambda h_{R_2}\right)R_n^{(2)}(y)=\lambda h_{R_2}\sum_{k=1}^{n}a_k R_{n-k}^{(2)}(y),\quad n\geq 1 \tag{20}$$

Under steady state, the boundary conditions are

$$P_n^{(1)}(0)=\mu_2 P_{n+1}^{(2)}+(1-r)\int_0^\infty P_{n+1}^{(1)}(x)\mu_1(x)dx+(n+1)\theta Q_{n+1},\ \ n\geq 1 \tag{21}$$

$$P_0^{(1)}(0)=\mu_2 P_1^{(2)}+(1-r)\int_0^\infty P_1^{(1)}(x)\mu_1(x)dx+\lambda\sum_{k=1}^{n}a_k Q_{n-k},\ \ n\geq 1 \tag{22}$$

$$S_n^{(1)}(x,0)=\alpha_1 P_n^{(1)}(x),\quad n\geq 1 \tag{23}$$

$$S_n^{(2)}(0)=\alpha_2 P_n^{(2)},\quad n\geq 1 \tag{24}$$

$$R_n^{(1)}(x,0)=\int_0^\infty S_n^{(1)}(x,y)\theta_1(y)dy,\quad n\geq 1 \tag{25}$$

$$R_n^{(2)}(0)=\int_0^\infty S_n^{(2)}(y)\theta_2(y)dy,\quad n\geq 1 \tag{26}$$

4. THE ANALYSIS

In order to provide the analytic solution, the following probability generating functions are defined as follows:

$$X(z)=\sum_{k=1}^{\infty}a_k z^k\ ,\ P_n^{(1)}(x,z)=\sum_{n=0}^{\infty}P_n^{(1)}(x)z^n\ ,\ P_n^{(2)}(z)=\sum_{n=0}^{\infty}P_n^{(2)}z^n\ ,\ S_n^{(1)}(x,y,z)=\sum_{n=0}^{\infty}S_n^{(1)}(x,y)z^n\ ,$$

$$S_n^{(2)}(y,z)=\sum_{n=0}^{\infty}S_n^{(2)}(y)z^n\ ,\ R_n^{(1)}(x,y,z)=\sum_{n=0}^{\infty}R_n^{(1)}(x,y)z^n\ ,\ R_n^{(2)}(y,z)=\sum_{n=0}^{\infty}R_n^{(2)}(y)z^n$$

Multiplying equations (14)-(26) by appropriate power of z^n and summing over n

$$(\lambda+n\theta)Q_n(z)=\mu_2 P_0^{(2)}(z)+(1-r)\int_0^\infty P_0^{(1)}(x,z)\mu_1(x)dx,\ \ n\geq 0 \tag{27}$$

$$\left(\frac{\partial}{\partial x}+\mu_1(x)+\lambda h_{P_1}+\alpha_1\right)P_n^{(1)}(x,z)=\lambda h_{P_1}X(z)P_n^{(1)}(x,z)\ +\int_0^\infty R_n^{(1)}(x,y,z)\beta_1(y)dy,\ \ n\geq 1 \tag{28}$$

$$\left(\frac{\partial}{\partial\varepsilon}+\theta_1(y)+\lambda h_{S_1}\right)S_n^{(1)}(x,y,z)=\lambda h_{S_1}X(z)S_n^{(1)}(x,y,z),\ \ n\geq 1 \tag{29}$$

$$\left(\frac{\partial 1}{\partial y}+\beta_1(y)+\lambda h_{R_1}\right)R_n^{(1)}(x,y,z)=\lambda h_{R_1}X(z)R_n^{(1)}(x,y,z),\ \ n\geq 1 \tag{30}$$

$$\left(\mu_2+\lambda h_{P_2}+\alpha_2\right)P_n^{(2)}(z)=\lambda h_{P_2}X(z)P_n^{(2)}+\int_0^\infty R_n^{(2)}(y,z)\beta_2(y)dy+r\int_0^\infty P_n^{(1)}(x,z)\mu_1(x)dx,\ \ n\geq 1 \tag{31}$$

$$\left(\frac{\partial}{\partial\varepsilon}+\theta_2(y)+\lambda h_{S_{21}}\right)S_n^{(2)}(y,z)=\lambda h_{S_2}X(z)S_n^{(2)}(y,z),\ \ n\geq 1 \tag{32}$$

$$\left(\frac{\partial}{\partial y}+\beta_2(y)+\lambda h_{R_2}\right)R_n^{(2)}(y,z)=\lambda h_{R_2}X(z)R_n^{(2)}(y,z),\ \ n\geq 1 \tag{33}$$

The boundary conditions yield

$$P_n^{(1)}(z)=\mu_2 P_n^{(2)}(z)z^{-1}+(1-r)z^{-1}\int_0^\infty P_1^{(1)}(x,z)\mu_1(x)dx+\lambda X(z)Q_n(z)+(n+1)\theta z^{-1}Q_n(z)$$
$$-(1-r)z^{-1}\int_0^\infty P_0^{(1)}(x,z)\mu_1(x)dx-\mu_2 z^{-1}P_0^{(2)}(z),\quad n\geq 1 \tag{34}$$

$$S_n^{(1)}(x,0,z)=\alpha_1 P_n^{(1)}(x,z),\quad n\geq 1 \tag{35}$$

$$S_n^{(2)}(0,z)=\alpha_2 P_n^{(2)}(z),\quad n\geq 1 \tag{36}$$

$$R_n^{(1)}(x,0,z)=\int_0^\infty S_n^{(1)}(x,y,z)\theta_1(y)dy,\quad n\geq 1 \tag{37}$$

$$R_n^{(2)}(0,z)=\int_0^\infty S_n^{(2)}(y)\theta_2(y,z)dy,\quad n\geq 1 \tag{38}$$

Theorem 1: The partial probability generating functions when the server is in busy state, under setup state and in repair state respectively, are given by

$$P_n^{(1)}(x,z)=P_n^{(1)}(0,z)e^{-\phi_1(z)x}\overline{W_1(x)} \tag{39}$$

$$P_n^{(2)}(z)=\frac{rw*\phi_1(z)}{\mu_2+\phi_1(z)}P_n^{(1)}(z) \tag{40}$$

$$S_n^{(1)}(x,y,z)=\alpha_1 P_n^{(1)}(x,z)e^{-\lambda h_{S_1}(1-X(z))y}\overline{S_1(y)} \tag{41}$$

$$S_n^{(2)}(y,z)=\alpha_2 P_n^{(2)}(z)e^{-\lambda h_{S_2}(1-X(z))y}\overline{S_2(y)} \tag{42}$$

$$R_n^{(1)}(x,y,z)=\alpha_1 P_n^{(1)}(x,z)\,s*(\lambda h_{s_1}(1-X(z))ye^{-\lambda h_{R_1}(1-X(z))y}\,\overline{B_1(y)} \tag{43}$$

$$R_n^{(2)}(y,z)=\alpha_2 P_n^{(2)}(z)\,s*(\lambda h_{s_2}(1-X(z))ye^{-\lambda h_{R_2}(1-X(z))y}\,\overline{B_2(y)} \tag{44}$$

where $\overline{W_i(x)}=1-W_i(x)$, $\overline{S_i(y)}=1-S_i(y)$ and $\overline{B_i(y)}=1-B_i(y)$, i=1,2.

Proof: For proof see appendix *A-I*.

Theorem 2: The marginal generating functions are obtained as

$$P_n^{(1)}(z)=\left\{\frac{1-w*\phi_1(z)}{\phi_1(z)}\right\}P_n^{(1)}(0,z) \tag{45}$$

$$P_n^{(2)}(z)=\left\{\frac{rw*\phi_1(z)}{\mu_2+\phi_2(z)}\right\}P_n^{(1)}(0,z) \tag{46}$$

$$S_n^{(1)}(z)=\alpha_1\left\{\frac{1-w*\phi_1(z)}{\phi_1(z)}\right\}\left\{\frac{1-s^*(\lambda h_{s_1}(1-X(z))y\}}{\lambda h_{s_1}(1-X(z))}\right\}P_n^{(1)}(0,z) \tag{47}$$

$$S_n^{(2)}(z)=\alpha_2\left\{\frac{rw*\phi_1(z)}{\mu_2+\phi_2(z)}\right\}\left\{\frac{1-s^*(\lambda h_{s_2}(1-X(z))y\}}{\lambda h_{s_2}(1-X(z))}\right\}P_n^{(1)}(0,z) \tag{48}$$

$$R_n^{(1)}(z)=\alpha_1\,s^*\{\lambda h_{S_1}(1-X(z))y\}\left\{\frac{1-w*\phi_1(z)}{\phi_1(z)}\right\}\left\{\frac{1-b^*(\lambda h_{R_1}(1-X(z))y\}}{\lambda h_{R_1}(1-X(z))}\right\}P_n^{(1)}(0,z) \tag{49}$$

$$R_n^{(2)}(z)=\alpha_2\,s^*\{\lambda h_{S_2}(1-X(z))y\}\left\{\frac{rw*\phi_1(z)}{\mu_2+\phi_2(z)}\right\}\left\{\frac{1-b^*(\lambda h_{R_2}(1-X(z))y\}}{\lambda h_{R_2}(1-X(z))}\right\}P_n^{(1)}(0,z) \tag{50}$$

Proof: For proof see appendix A-*II*.

Theorem 3: The probability generating functions for the number of customers in the retrial queue and in the system are

$$R(z)=\frac{[\phi_1 B+h_{R_2}h_{S_2}A_1(\lambda X(z)-\theta-\lambda)(1-w*\phi_1(z))(\mu_2+\phi_2(z))]Q_n(z)}{\phi_1(z)B}+\frac{[rh_{R_1}h_{S_1}A_2 w*\phi_1(z)]Q_n(z)}{\phi_1(z)B} \tag{51}$$

$$L(z)=\frac{[\phi_1 B+z\,h_{R_2}h_{S_2}A_1(\lambda X(z)-\theta-\lambda)(1-w*\phi_1(z))(\mu_2+\phi_2(z))]Q_n(z)}{\phi_1(z)B}+\frac{[zrh_{R_1}h_{S_1}A_2 w*\phi_1(z)]Q_n(z)}{\phi_1(z)B} \tag{52}$$

where

$$A_i=h_{R_i}h_{S_i}+\alpha_i h_{R_i}(1-S^*(\lambda h_{s_i}(1-X(z))y\})+\alpha_i h_{S_i}(1-S^*(\lambda h_{s_i}(1-X(z))y\})(1-B^*(\lambda h_{R_i}(1-X(z))y)$$
$$B=h_{R_1}h_{R_2}h_{S_1}h_{S_2}(1-X(z))\left\{z-w^*\phi_1(z))(\mu_2+\phi_2(z))+r\phi_2(z)w^*\phi_1(z)\right\}$$

Proof: We use the marginal probabilities and the following relation to obtain R(z) and L(z)

$$R(z)=Q_n(z)+P_n^{(1)}(z)+P_n^{(2)}(z)+S_n^{(1)}(z)+S_n^{(2)}(z)+R_n^{(1)}(z)+R_n^{(2)}(z)$$

$$L(z)=Q_n(z)+zP_n^{(1)}(z)+zP_n^{(2)}(z)+zS_n^{(1)}(z)+zS_n^{(2)}(z)+zR_n^{(1)}(z)+zR_n^{(2)}(z)$$

Theorem 4: The expected number of customers in the retrial queue is

$$E[L]=\frac{h_{R_1}h_{R_2}h_{S_1}h_{S_2}}{\theta(1+r\lambda\xi\mu_2 E[X]E[W_1])}\{\xi E[W_1](r-\theta+\lambda)+r\theta\alpha_2(1+E[B_1])+\{\lambda E[X(X-1)]\xi+\lambda\alpha_1 E[X]h_{S_1}\}$$
$$\{E^2[S_1]+E^2[B_1]\}+2\alpha_1\lambda^2 h_{R_1}h_{S_1}E^2[X]E[B_1]E[S_2] \tag{53}$$

where $\xi=h_{P_1}+\alpha_1 h_{S_1}E[S_1]+\alpha_1 h_{r_1}E[B_1]$

The expected number of customers in the system is

$$E[R]=E[L]+\frac{h_{R_1}h_{R_2}h_{S_1}h_{S_2}E[W_1](r+\mu_2\theta)}{2E[X]\xi\{(1+r\mu_2 E[W_1]E[W_1])\xi E[X]\}} \tag{54}$$

Proof: The expressions for the expected customers in the retrial queue and in the system are obtained by using

$E[L]=\left.\frac{\partial L(z)}{\partial z}\right|_{z=1}$ and $E[R]=\left.\frac{\partial R(z)}{\partial z}\right|_{z=1}$ where L(z) and R(z) are given in equations (51) and (52).

5. PERFORMANCE MEASURES

Theorem 5: The probability that the server is in idle, busy with i[th] (i=1,2) phase of service, setup and under repair, respectively are given by

$$P_{B_1}=\frac{\mu_2\psi}{1+\psi(\mu_2+r\mu_1)\sum_{i=1}^{2}(1+\frac{\alpha_i}{\theta_i}+\frac{\alpha_i}{\beta_i})} \tag{55}$$

$$P_{B_2}=\frac{r\mu_1\psi}{1+\psi(\mu_2+r\mu_1)\sum_{i=1}^{2}(1+\frac{\alpha_i}{\theta_i}+\frac{\alpha_i}{\beta_i})}$$

(56)

$$P_{S_1} = \frac{\alpha_1 \mu_2 \psi}{\theta_1 \left\{ 1 + \psi(\mu_2 + r\mu_1) \sum_{i=1}^{2} (1 + \frac{\alpha_i}{\theta_i} + \frac{\alpha_i}{\beta_i}) \right\}} \qquad (57)$$

$$P_{S_2} = \frac{r\alpha_2 \mu_1 \psi}{\theta_2 \left\{ 1 + \psi(\mu_2 + r\mu_1) \sum_{i=1}^{2} (1 + \frac{\alpha_i}{\theta_i} + \frac{\alpha_i}{\beta_i}) \right\}} \qquad (58)$$

$$P_{R_1} = \frac{\alpha_1 \mu_2 \psi}{\beta_1 \left\{ 1 + \psi(\mu_2 + r\mu_1) \sum_{i=1}^{2} (1 + \frac{\alpha_i}{\theta_i} + \frac{\alpha_i}{\beta_i}) \right\}} \qquad (59)$$

$$P_{R_2} = \frac{r\alpha_2 \mu_1 \psi}{\beta_2 \left\{ 1 + \psi(\mu_2 + r\mu_1) \sum_{i=1}^{2} (1 + \frac{\alpha_i}{\theta_i} + \frac{\alpha_i}{\beta_i}) \right\}} \qquad (60)$$

where $\psi = \frac{\lambda E[X]}{(\mu_2 + r\varepsilon_2)\mu_1 - \mu_2 \varepsilon_1}$.

Proof: For proof see appendix A-*III*.

6. RELIABILITY INDICES

In order to analyze reliability indices, we consider setup and breakdown states as absorbing states. By using the same notations as in the previous section, we can get the following set of equations:

$$\left(\frac{d}{dt} + \lambda + n\theta \right) Q_n(t) = \mu_2 P_0^{(2)}(t) + (1-r) \int_0^\infty P_0^{(1)}(t,x)\mu_1(x)dx \qquad (61)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_1(x) + \lambda h_{P_1} + \alpha_1 \right) P_n^{(1)}(t,x) = \lambda h_{P_1} \sum_{k=1}^{n} a_i P_{n-k}^{(1)}(x),\ n \ge 1 \qquad (62)$$

$$\left(\frac{d}{dt} + \mu_2 + \lambda h_{P_2} + \alpha_2 \right) P_n^{(2)}(t) = \lambda h_{P_2} \sum_{k=1}^{n} a_k P_{n-k}^{(2)}(t) + r\int_0^\infty P_n^{(1)}(t,x)\mu_1(x)dx,\ n \ge 1 \qquad (63)$$

The boundary conditions are:

$$P_n^{(1)}(t,0) = \mu_2 P_{n+1}^{(2)}(t) + (1-r)\int_0^\infty P_{n+1}^{(1)}(t,x)\mu_1(x)dx + (n+1)\theta Q_{n+1}(t),\ n \ge 1 \qquad (64)$$

$$P_0^{(1)}(t,0) = \mu_2 P_1^{(2)}(t) + (1-r)\int_0^\infty P_1^{(1)}(t,x)\mu_1(x)dx + \lambda \sum_{k=1}^{n} a_k Q_{n-k}(t) \qquad (65)$$

Taking Laplace transform, we get

$$(s + \lambda + n\theta) Q^*_n(s) - 1 = \mu_2 P^{*(2)}_0(s) + (1-r)\int_0^\infty P^{*(1)}_0(x,s)\mu_1(x)dx \qquad (66)$$

$$\frac{\partial}{\partial x} P^{*(1)}_n(s,x) + (s + \mu_1(x) + \lambda h_{P_1} + \alpha_1) P^{*(1)}_n(s,x) = \lambda h_{P_1} \sum_{k=1}^{n} a_k P^{*(1)}_{n-k}(s,x) \qquad (67)$$

Now the boundary conditions yield

$$(s + \mu_2 + \lambda h_{P_2} + \alpha_2) P^{*(2)}_n(s) = \lambda h_{P_2} \sum_{k=1}^{n} a_k P^{*(2)}_{n-k}(s) + r\int_0^\infty P^{*(1)}_n(s,x)\mu_1(x)dx \qquad (68)$$

$$P^{*(1)}_n(s,0) = \mu_2 P^{*(2)}_{n+1}(s) + (1-r)\int_0^\infty P^{*(1)}_{n+1}(s,x)\mu_1(x)dx + (n+1)\theta Q^*_{n+1}(s) \qquad (69)$$

The probability generating functions in the form of Laplace transformation can be written as

$$Q_n^*(s,z)=\sum_{n=0}^{\infty}Q_n^*(s)z^n \quad P_n^{*(1)}(s,x,z)=\sum_{n=0}^{\infty}P_n^{*(1)}(s,x)z^n\ ,\ \ P_n^{*(1)}(s,0,z)=\sum_{n=0}^{\infty}P_n^{*(1)}(s,0)z^n \quad P_n^{*(2)}(s,z)=\sum_{n=0}^{\infty}P_n^{*(2)}(s)z^n$$

Multiplying equations (66)-(65) by suitable powers of z and summing, we get

$$Q^*(s)=\frac{1}{\lambda+s+\theta-\lambda X(z_s)} \tag{70}$$

$$P_n^{*(1)}(s,x,z)=\left\{\frac{1-w^*a_1(s,z)}{a_1(s,z)}\right\}P_n^{*(1)}(s,0,z) \tag{71}$$

$$P_n^{*(2)}(s,z)=\left\{\frac{rw^*a_1(s,z)}{a_2(s,z)+\mu_2}\right\}P_n^{*(1)}(s,0,z) \tag{72}$$

where

$$a_i(s,z)=s+\alpha_i-\lambda h_{P_i}-\lambda h_{P_i}X(z),\ i=1,2$$

$$P_n^{*(1)}(s,0,z)=\left\{\frac{(a_2+\mu_2)\{1+\lambda X(z)-\lambda-s-\theta\}}{(a_2+\mu_2)(z-w^*a_1(s,z))-ra_1(s,z)w^*a_1(s,z)}\right\}Q_n^*(s)$$

Let z_s be the root of equation

$$x=w^*a_1(s,z)\left\{1+\frac{ra_1(s,z)}{a_2(s,z)+\mu_2}\right\}.$$

Now we derive some reliability indices as follows:

(i) The availability of the server under steady state is

$$\begin{aligned}A&=\sum_{n=0}^{\infty}Q_n+\sum_{n=0}^{\infty}P_n^{(1)}(1)dx+\sum_{n=0}^{\infty}P_n^{(2)}(1)dx\\ &=\frac{1+\psi(\mu_2+r\mu_1)}{1+\psi(\mu_2+r\mu_1)\sum_{i=1}^{2}\left\{1+\frac{\alpha_i}{\theta_i}+\frac{\alpha_i}{\beta_i}\right\}}\end{aligned} \tag{73}$$

(ii) The failure frequency of the server under steady state can be obtained as

$$\begin{aligned}f&=\sum_{n=0}^{\infty}\left\{\int_0^{\infty}\alpha_1P_n^{(1)}(x)dx+\int_0^{\infty}\alpha_2P_n^{(2)}(x)dx\right\}\\ &=\lim_{z\to1}\left\{\int_0^{\infty}\alpha_1P_n^{(1)}(x,z)dx+\int_0^{\infty}\alpha_2P_n^{(2)}(z)dx\right\}\\ &=\frac{(\alpha_1\mu_2+\alpha_2\mu_1)\psi}{1+\psi(\mu_2+r\mu_1)\sum_{i=1}^{2}\left\{1+\frac{\alpha_i}{\theta_i}+\frac{\alpha_i}{\beta_i}\right\}}\end{aligned} \tag{74}$$

(iii) The mean time to failure is given by

$$\begin{aligned}MTTF&=\int_0^{\infty}R(t)dt=[R^*(s)]_{s=0}\\ &=Q^*(0)+(1-\theta)Q^*(0)\left\{\frac{(1-w^*(\alpha_1))(\alpha_2+\mu_2)+r\alpha_1w^*(\alpha_1)}{\{(\alpha_2+\mu_2)(1-w^*(\alpha_1))-r\alpha_1w^*(\alpha_1)\}}\right\}\end{aligned} \tag{75}$$

7. SENSITIVITY ANALYSIS

The graphical presentations of E[R] and E[L] have been done in figures 1-6. For numerical results summarized in tables 1-4, we set default parameters as

We have plotted the graphs in figs 1-6 for different service time distributions, namely (a) $M/E_4/1$ (b) M/D/1 (c) M/γ/1. Figures 1-4 depict the variation in the expected number of customers in the retrial queue E[R] and in the system E[L] for both homogeneous and heterogeneous failure rates by continuous and discrete curves, respectively, for different values of repair rates and different sets for balking probabilities, respectively by varying the arrival rate λ. The balking parameters (i.e. joining probability) chosen for different sets are as follows:

Set I: $h_{P1}= h_{P2} =0.7$, $hs1=h_{s2}=0.8$, $h_{R1}=h_{R2}=0.9$

Set II: $h_{P1}= h_{P2} =0.3$, $hs_1=hs_2=0.4$, $h_{R1}=h_{R2}=0.6$

Set III: $h_{P1}= h_{P2} =0.1$, $hs_1=hs_2=0.3$, $h_{R1}=h_{R2}=0.4$

Set IV: $h_{P1}= h_{P2} =1$, $hs_1=hs_2=1$, $h_{R1}=h_{R2}=1$

From figures 1-6, it can be seen that E[R] and E[L] increase first gradually and then significantly with the increase in arrival rate λ. The expected number of customers in the retrial queue E[R] and in the system queue E[L] increase almost linearly for lower values of arrival rates and then a sharp increment can be found. It can be noticed in figures 1 (a-c) and 2 (a-c) that E[L] and E[R] both increase with the increment in the failure rates. There is noteworthy effect of repair rates on E[R] and E[L] as can be seen in figures 3(a-c) and 4(a-c); as we increase the repair rate, the expected number of customers in the retrial queue E[R] and in the system queue E[L] demonstrate the decreasing trends. A notable increasing effect of joining probabilities on E[R] and E[L] can be noticed from figures 5 (a-c) and 6 (a-c).

In all figs, we see that E[R] and E[L] are higher for heterogeneous arrival rates in comparison to homogeneous arrival rates. The increasing (decreasing) pattern of E[R] and E[L] for increasing values of arrival rate, failure rate, repair rate and joining probabilities tally with physical experiences. For heavy traffic, the effects are more prominent which is same as we expect for the real time system.

8. CONCLUDING REMARKS

In this work, we have studied balking aspects while predicting the performance of unreliable $M^x/G/1$ queueing systems with second optional service. By using the supplementary variable method, we have modeled the system as a Markov chain and obtained stationary queueing and reliability measures of interest.

Batch arrival queueing model with retrials has potential applicability in many real world congestion situations. In our investigations we have incorporated the server breakdown which is an unavoidable phenomenon for any queueing systems. Moreover, the optional services and retrial attempts considered can be realized in queueing models while modeling many practical applications related to computer sciences, communication, production, human- computer interactions and so on. The incorporation of more realistic assumptions namely (i) bulk arrival (ii) retrial attempts (iii) unreliable server (iv) balking, altogether make our model more versatile and robust than previous models. The numerical illustrations given provide an insight regarding computational tractability of the analytical results established for the concerned model.

APPENDIX

A-I: Proof of theorem 1:

Solution of eq. (29) gives

$$S_n^{(1)}(x,y,z)=\alpha_1 S_n^{(1)}(x,0,z)\,e^{-\lambda h_{S_1}(1-X(z))y}\,\overline{S_1(y)} \qquad \text{(A.1)}$$

By using (35) in (29), we get

$$S_n^{(1)}(x,y,z)=\alpha_1 P_n^{(1)}(x,z)\,e^{-\lambda h_{S_1}(1-X(z))y}\,\overline{S_1(y)} \qquad (A.2)$$

Similarly from equations (32) and (40)

$$S_n^{(2)}(y,z)=\alpha_2 P_n^{(2)}(z)\,e^{-\lambda h_{S_2}(1-X(z))y}\,\overline{S_2(y)} \qquad (A.3)$$

Equations (30) and (33) give

$$R_n^{(1)}(x,y,z)=R_n^{(1)}(x,0,z)e^{-\lambda h_{R_1}(1-X(z))y}\,\overline{B_1(y)} \qquad (A.4)$$

$$R_n^{(2)}(y,z)=R_n^{(2)}(0,z)e^{-\lambda h_{R21}(1-X(z))y}\,\overline{B_2(y)} \qquad (A.5)$$

From equations (37), (A.1) and (A.4), we obtain

$$R_n^{(1)}(x,y,z)=\alpha_1 P_n^{(1)}(x,z)\,s*(\lambda h_{s_1}(1-X(z))ye^{-\lambda h_{R_1}(1-X(z))y}\,\overline{B_1(y)} \qquad (A.6)$$

where $\int_0^\infty e^{-\lambda h_{S_i}(1-X(z))y}ds_i(y)=S_i^*(\lambda h_{S_i}(1-X(z)))$.

From equations (A.3), (A.4) and (38), we have

$$R_n^{(2)}(y,z)=\alpha_2 P_n^{(2)}(z)\,s*(\lambda h_{s_2}(1-X(z))ye^{-\lambda h_{R21}(1-X(z))y}\,\overline{B_2(y)} \qquad (A.7)$$

By using equations (28) in (A.7), we get

$$P_n^{(1)}(x,z)=P_n^{(1)}(0,z)\,e^{-\phi_1(z)x}\,\overline{W_1(x)} \qquad (A.8)$$

A-II: Proof of theorem 2:

Equations (31), (A.7) and (A.8) give

$$P_n^{(2)}(z)=\left\{\frac{rw*\phi_1(z)}{\mu_2+\phi_2(z)}\right\}P_n^{(1)}(0,z) \qquad (A.9)$$

Integrating equation (A.8) by parts, we get

$$P_n^{(1)}(z)=\left\{\frac{1-w*\phi_1(z)}{\phi_1(z)}\right\}P_n^{(1)}(0,z) \qquad (A.10)$$

Again from equation (A.1) and (A.9), we get

$$S_n^{(1)}(z)=\alpha_1\left\{\frac{1-w*\phi_1(z)}{\phi_1(z)}\right\}\left\{\frac{1-S^*(\lambda h_{s_1}(1-X(z))y\}}{\lambda h_{s_1}(1-X(z))}\right\}P_n^{(1)}(0,z) \qquad (A.11)$$

On integrating equation (A.3) by parts and using equation (A.9), we obtain

$$S_n^{(2)}(z)=\alpha_2\left\{\frac{rw*\phi_1(z)}{\mu_2+\phi_2(z)}\right\}\left\{\frac{1-S^*(\lambda h_{s_2}(1-X(z))y\}}{\lambda h_{s_2}(1-X(z))}\right\}P_n^{(1)}(0,z) \qquad (A.12)$$

Now from equations (A.7) and (A.9), we get

$$R_n^{(1)}(z)=\alpha_1\,S^*\{\lambda h_{S_1}(1-X(z))y\}\left\{\frac{1-w*\phi_1(z)}{\phi_1(z)}\right\}\left\{\frac{1-B^*(\lambda h_{R_1}(1-X(z))y\}}{\lambda h_{R_1}(1-X(z))}\right\}P_n^{(1)}(0,z) \qquad (A.13)$$

$$R_n^{(2)}(z)=\alpha_2\,S^*\{\lambda h_{S_2}(1-X(z))y\}\left\{\frac{rw*\phi_1(z)}{\mu_2+\phi_2(z)}\right\}\left\{\frac{1-B^*(\lambda h_{R_2}(1-X(z))y\}}{\lambda h_{R_2}(1-X(z))}\right\}P_n^{(1)}(0,z) \qquad (A.14)$$

A-III Proof of theorem 4:

To obtain required probabilities, we use

$$P_{B_i} = \lim_{z\to 1} P_n^{(i)}(z),\ i = 1,2 \quad (A.15)$$

$$P_{S_i} = \lim_{z\to 1} S_n^{(i)}(z),\ i = 1,2 \quad (A.16)$$

$$P_{R_i} = \lim_{z\to 1} R_n^{(i)}(z),\ i = 1,2 \quad (A.17)$$

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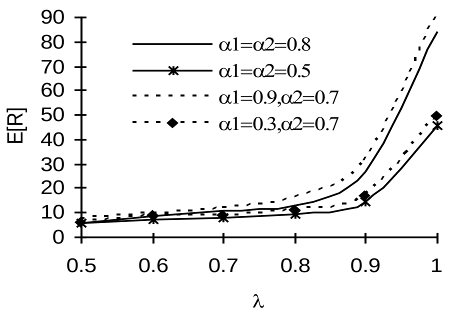

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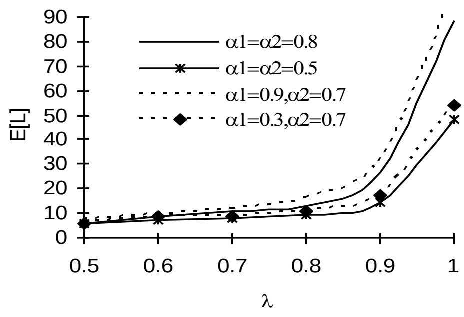

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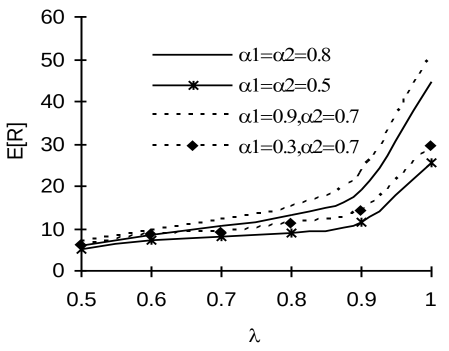

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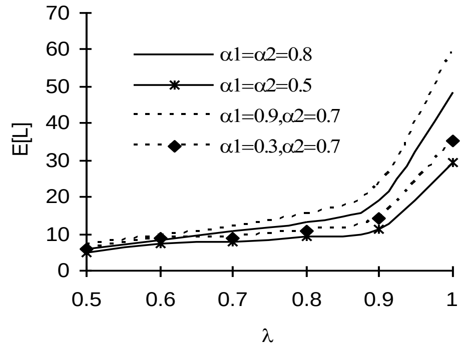

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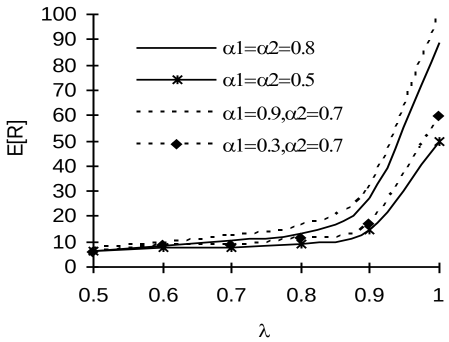

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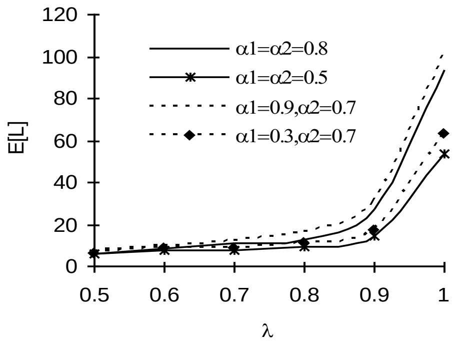

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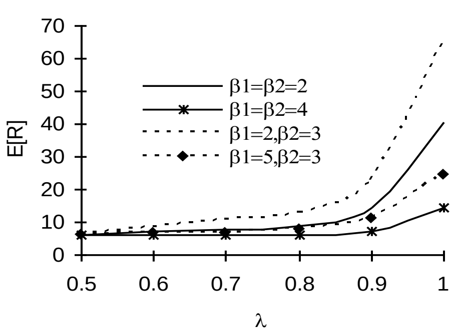
Fig 1: E[R] vs λ for (a) $M/E_4/1$ (b) M/D/1 (c) M/γ/1

Fig 2: E[L] vs λ for (a) $M/E_4/1$ (b) M/D/1 (c) M/γ/1



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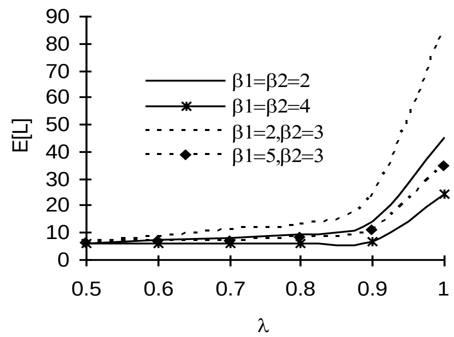

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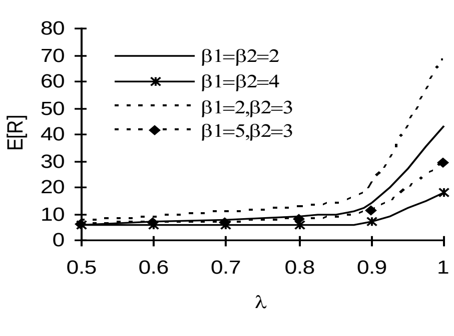

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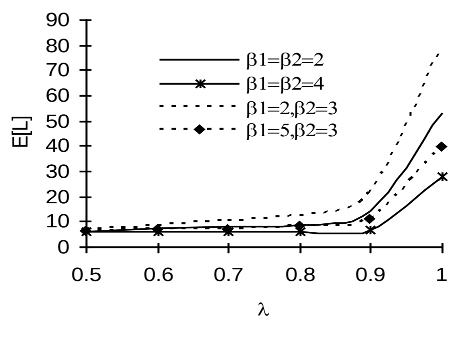

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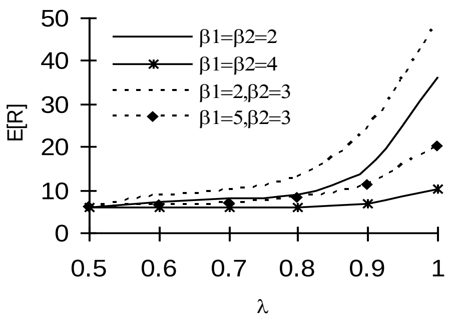

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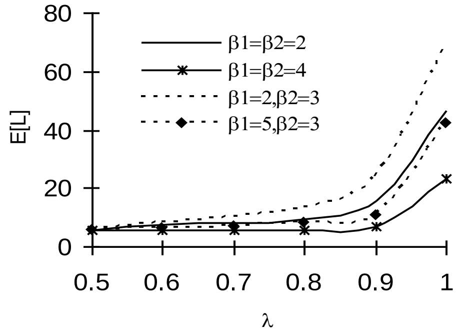
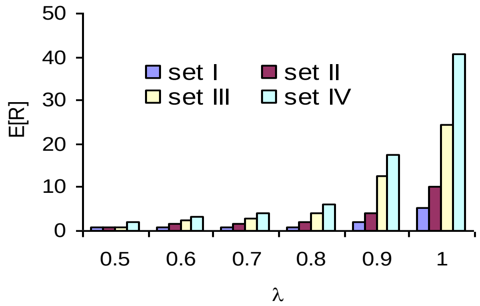


(c)

Fig 3: E[R] vs λ for (a) $M/E_4/1$
(b) M/D/1 (c) M/γ/1

Fig 4: E[L] vs λ for (a) $M/E_4/1$
(b) M/D/1 (c) M/γ/1



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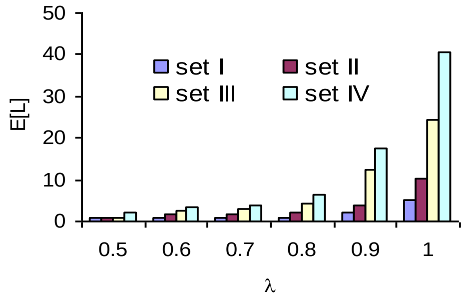

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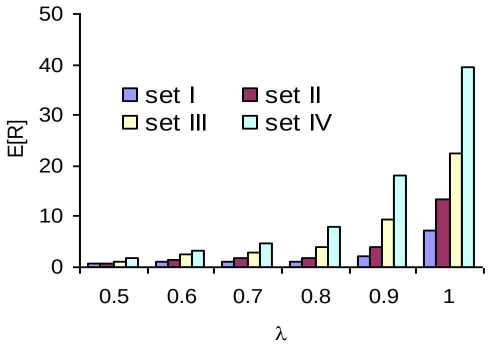

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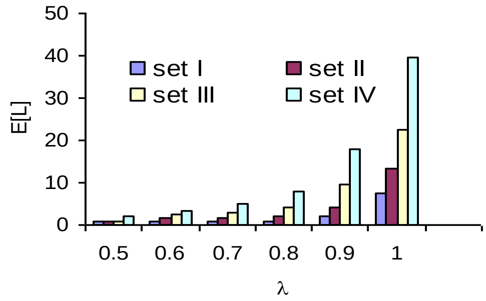

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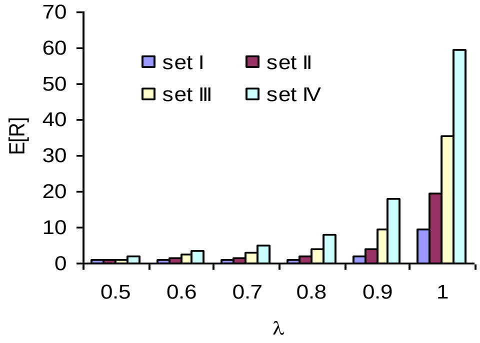

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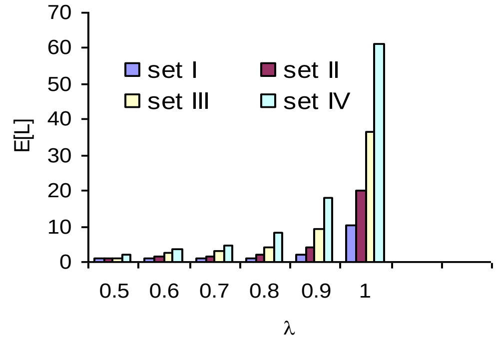

(c)

Fig 5: E[R] vs λ for (a) $M/E_4/1$

Fig 6: E[L] vs λ for (a) $M/E_4/1$

(b) M/D/1 (c) M/γ/1

(b) M/D/1 (c) M/γ/1